On the relation between 2 and ∞ in Galois cohomology of number fields

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1 Introduction

Number theorist's nightmare, the prime number 2, frequently causes technical problems and requires additional efforts. In Galois cohomology the problems with $p = 2$ are essentially due to the fact that the decomposition groups of the real places are 2-groups and so the case of a totally imaginary number field is comparatively easier to deal with.

A classical object of study in number theory is Galois groups with restricted ramification. For a number field k, a set S of primes of k and a prime number p, one is interested in the Galois group $G_S(p) = G(k_S(p)|k)$ of the maximal p-extension $k_S(p)$ of k which is unramified outside S. If S is empty, then $G_S(p)$ is the Galois group of the so-called p-class field tower of k and, besides the fact that it can be infinite (Golod-Šafarevič), not much is known about this group. The situation is easier in the case that S contains the set S_p of primes dividing p, where the cohomological dimension of $G_S(p)$ is known to be less than or equal to two (cf. [9], (8.3.17), (10.4.9)). However, there is an exception: if $p = 2$ and k has at least one real place. If, in this exceptional case, S contains all real places, then these places become complex in $k_S(2)$ and therefore $G_S(2)$, containing involutions, has infinite cohomological dimension. Furthermore, the virtual cohomological dimension $\operatorname{vcd} G_S(2)$ is less than or equal to two in this case, i.e. $G_S(2)$ has an open subgroup U with $\operatorname{cd} U \leq 2$. The case when not all real places are in S has been open so far and is the subject of this paper.

Theorem 1 *Let k be a number field and let S be a set of primes of k which contains all primes dividing 2. If no real prime is in S, then $\operatorname{cd} G_S(2) \leq 2$. If S contains real primes, then they become complex in $k_S(2)$ and $\operatorname{cd} G_S(2) = \infty$, $\operatorname{vcd} G_S(2) \leq 2$.*

If S is finite, then $H^i(G_S(2)) := H^i(G_S(2), \mathbb{Z}/2\mathbb{Z})$ is finite for all i and

$$\chi_2(G_S(2)) = -r_2,$$

where $\chi_2(G_S(2)) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_2} H^i(G_S(2))$ is the second partial Euler characteristic and r_2 is the number of complex places of k.

The key for the proof of theorem 1 is the following theorem 2 in the case $p = 2$ and $T = S \cup S_{\mathbb{R}}$, where $S_{\mathbb{R}}$ is the set of real places of k. Theorem 2 is the number theoretical analogue of Riemann's existence theorem and was previously known under the assumption that p is odd or that S contains $S_{\mathbb{R}}$ (see [9], (10.5.1)).

Theorem 2 *Let k be a number field, p a prime number and $T \supset S \supseteq S_p$ sets of primes of k. Then the canonical homomorphism*

$$\mathop{\ast}_{\mathfrak{p} \,\in\, T \smallsetminus S(k_S(p))} T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) \longrightarrow G(k_T(p)|k_S(p))$$

is an isomorphism. Here $T(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}) \subset G(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}})$ is the inertia group and $\ast$ denotes the free pro-p-product.

Since the cyclotomic $\mathbb{Z}_2$-extension $k_\infty(2)$ of k is contained in $k_{S_2}(2)$, the group $G_{S_2}(2)$ is infinite, in particular, it is nontrivial. Hence, for $S \supseteq S_2$ and $S \cap S_{\mathbb{R}} = \varnothing$, the group $G_S(2)$ is of cohomological dimension 1 or 2. The next theorem gives a criterion for which case occurs. In condition (3) below, $\mathrm{Cl}_S^0(k)(2)$ denotes the 2-torsion part of the S-ideal class group in the narrow sense of k.

Theorem 3 *Assume that $S \supseteq S_2$ and $S \cap S_{\mathbb{R}} = \varnothing$. Then $\mathrm{cd}\, G_S(2) = 1$ if and only if the following conditions (1)–(3) hold.*

(1) $S_2 = \{\mathfrak{p}_0\}$, *i.e. there exist exactly one prime dividing 2 in k.*
(2) $S = \{\mathfrak{p}_0\} \cup \{\text{complex places}\}$.
(3) $\mathrm{Cl}_S^0(k)(2) = 0$.

In this case, $G_S(2)$ is a free pro-2-group of rank $r_2 + 1$ and $\mathfrak{p}_0$ does not split in $k_{S \cup S_{\mathbb{R}}}(2)$. In particular, if k is totally real and $G_S(2)$ is free, then $k_S(2) = k_\infty(2)$.

Let k be a number field, p a prime number and $S \supseteq S_p$ a set of places of k. A (necessarily infinite) extension $K|k$ is called p-S-closed if it has no p-extension which is unramified outside S. If p is odd and K is p-S-closed, then the group $\mathrm{Cl}_S(K(\mu_p))(p)(j)^{G(K(\mu_p)|K)}$ is trivial for $j = 0, -1$, where μ_p is the group of p-th roots of unity, (p) denotes the p-torsion part and (j) the j-th Tate-twist (see [9], (10.4.7)). The corresponding result for $p = 2$ is the following

Theorem 4 *Let k be a number field, $S \supseteq S_2$ a set of primes of k and K a 2-S-closed extension of k. Then the following holds.*

(i) $\mathrm{Cl}_S(K(\mu_4))(2) = 0$.

(ii) $\mathrm{Cl}_S^0(K)(2) = 0$.

Remarks: 1. The triviality of $\mathrm{Cl}(K)(2)$, and hence also that of $\mathrm{Cl}_S(K)(2)$, follows easily from the principal ideal theorem; assertions (i) and (ii) do not.

2. In (i) one can replace $K(\mu_4)$ by any totally imaginary extension of degree 2 of K in $K_S(2)$.

Finally, we consider the full extension k_S, i.e. the maximal extension of k which is unramified outside S, and its Galois group $G_S = G(k_S|k)$.

Theorem 5 *Let k be a number field and S a set of primes of k containing all primes dividing 2. Then $\mathrm{vcd}_2 G_S \leq 2$ and $\mathrm{cd}_2 G_S \leq 2$ if and only if S contains no real primes. For every discrete $G_S(2)$-module A the inflation maps*

$$inf : H^i(G_S(2), A) \longrightarrow H^i(G_S, A)(2)$$

are isomorphisms for all $i \geq 1$.

Remark: If $\mathrm{cd}\, G_S(K)(2) = 2$ (e.g. if K contains at least two primes dividing 2) for some finite subextension K of k in k_S, then $\mathrm{vcd}_2 G_S = 2$. This is always the case if $S \supset S_{\mathbb{R}}$ because the class numbers of the cyclotomic fields $\mathbb{Q}(\mu_{2^n})$ are nontrivial for $n \gg 0$. But, for example, we do not know whether $\mathrm{cd}_2 G(\mathbb{Q}_{S_2}|\mathbb{Q})$ equals 1 or 2. The answer would be '2' if at least one of the real cyclotomic fields $\mathbb{Q}(\mu_{2^n})^+$, $n = 2, 3, \ldots$, would have a nontrivial class number. But this is unknown.

In section 5 we investigate the relation between the cohomology of the group $G_S(k)$ and the modified étale cohomology of the scheme $\mathrm{Spec}(\mathcal{O}_{k,S})$. A discrete $G_S(k)$-module A induces a locally constant sheaf on $\mathrm{Spec}(\mathcal{O}_{k,S})_{et,mod}$, which we will denote by the same letter. We show the following theorem which is well-known if S contains all real primes (and also for odd p).

Theorem 6 *Let k be a number field and S a finite set of primes of k containing all primes dividing 2. Then for every 2-primary discrete $G_S(k)$-module A the natural comparison maps*

$$H^i(G_S(k), A) \longrightarrow H^i_{et,mod}(\mathrm{Spec}(\mathcal{O}_{k,S}), A)$$

are isomorphisms for all $i \geq 0$.

For finite A it is not difficult to show that the modified étale cohomology groups on the right hand side of the comparison map are finite and that they vanish for $i \geq 3$ if S contains no real primes. Therefore one could deduce theorem 1 (with $G_S(k)(2)$ replaced by $G_S(k)$) from theorem 6. However, in order to prove theorem 6, one needs information on the interaction between the decomposition groups of the real primes and so theorem 1 and theorem 6 are both consequences of theorem 2.

The main ingredients in the proofs of theorems 1–5 are Poitou-Tate duality, the validity of the weak Leopoldt-conjecture for the cyclotomic $\mathbb{Z}_p$-extension and, most essential, the systematic use of free products of bundles of profinite groups over a topological base. The reason that the above theorems had not been proven earlier seems to be a psychological one. At least the author always thought that one has to prove theorem 1 first, before showing the other assertions. For example, theorem 2 for $p = 2$, $T = S_2 \cup S_{\mathbb{R}}$ and $S = S_2$ was known if $k_{S_2}(2) = k_\infty(2)$ (see [12], §4.2 for the case $k = \mathbb{Q}$ and [15], Satz 1.4 for the general case). But now it is theorem 2 which is used in the proof of theorem 1.

Finally, we should mention that theorem 1 was formulated as a conjecture in O. Neumann's article [10].

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2 Free products of inertia groups

In this section we briefly collect some facts on free products of profinite groups and how they naturally occur in number theory. For a more detailed presentation and for proofs of the facts cited below we refer the reader to [9], chap. IV and chap. X, §1.

A profinite space is a topological space which is compact and totally disconnected. Equivalently, a profinite space is a topological inverse limit of finite discrete spaces. A profinite group is a group object in the category of profinite spaces. It can be shown that a profinite group is the inverse limit of finite groups. A full class of finite groups $\mathfrak{c}$ is a full subcategory of the category of all finite groups which is closed under taking subgroups, quotients and extensions. A pro-$\mathfrak{c}$-group is a profinite group which is the inverse limit of groups in $\mathfrak{c}$.

Let T be a profinite space. A *bundle of profinite groups* $\mathcal{G}$ over T is a group object in the category of profinite spaces over T. We say that $\mathcal{G}$ is a bundle of pro-$\mathfrak{c}$-groups if the fibre $\mathcal{G}_t$ of $\mathcal{G}$ over every point $t \in T$ is a pro-$\mathfrak{c}$-group. The functor "constant bundle", which assigns to a pro-$\mathfrak{c}$-group G the bundle $pr_2 : G \times T \to T$ has a left adjoint

$$\begin{array}{ccc} \{\text{bundles of pro-}\mathfrak{c}\text{-groups over } T\} & \longrightarrow & \{\text{pro-}\mathfrak{c}\text{-groups}\} \\ \mathcal{G} & \longmapsto & \underset{T}{*}\,\mathcal{G}. \end{array}$$

The image $*_T\mathcal{G}$ of a bundle $\mathcal{G}$ under this functor is called its free pro-$\mathfrak{c}$-product. It satisfies a universal property which is determined by the functor adjunction. Bundles of pro-$\mathfrak{c}$-groups often arise in the following way:

Let G be a pro-$\mathfrak{c}$-group and assume we are given a continuous family of closed subgroups of G, i.e. a family of closed subgroups $\{G_t\}_{t\in T}$ indexed by the points of a profinite space T which has the property that for every open subgroup $U \subset G$ the set $T(U) = \{t \in T \,|\, G_t \subseteq U\}$ is open in T. Then

$$\mathcal{G} = \{(g,t) \in G \times T \,|\, g \in G_t\}$$

is in a natural way a bundle of pro-$\mathfrak{c}$-groups over T. We have a canonical homomorphism

$$\phi : \underset{T}{*}\,\mathcal{G} \longrightarrow G$$

and we say that G is the free product of the family $\{G_t\}_{t\in T}$ if ϕ is an isomorphism.

The usual free pro-$\mathfrak{c}$-product of a discrete family of pro-$\mathfrak{c}$-groups as defined in various places in the literature (e.g. [8]) fits into the picture as follows. For a family $\{G_i\}_{i\in I}$ we consider the disjoint union $(\dot\cup_i G_i)\dot\cup\{*\}$ of the G_i and one external point $*$. Equipped with a suitable topology, this is a bundle of pro-$\mathfrak{c}$-groups over the one-point compactification $\bar I = I\dot\cup\{*\}$ of I and the free pro-$\mathfrak{c}$-product of the family $\{G_i\}_{i\in I}$ coincides with that of the bundle (cf. [9], chap.IV, §3, examples 2 and 4). For the free product of a discrete family of pro-$\mathfrak{c}$-groups we have the following profinite version of Kurosh's subgroup theorem (see [2] or [9], (4.2.1)).

Theorem 2.1 *Let $G = \mathop{*}_{i\in I} G_i$ be the free pro-$\mathfrak{c}$-product of the discrete family G_i and let H be an open subgroup of G. Then there exist systems S_i of representatives s_i of the double coset decomposition $G = \dot\bigcup_{s_i\in S_i} Hs_iG_i$ for all i and a free pro-$\mathfrak{c}$-group $F \subseteq G$ of finite rank*

$$\mathrm{rk}(F) = \sum_{i\in I}[(G:H) - \#S_i] - (G:H) + 1,$$

such that the natural inclusions induce a free product decomposition

$$H = \mathop{*}_{i,s_i}(G_i^{s_i}\cap H) * F,$$

where $G_i^{s_i}$ $(= s_iG_is_i^{-1})$ denotes the conjugate subgroup.

In number theory, continuous families of pro-$\mathfrak{c}$-groups occur in the following way. For a number field k we denote the one-point compactification of the set of all places of k by $\mathrm{Sp}(k)$. The compactifying point will be denoted by η_k and should be thought as the generic point of the scheme $\mathrm{Spec}(\mathcal{O}_k)$ in the sense of algebraic geometry or as the trivial valuation of k from the point of view of valuation theory. For an infinite extension $K|k$, we set

$$\mathrm{Sp}(K) = \varprojlim_{k'} \mathrm{Sp}(k'),$$

where k' runs through all finite subextensions of k in K. The complement of the (closed and open) subset of all archimedean places of K in $\mathrm{Sp}(K)$ is naturally isomorphic to $\mathrm{Spec}(\mathcal{O}_K)$ endowed with the constructible topology (see [6], chap.I, §7, (7.2.11) for the definition of the constructible topology of a scheme). Let S be a set of primes of k and $\bar S$ its closure in $\mathrm{Sp}(k)$ ($\bar S = S$ if S is finite, $\bar S = S\cup\{\eta_k\}$ if S is infinite). The pre-image $\bar S(K)$ of $\bar S$ under the natural projection $\mathrm{Sp}(K)\to\mathrm{Sp}(k)$ is the closure of the set $S(K)$ of all prolongations of primes in S to K in $\mathrm{Sp}(K)$.

Now assume that $M\supset K\supset k$ are possibly infinite extensions of k such that $M|K$ is Galois and $G(M|K)$ is a pro-$\mathfrak{c}$-group. The natural projection $\bar S(M)\to\bar S(K)$ has a section (in fact, there are many of them). For a fixed section $s:\bar S(K)\to\bar S(M)$ we consider the family of inertia groups $\{T_{s(\mathfrak{p})}(M|K)\}_{\mathfrak{p}\in\bar S(K)}$, where by convention $T_{\eta_M} = \{1\}$. Since a finite extension of number fields is ramified only at finitely many primes, this is a continuous family of subgroups of $G(M|K)$ indexed by $\bar{S}(K)$. We obtain a natural homomorphism

$$\phi : \mathop{*}_{\bar{S}(K)} T_{s(\mathfrak{p})}(M|K) \longrightarrow G(M|K),$$

which we also write in the form

$$\phi : \mathop{*}_{\mathfrak{p}\in S(K)} T_{\mathfrak{p}}(M|K) \longrightarrow G(M|K).$$

The cohomology groups of the free product on the left hand side with coefficients in a trivial module do not depend on the particularly chosen section s. The question, however, whether the homomorphism ϕ is an isomorphism *does* depend on s. Moreover, if s is a section for which ϕ is an isomorphism, we always find a section s' for which it is not, at least if $\mathfrak{c}$ is not the class of p-groups, where p is a prime number. In the case of pro-p-groups this pathology does not occur because of the following easy and well-known

Lemma 2.2 *Let p be a prime number and let $\phi : G' \longrightarrow G$ be a (continuous) homomorphism of pro-p-groups. Let A be $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial action. Then ϕ is an isomorphism if and only if the induced homomorphism*

$$H^i(\phi, A) : H^i(G, A) \longrightarrow H^i(G', A)$$

is an isomorphism for $i = 1$ and injective for $i = 2$.

In the number theoretical situation above, we have the following formula for the cohomology of the free product with values in a torsion group A (considered as a module with trivial action) and for $i \geq 1$:

$$H^i\big(\mathop{*}_{\mathfrak{p}\in S(K)} T_{\mathfrak{p}}(M|K), A\big) = \varinjlim_{k'} \bigoplus_{\mathfrak{p}\in S(k')} H^i(T_{\mathfrak{p}}(M'|k'), A),$$

where k' runs through all finite subextensions of k in K and M' is the maximal pro-$\mathfrak{c}$ Galois subextension of $M|k'$ (so $M = \varinjlim M'$). The limit on the right hand side depends on K and not on k and we denote it by

$$\bigoplus_{\mathfrak{p}\in S(K)}{}' \, H^i(T_{\mathfrak{p}}(M|K), A).$$

If $K|k$ is Galois, then this limit is the maximal discrete $G(K|k)$-submodule of the product $\prod_{\mathfrak{p}\in S(K)} H^i(T_{\mathfrak{p}}(M|K), A)$.

3 Proof of theorem 2

Let us first remark that for $\mathfrak{p} \in T \setminus S(k)$ the inertia group has the following structure:

- if $\mathfrak{p}$ is nonarchimedean and $N(\mathfrak{p}) \equiv 1 \bmod p$ (i.e. if there is a primitive p-th root of unity in $k_\mathfrak{p}$), then $T(k_\mathfrak{p}(p)|k_\mathfrak{p})$ is a free pro-p-group of rank 1, i.e. isomorphic to $\mathbb{Z}_p$.
- if $\mathfrak{p}$ is nonarchimedean and $N(\mathfrak{p}) \not\equiv 1 \bmod p$, then $T(k_\mathfrak{p}(p)|k_\mathfrak{p}) = \{1\}$.
- if $\mathfrak{p}$ is real and $p = 2$, then $T(k_\mathfrak{p}(p)|k_\mathfrak{p}) \cong \mathbb{Z}/2\mathbb{Z}$.
- if $\mathfrak{p}$ is real and $p \neq 2$ or if $\mathfrak{p}$ is complex, then $T(k_\mathfrak{p}(p)|k_\mathfrak{p}) = \{1\}$.

If p is odd or if $p = 2$ and $S \supset S_\mathbb{R}$, then theorem 2 is known (see [9], (10.5.1)). So we assume that $p = 2$ and $S \not\supset S_\mathbb{R}$. For a pro-2-group G we use the notation $H^i(G)$ for $H^i(G, \mathbb{Z}/2\mathbb{Z})$. We start with the following

Lemma 3.1 *Let G and G' be pro-2-groups which are generated by involutions and assume that $H^2(G, \mathbb{Q}_2/\mathbb{Z}_2) = 0 = H^2(G', \mathbb{Q}_2/\mathbb{Z}_2)$. Let $\phi : G' \to G$ be a (continuous) homomorphism. Then the following assertions are equivalent.*

(i) *ϕ is an isomorphism.*

(ii) *$H^1(\phi) : H^1(G) \to H^1(G')$ is an isomorphism.*

(iii) *$H^2(\phi) : H^2(G) \to H^2(G')$ is an isomorphism.*

Proof: Clearly, (i) implies (ii) and (iii) and, by lemma 2.2, (ii) and (iii) together imply (i). So it remains to show that (ii) and (iii) are equivalent. Since $H^2(G, \mathbb{Q}_2/\mathbb{Z}_2) = 0$, the exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}_2/\mathbb{Z}_2 \to \mathbb{Q}_2/\mathbb{Z}_2 \to 0$ induces the four term exact sequence

$$0 \to H^1(G) \xrightarrow{\alpha} H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \xrightarrow{\beta} H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \xrightarrow{\gamma} H^2(G) \to 0.$$

Since G is generated by involutions, α is an isomorphism. Hence β is zero and γ is an isomorphism. The same argument also applies to G' and therefore (ii) and (iii) are both equivalent to

(iv) $H^1(\phi, \mathbb{Q}_2/\mathbb{Z}_2) : H^1(G, \mathbb{Q}_2/\mathbb{Z}_2) \to H^1(G', \mathbb{Q}_2/\mathbb{Z}_2)$ is an isomorphism.

This concludes the proof. □

We show theorem 2 first in the special case $T = S_2 \cup S_\mathbb{R}$, $S = S_2$. The groups $\ast_{\mathfrak{p} \in S_\mathbb{R}(k_{S_2}(2))} T(k_\mathfrak{p}(2)|k_\mathfrak{p})$ and $G(k_{S_2 \cup S_\mathbb{R}}(2)|k_{S_2}(2))$ are both generated by involutions. Since $H^2(T(k_\mathfrak{p}(2)|k_\mathfrak{p}), \mathbb{Q}_2/\mathbb{Z}_2) = 0$ for every $\mathfrak{p} \in S_\mathbb{R}(k_{S_2}(2))$, we have

$$H^2(\mathop{\ast}_{\mathfrak{p} \in S_\mathbb{R}(k_{S_2}(2))} T(k_\mathfrak{p}(2)|k_\mathfrak{p}), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

By [9], (10.4.8), the inflation map

$$H^2\big(G(k_{S_2 \cup S_\mathbb{R}}(2)|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2\big) \longrightarrow H^2\big(G(k_{S_2 \cup S_\mathbb{R}}|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2\big)$$

is an isomorphism and, since $k_{S_2}(2)$ contains the cyclotomic $\mathbb{Z}_2$-extension $k_\infty(2)$ of k, the validity of the weak Leopoldt-conjecture for the cyclotomic $\mathbb{Z}_p$-extension (see [9], (10.3.25)) implies (by [9], (10.3.22)) that

$$H^2(G(k_{S_2\cup S_\mathbb{R}}(2)|k_{S_2}(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

By lemma 3.1 and the calculation of the cohomology of free products (see §1), it therefore suffices to show that the natural map

$$H^2(\phi) : H^2\big(G(k_{S_2\cup S_\mathbb{R}}(2)|k_{S_2}(2)\big) \to \bigoplus_{\mathfrak{p}\in S_\mathbb{R}(k_{S_2}(2))}{}' H^2\big(T(k_\mathfrak{p}(2)|k_\mathfrak{p})\big)$$

is an isomorphism. Now let K be a finite extension of k inside $k_S(2)$. The 9-term exact sequence of Poitou-Tate induces the exact sequence

$$0 \to Ш^2(K_{S_2\cup S_\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to H^2(G(k_{S_2\cup S_\mathbb{R}}|K), \mathbb{Z}/2\mathbb{Z}) \to$$
$$\bigoplus_{\mathfrak{p}\in S_2\cup S_\mathbb{R}(K)} H^2(G(\bar{k}_\mathfrak{p}|K_\mathfrak{p}), \mathbb{Z}/2\mathbb{Z}) \to H^0(G(k_{S_2\cup S_\mathbb{R}}|K), \mu_2)^\vee \to 0,$$

where $\vee$ denotes the Pontryagin dual. Furthermore, we have

$$Ш^2(K_{S_2\cup S_\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \cong Ш^1(K_{S_2\cup S_\mathbb{R}}, \mu_2)^\vee = Ш^1(K_{S_2\cup S_\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})^\vee = \mathrm{Cl}_{S_2}(K)/2.$$

For a finite, nontrivial extension K' of K inside $k_{S_2}(2)$ the corresponding homomorphism $H^0(G(k_{S_2\cup S_\mathbb{R}}|K), \mu_2)^\vee \to H^0(G(k_{S_2\cup S_\mathbb{R}}|K'), \mu_2)^\vee$ is the dual of the norm map, hence trivial. Furthermore, $H^2\big(G(\bar{k}_\mathfrak{p}|(k_{S_2}(2))_\mathfrak{p}), \mathbb{Z}/2\mathbb{Z}\big) = 0$ for $\mathfrak{p} \in S_2(k_{S_2}(2))$ (see [9], (7.1.8)(i)). Therefore we obtain the following exact sequence in the limit over all finite subextensions $K|k$ in $k_{S_2}(2)|k$ (the omitted coefficients are $\mathbb{Z}/2\mathbb{Z}$):

$$\mathrm{Cl}_{S_2}(k_{S_2}(2))/2 \hookrightarrow H^2\big(G(k_{S_2\cup S_\mathbb{R}}|k_{S_2}(2))\big) \twoheadrightarrow \bigoplus_{\mathfrak{p}\in S_\mathbb{R}(k_{S_2}(2))}{}' H^2\big(G(\bar{k}_\mathfrak{p}|k_\mathfrak{p})\big).$$

The principal ideal theorem implies that $\mathrm{Cl}(k_{S_2}(2))(2) = 0$, and therefore also $\mathrm{Cl}_{S_2}(k_{S_2}(2))/2 = 0$. Furthermore, $G(\bar{k}_\mathfrak{p}|k_\mathfrak{p}) = T(k_\mathfrak{p}(2)|k_\mathfrak{p})$ for $\mathfrak{p} \in S_\mathbb{R}(k_{S_2}(2))$ and the inflation map

$$H^2\big(G(k_{S_2\cup S_\mathbb{R}}(2)|k_{S_2}(2))\big) \longrightarrow H^2\big(G(k_{S_2\cup S_\mathbb{R}}|k_{S_2}(2))\big)$$

is an isomorphism (see [9], (10.4.8)). This concludes the proof of theorem 2 in the case $T = S_2 \cup S_\mathbb{R}$, $S = S_2$. For the proof in the general case we need the

Proposition 3.2 *Let k be a number field, p a prime number and $T \supset S \supseteq S_p$ sets of primes in k. Let K be a p-S_p-closed extension of k. Then the following assertions are equivalent.*

(i) *The natural homomorphism*

$$\phi_{T,S_p} : \mathop{\ast}_{\mathfrak{p}\in T\smallsetminus S_p(K)} T(K_\mathfrak{p}(p)|K_\mathfrak{p}) \to G(K_T(p)|K)$$

is an isomorphism.

(ii) *The natural homomorphisms*

$$\phi_{T,S} : \mathop{\ast}_{\mathfrak{p}\in T\smallsetminus S(K_S(p))} T(K_\mathfrak{p}(p)|K_\mathfrak{p}) \to G(K_T(p)|K_S(p))$$

and

$$\phi_{S,S_p} : \mathop{\ast}_{\mathfrak{p}\in S\smallsetminus S_p(K)} T(K_\mathfrak{p}(p)|K_\mathfrak{p}) \to G(K_S(p)|K)$$

are isomorphisms.

Here $\ast$ denotes the free pro-p-product.

Proof: If ϕ_{T,S_p} is an isomorphism, then also ϕ_{S,S_p} is an isomorphism. Furthermore, a straightforward application of theorem 2.1 shows that also $\phi_{T,S}$ is an isomorphism in this case. Let us show the converse statement. Assume that $\phi_{T,S}$ and ϕ_{S,S_p} are isomorphisms. Note that all primes in $S \smallsetminus S_p(K_S(p))$ split completely in $K_T(p)|K_S(p)$. Therefore the extension of pro-p-groups

(1) $$1 \to G(K_T(p)|K_S(p)) \to G(K_T(p)|K) \to G(K_S(p)|K) \to 1$$

splits. By lemma 2.2, we have to show that the induced homomorphism

$$H^i(\phi_{T,S_p}) : H^i\big(G(K_T(p)|K)\big) \longrightarrow \bigoplus_{\mathfrak{p}\in T\smallsetminus S_p(K)}{}' H^i\big(T(K_\mathfrak{p}(p)|K_\mathfrak{p})\big)$$

is an isomorphism for $i = 1$ and injective for $i = 2$ (coefficients $\mathbb{Z}/p\mathbb{Z}$). This follows easily from the Hochschild-Serre spectral sequence associated to the split exact sequence (1):

$$E_2^{ij} = H^i\big(G(K_S(p)|K), H^j(G(K_T(p)|K_S(p)))\big) \Longrightarrow H^{i+j}(G(K_T(p)|K)).$$

First of all, the differentials d_2 are zero ($-d_2$ is the cup-product with the extension class, see [9], (2.1.8)). Furthermore, every prime in $T \smallsetminus S(K)$ splits completely in $K_S(p)|K$ because these primes are unramified in $K_S(p)|K$ and K contains $K_\infty(p)$. Since $\phi_{T,S}$ is an isomorphism, the $G(K_S(p)|K)$-module ($j \geq 1$)

$$\begin{aligned} H^j(G(K_T(p)|K_S(p))) &= \bigoplus_{\mathfrak{p}\in T\smallsetminus S(K_S(p))}{}' H^j(T(K_\mathfrak{p}(p)|K_\mathfrak{p})) \\ &= \mathrm{Ind}_{G(K_S(p)|K)} \bigoplus_{\mathfrak{p}\in T\smallsetminus S(K)}{}' H^j(T(K_\mathfrak{p}(p)|K_\mathfrak{p})) \end{aligned}$$

is cohomologically trivial. Therefore we obtain short exact sequences

$$0 \to H^i(K_S(p)|K) \to H^i(K_T(p)|K) \to \bigoplus_{\mathfrak{p}\in T\smallsetminus S(K)}{}' H^i(T(K_\mathfrak{p}(p)|K_\mathfrak{p})) \to 0$$

for $i = 1, 2$, and the result follows from the five-lemma. □

Now we can prove theorem 2 in the general case. It is true for odd p and for $p = 2$ in the special cases $T = S_2 \cup S_\mathbb{R}$, $S = S_2$ and $T = \{\text{all primes}\}$, $S = S_2 \cup S_\mathbb{R}$. Applying proposition 3.2 in the situation $p = 2$, $T = \{\text{all primes}\}$, $S = S_2 \cup S_\mathbb{R}$ and $K = k_{S_2}(2)$, we obtain theorem 2 in the 'extremal' case $T = \{\text{all primes}\}$, $S = S_2$. Applying proposition 3.2 again, we obtain the case $T = \{\text{all primes}\}$ and S arbitrary and then the general case. This concludes the proof of theorem 2.

A straightforward limit process shows the following variant of theorem 2.

Theorem 2' *Let k be a number field, p a prime number and $T \supset S \supseteq S_p$ sets of primes of k. Let K be a p-S-closed extension field of k. Then the canonical homomorphism*

$$\mathop{*}_{\mathfrak{p} \in T \smallsetminus S(K)} T(K_\mathfrak{p}(p)|K_\mathfrak{p}) \longrightarrow G(K_T(p)|K)$$

is an isomorphism.

4 Proofs of the remaining statements

In order to prove theorem 1, we may assume that $S \not\supset S_\mathbb{R}$ and we investigate the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i\big(G_S(2), H^j(G(k_{S\cup S_\mathbb{R}}(2)|k_S(2))\big) \Longrightarrow H^{i+j}(G_{S\cup S_\mathbb{R}}(2)),$$

where the omitted coefficient are $\mathbb{Z}/2\mathbb{Z} = \mu_2$. By theorem 2, we have complete control over the $G_S(2)$-modules $H^j(G(k_{S\cup S_\mathbb{R}}(2)|k_S(2))$, which are for $j \geq 1$ isomorphic to

$$\mathrm{Ind}_{G_S(2)} \bigoplus_{\mathfrak{p}\in S_\mathbb{R} \smallsetminus S(k)} H^j(G(\mathbb{C}|\mathbb{R})).$$

In particular, $E_2^{ij} = 0$ for $ij \neq 0$. Therefore the spectral sequence induces an exact sequence

$$(2) \qquad 0 \to H^1(G_S(2)) \to H^1(G_{S\cup S_\mathbb{R}}(2)) \to \bigoplus_{\mathfrak{p}\in S_\mathbb{R} \smallsetminus S(k)} H^1(G(\mathbb{C}|\mathbb{R})) \to$$
$$H^2(G_S(2)) \to H^2(G_{S\cup S_\mathbb{R}}(2)) \to \bigoplus_{\mathfrak{p}\in S_\mathbb{R} \smallsetminus S(k)} H^2(G(\mathbb{C}|\mathbb{R})) \to 0$$

and exact sequences

$$(3) \qquad 0 \to H^i(G_S(2)) \to H^i(G_{S\cup S_\mathbb{R}}(2)) \to \bigoplus_{\mathfrak{p}\in S_\mathbb{R} \smallsetminus S(k)} H^i(G(\mathbb{C}|\mathbb{R})) \to 0.$$

for $i \geq 3$. If S is finite, this shows the finiteness statement on the cohomology of $G_S(2)$ and that

$$\chi_2(G_S(2)) = \chi_2(G_{S\cup S_\mathbb{R}}(2)).$$

But $\chi_2(G_{S\cup S_\mathbb{R}}(2)) = \chi_2(G_{S\cup S_\mathbb{R}}) = -r_2$ (see [9], (8.6.16) and (10.4.8)).

For arbitrary S and $i \geq 3$ the restriction map

$$H^i(G_{S\cup S_\mathbb{R}}(2)) \to \bigoplus_{\mathfrak{p}\in S_\mathbb{R}(k)} H^i(G(\mathbb{C}|\mathbb{R}))$$

is an isomorphism (see [9], (8.6.13)(ii) and (10.4.8)). This together with (3) shows that the natural homomorphism

$$H^i(G_S(2)) \to \bigoplus_{\mathfrak{p}\in S\cap S_\mathbb{R}(k)} H^i(G(\mathbb{C}|\mathbb{R}))$$

is an isomorphism for $i \geq 3$. Therefore cd $G_S(2) \leq 2$ if $S \cap S_\mathbb{R} = \varnothing$. For later use we formulate the last result as a proposition.

Proposition 4.1 *Let k be a number field and $S \supset S_2$ a set of primes. Then the natural homomorphism*

$$H^i(G_S(2), \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{\mathfrak{p}\in S\cap S_\mathbb{R}(k)} H^i(G(\mathbb{C}|\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism for $i \geq 3$.

In order to conclude the proof of theorem 1, it remains to show that every real prime in S ramifies in $k_S(2)$. Let S^f be the subset of nonarchimedean primes in S. Then theorem 2 yields an isomorphism

$$\mathop{\ast}_{\mathfrak{p}\in S_\mathbb{R}(k_{S^f}(2))} T(k_\mathfrak{p}(2)|k_\mathfrak{p}) \cong G(k_S(2)|k_{S^f}(2))$$

which shows the required assertion. This finishes the proof of theorem 1.

Now we prove theorem 3. To fix conventions, we recall the following definitions. For a set S of primes of k the group $\mathcal{O}_{k,S}^\times$ of S-units is defined as the subgroup in $k^\times$ of those elements which are units at every finite prime not in S and positive at every real prime not in S. The S-ideal class group $\mathrm{Cl}_S^0(k)$ in the narrow sense of k is the quotient of the group of fractional ideals of k by the subgroup generated by the nonarchimedean primes in S and the principal ideals (a) with a positive at every real place of k not contained in S. In particular, $\mathrm{Cl}_\varnothing^0(k) = \mathrm{Cl}^0(k)$ is the ideal class group in the narrow sense and $\mathrm{Cl}_{S\cup S_\mathbb{R}}^0(k) = \mathrm{Cl}_S(k)$ is the usual S-ideal class group. By class field theory, $\mathrm{Cl}_S^0(k)$ is isomorphic to the Galois group of the maximal abelian extension of k which is unramified outside $S_\mathbb{R}$ and in which every prime in S splits completely. By Kummer theory, we can replace condition (3) of theorem 3 by the following condition

(3') $\{x \in k^\times \mid x \in k_{\mathfrak{p}_0}^{\times 2} \text{ and } 2 \mid v_\mathfrak{p}(x) \text{ for every finite prime } \mathfrak{p}\} = k^{\times 2}$.

Lemma 4.2 *If $S \supseteq S_2$ and $\operatorname{cd} G_{S_2}(2) = 1$, then $S = S_2$.*

Proof: By theorem 2, we have an isomorphism

$$\mathop{\ast}_{\mathfrak{p} \in S \smallsetminus S_2(k_{S_2}(2))} T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}) \xrightarrow{\sim} G(k_S(2)|k_{S_2}(2))$$

Since for nonarchimedean primes $\mathfrak{p} \notin S_2$ the maximal unramified 2-extension of $k_{\mathfrak{p}}$ is realized by $k_\infty(2) \subset k_{S_2}(2)$, this shows that for $\mathfrak{p} \in S \smallsetminus S_2$ the maximal 2-extension of the local field $k_{\mathfrak{p}}$ is realized by $k_S(2)$ or, in other words, the natural homomorphism

$$G(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}) \longrightarrow G_S(2)$$

is injective. But for these primes we have $\operatorname{cd} G(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}}) \geq 2$ which shows that $S \smallsetminus S_2 = \varnothing$. □

Now assume that $G_{S_2}(2)$ is free. For a prime $\mathfrak{p}$ we denote the local group $G(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}})$ by $\mathcal{G}_{\mathfrak{p}}$ and the inertia group $T(k_{\mathfrak{p}}(2)|k_{\mathfrak{p}})$ by $\mathcal{T}_{\mathfrak{p}}$. By Čebotarev's density theorem, we find a finite set of nonarchimedean primes $T \supset S_2$ such that the natural homomorphism

$$H^1(G_{S_2}) \longrightarrow \bigoplus_{\mathfrak{p} \in T \smallsetminus S} H^1(\mathcal{G}_{\mathfrak{p}}/\mathcal{T}_{\mathfrak{p}})$$

is an isomorphism. It is then an easy exercise using lemma 2.2 to show that the natural homomorphism

$$\mathop{\ast}_{\mathfrak{p} \in T \smallsetminus S_2} \mathcal{G}_{\mathfrak{p}}/\mathcal{T}_{\mathfrak{p}} \longrightarrow G_{S_2}(2)$$

is an isomorphism. Theorem 2 for $T = S_2 \cup S_{\mathbb{R}}$ and $S = S_2$ and the same arguments as in the proof of proposition 3.2 show that the natural homomorphism

$$\mathop{\ast}_{\mathfrak{p} \in T \smallsetminus S_2} \mathcal{G}_{\mathfrak{p}}/\mathcal{T}_{\mathfrak{p}} \ast \mathop{\ast}_{\mathfrak{p} \in S_{\mathbb{R}}} \mathcal{G}_{\mathfrak{p}} \longrightarrow G_{S_2 \cup S_{\mathbb{R}}}(2)$$

is an isomorphism. Then, by ([16], Theorem 6) or ([9], (10.7.2)), we obtain the conditions (1)–(3) and that the unique prime $\mathfrak{p}_0$ dividing 2 in k does not split in $k_{S_2 \cup S_{\mathbb{R}}}$. If, on the other hand, conditions (1)–(3) of theorem 3 are satisfied, then we obtain (loc. cit.) the above isomorphism and deduce that $G_{S_2}(2)$ is free. The statement on the rank of $G_{S_2}(2)$ follows from $\chi_2(G_{S_2}(2)) = -r_2$. If k is totally real, then the homomorphism

$$G_{S_2}(2) \longrightarrow G(k_\infty(2)|k)$$

is a surjection of free pro-2-groups of rank 1 and hence an isomorphism. This concludes the proof of theorem 3.

Next we show theorem 4. Let S be a set of finite primes of k and $\Sigma = S \cup S_{\mathbb{R}}$. If S is finite, then the image of the group of Σ-units of k under the logarithm map $Log : \mathcal{O}_{k,\Sigma}^{\times} \longrightarrow \bigoplus_{v\in\Sigma}\mathbb{R}$, $a \mapsto (\log|a|_v)_{v\in S}$ is a lattice of rank equal to $\#S + r_1 + r_2 - 1$ (Dirichlet's unit theorem). Complementary to this map is the signature map (which is also defined for infinite S)

$$Sign_{k,S} : \mathcal{O}_{k,\Sigma}^{\times} \longrightarrow \bigoplus_{v\in S_{\mathbb{R}}} \mathbb{R}^{\times}/\mathbb{R}^{\times 2}.$$

More or less by definition, there exists a five-term exact sequence

$$0 \to \mathcal{O}_{k,S}^{\times} \to \mathcal{O}_{k,\Sigma}^{\times} \to \bigoplus_{v\in S_{\mathbb{R}}(k)} \mathbb{R}^{\times}/\mathbb{R}^{\times 2} \to \mathrm{Cl}_S^0(k) \to \mathrm{Cl}_{\Sigma}^0(k) \to 0,$$

and so the cokernel of $Sign_{k,S}$ measures the difference between the usual S-ideal class group $\mathrm{Cl}_S(k) = \mathrm{Cl}_{\Sigma}^0(k)$ and that in the narrow sense. Of course this discussion is void if k is totally imaginary. If K is an infinite extension of k, we define the signature map

$$Sign_{K,S} : \mathcal{O}_{K,\Sigma}^{\times} \longrightarrow \varinjlim_{k'} \bigoplus_{v\in S_{\mathbb{R}}(k')} \mathbb{R}^{\times}/\mathbb{R}^{\times}$$

as the limit over the signature maps $Sign_{k',S}$, where k' runs through all finite subextension $k'|k$ of $K|k$. If K is 2-S-closed, then $\mathrm{Cl}_S(K)(2) = 0$ and so statement (ii) of theorem 4 is equivalent to the statement that $Sign_K$ is surjective.

Now assume that k, S, K are as in theorem 4. By theorem 1, all real places in S become complex in K. By the principal ideal theorem, $\mathrm{Cl}(K)(2) = 2$ and so statement (i) and (ii) are trivial if K is totally imaginary (note that $K = K(\mu_4)$ in this case). So we may assume that $S_{\mathbb{R}}(K) \neq \varnothing$ and, by theorem 1, we may suppose $S \cap S_{\mathbb{R}} = \varnothing$.

Let $K' = K(\mu_4)$. Then K' is totally imaginary and $G = G(K'|K)$ is cyclic of order 2. Let $\Sigma = S \cup S_{\mathbb{R}}$ and let K_{Σ} be the maximal (not just the pro-2) extension of K which is unramified outside Σ. Inspecting the Hochschild-Serre spectral sequence associated to $K_{\Sigma}|K_{\Sigma}(2)|K$ and using the well-known calculation of $H^i(G(K_{\Sigma}|K), \mathcal{O}_{K_{\Sigma},\Sigma}^{\times})$ (cf. [9], (10.4.8)) we see that

$$\begin{aligned} H^1\big(G(K_{\Sigma}(2)|K), \mathcal{O}_{K_{\Sigma}(2),\Sigma}^{\times}\big) &= H^1\big(G(K_{\Sigma}|K), \mathcal{O}_{K_{\Sigma},\Sigma}^{\times}\big)(2) \\ &= \mathrm{Cl}_S(K)(2) = 0 \end{aligned} \tag{4}$$

and the same argument shows that

$$H^1\big(G(K_{\Sigma}(2)|K'), \mathcal{O}_{K_{\Sigma}(2),\Sigma}^{\times}\big) \cong \mathrm{Cl}_S(K')(2). \tag{5}$$

Next we consider the Hochschild-Serre spectral sequence for the extension $K_{\Sigma}(2)|K'|K$ and the module $\mathcal{O}_{K_{\Sigma}(2),\Sigma}^{\times}$. By (4) and (5), we obtain an exact sequence

$$0 \to \mathrm{Cl}_S(K')(2)^G \to H^2(G, \mathcal{O}_{K',\Sigma}^{\times}) \xrightarrow{\phi} H^2\big(G(K_{\Sigma}(2)|K), \mathcal{O}_{K_{\Sigma}(2),\Sigma}^{\times}\big).$$

Since G is a 2-group, in order to prove assertion (i), it suffices to show that ϕ is injective. Let c be a generator of the cyclic group $H^2(G,\mathbb{Z})$. For each prime $\mathfrak{p} \in S_{\mathbb{R}}(K)$ (respectively for the chosen prolongation of $\mathfrak{p}$ to $K_\Sigma(2)$, cf. the discussion in section 1), the composition $T_{\mathfrak{p}}(K_\Sigma(2)|K) \to G(K_\Sigma(2)|K) \to G$ is an isomorphism and we denote the image of c in $H^2(T_{\mathfrak{p}}(K_\Sigma(2)|K),\mathbb{Z})$ by $c_{\mathfrak{p}}$. As is well known, the cup-product with c induces an isomorphism $\hat{H}^0(G,\mathcal{O}^\times_{K',\Sigma}) \xrightarrow{\sim} H^2(G,\mathcal{O}^\times_{K',\Sigma})$ and the similar statement holds for each $c_{\mathfrak{p}}$, $\mathfrak{p} \in S_{\mathbb{R}}(K)$.

The quotient $\mathcal{O}^\times_{K_\Sigma(2),\Sigma}/\mu_{2^\infty}$ is uniquely 2-divisible, and so we obtain a natural isomorphism

$$H^2\big(G(K_\Sigma(2)|K),\mu_{2^\infty}\big) \xrightarrow{\sim} H^2\big(G(K_\Sigma(2)|K),\mathcal{O}^\times_{K_\Sigma(2),\Sigma}\big).$$

Furthermore, for each $\mathfrak{p} \in S_{\mathbb{R}} \smallsetminus S$ we obtain an isomorphism

$$\begin{array}{rcl} H^2(T_{\mathfrak{p}}(K_\Sigma(2)|K),\mu_{2^\infty}) & \xrightarrow{\sim} & H^2(T_{\mathfrak{p}}(K_\Sigma(2)|K),\mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \\ & \cong & H^2(G(\bar{K}_{\mathfrak{p}}|K_{\mathfrak{p}}),\bar{K}^\times_{\mathfrak{p}}). \end{array}$$

Therefore, the calculation of the cohomology in dimension $i \geq 2$ of free products with values in torsion modules (see [10], Satz 4.1 or [9], (4.1.4)) and theorem 2 for the pair Σ, S show that we have a natural isomorphism

$$H^2(G(K_\Sigma(2)|K),\mathcal{O}^\times_{K_\Sigma(2),\Sigma}) \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}}(K)}{}' H^2(G(\bar{K}_{\mathfrak{p}}|K_{\mathfrak{p}}),\bar{K}^\times_{\mathfrak{p}}).$$

(Alternatively, we could have obtained this isomorphism from the calculation of the cohomology of the Σ-units, cf. ([9], (8.3.10)(iii)) by passing to the limit over all finite subextensions of k in K). We obtain the following commutative diagram

$$\begin{array}{ccccc} \hat{H}^0(G,\mathcal{O}^\times_{K',\Sigma}) & & \overset{\psi}{\dashrightarrow} & & \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}}(K)}{}' \hat{H}^0(G(\bar{K}_{\mathfrak{p}}|K_{\mathfrak{p}}),\bar{K}^\times_{\mathfrak{p}}) \\ \wr\downarrow \cup c & & & & \wr\downarrow \bigoplus{}' \cup c_{\mathfrak{p}} \\ H^2(G,\mathcal{O}^\times_{K',\Sigma}) & \xrightarrow{\phi} & H^2(G(K_\Sigma(2)|K),\mathcal{O}^\times_{K_\Sigma(2),\Sigma}) & \xrightarrow{\sim} & \bigoplus_{\mathfrak{p}\in S_{\mathbb{R}}(K)}{}' H^2(G(\bar{K}_{\mathfrak{p}}|K_{\mathfrak{p}}),\bar{K}^\times_{\mathfrak{p}}). \end{array}$$

Hence $\ker(\phi) \cong \ker(\psi)$ and $\mathrm{coker}(\phi) \cong \mathrm{coker}(\psi)$. Since $\hat{H}^0(G,\mathcal{O}^\times_{K',\Sigma}) = \mathcal{O}^\times_{K,\Sigma}/N_{K'|K}(\mathcal{O}^\times_{K',\Sigma})$, each element in $\ker(\psi)$ is represented by an S-unit in K and we have to show that all these are norms of Σ-units in K'. Let $e \in \mathcal{O}^\times_{K,S}$. Then $K(\sqrt{e})|K$ is a 2-extension which is unramified outside S, hence trivial. Therefore e is a square in K and if $f^2 = e$, then $f \in \mathcal{O}^\times_{K,\Sigma}$ and $e = N_{K'|K}(f)$. This concludes the proof of assertion (i).
To show assertion (ii), it remains to show that $\mathrm{coker}(Sign_{K,S}) = \mathrm{coker}(\psi) \cong \mathrm{coker}(\phi)$ is trivial. Using the same spectral sequence as before, in order to see that $\mathrm{coker}(\phi) = 0$, it suffices to show that the spectral terms

- $E_2^{02} = H^0\big(G, H^2(G(K_\Sigma(2)|K'), \mathcal{O}^\times_{K_\Sigma(2),\Sigma})\big)$ and

- $E_2^{11} = H^1(G, \mathrm{Cl}_S(K')(2))$

are trivial. The first assertion is easy, because K' is totally imaginary and contains $k_\infty(2)$ and so $H^2(G(K_\Sigma(2)|K'), \mathcal{O}^\times_{K_\Sigma(2),\Sigma}) = 0$. That the second spectral term is trivial follows from (i). This completes the proof of theorem 4.

Finally, we prove theorem 5. The statement on $\mathrm{cd}_2 G_S$ and $\mathrm{vcd}_2 G_S$ follows by choosing a 2-Sylow subgroup $H \subset G_S$ and applying theorem 1 to all finite subextensions of k in $(k_S)^H$. It remains to show the statement on the inflation map. It is equivalent to the statement that

$$\mathit{inf} \otimes \mathbb{Z}_{(2)} : H^i(G_S(2), A) \otimes \mathbb{Z}_{(2)} \longrightarrow H^i(G_S, A) \otimes \mathbb{Z}_{(2)}$$

is an isomorphism for every discrete $G_S(2)$-module A and all $i \geq 0$, where $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (2).

Since cohomology commutes with inductive limits, we may assume that A is finitely generated (as a $\mathbb{Z}$-module). Using the exact sequences

$$0 \longrightarrow \mathrm{tor}(A) \longrightarrow A \longrightarrow A/\mathrm{tor}(A) \longrightarrow 0,$$

$$0 \longrightarrow A/\mathrm{tor}(A) \longrightarrow \big(A/\mathrm{tor}(A)\big) \otimes \mathbb{Q} \longrightarrow \big(A/\mathrm{tor}(A)\big) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and using the limit argument for $\big(A/\mathrm{tor}(A)\big) \otimes \mathbb{Q}/\mathbb{Z}$ again, we are reduced to the case that A is finite. Every finite $G_S(2)$-module is the direct sum of its 2-part and its prime-to-2-part. The statement is obvious for the prime-to-2-part and every finite 2-primary $G_S(2)$-module has a composition series whose quotients are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore we are reduced to showing the statement on the inflation map for $A = \mathbb{Z}/2\mathbb{Z}$. But it is more convenient to work with $A = \mathbb{Q}_2/\mathbb{Z}_2$ (with trivial action) which is possible by the exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}_2/\mathbb{Z}_2 \longrightarrow \mathbb{Q}_2/\mathbb{Z}_2 \longrightarrow 0.$$

Using the Hochschild-Serre spectral sequence for the extensions $k_S|k_S(2)|k$, we thus have to show that

$$H^i(G(k_S|k_S(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0$$

for $i \geq 1$. The case $i = 1$ is obvious by the definition of the field $k_S(2)$. By theorem 1, every real prime in S becomes complex in $k_S(2)$ and therefore $\mathrm{cd}_2 G(k_S|k_S(2)) \leq 2$. It remains to show that $H^2(G(k_S|k_S(2)), \mathbb{Q}_2/\mathbb{Z}_2) = 0$. Therefore the next proposition implies the remaining statement of theorem 5.

Proposition 4.3 *Let k be a number field, $S \supseteq S_2$ a set of primes in k and $K \supseteq k_\infty(2)$ an extension of K in k_S. Then*

$$H^2(G(k_S|K), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

Proof: Let H be a 2-Sylow subgroup in $G(k_S|K)$ and $L = (k_S)^H$. Then the restriction map

$$H^2(G(k_S|K), \mathbb{Q}_2/\mathbb{Z}_2) \longrightarrow H^2(G(k_S|L), \mathbb{Q}_2/\mathbb{Z}_2)$$

is injective and so, replacing K by L, we may suppose that $k_S = K_S(2)$. Applying theorem 2' to the 2-S-closed field $K_S(2)$, we obtain an isomorphism

$$G(K_{S\cup S_{\mathbb{R}}}(2)|K_S(2)) \cong \mathop{\ast}_{\mathfrak{p}\in S_{\mathbb{R}}(K_S(2))} T_{\mathfrak{p}}(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}).$$

Hence we have complete control over the Hochschild-Serre spectral sequence associated to $K_{S\cup S_{\mathbb{R}}}(2)|K_S(2)|K$. Furthermore, the weak Leopoldt conjecture holds for the cyclotomic $\mathbb{Z}_2$-extension and $K \supseteq k_\infty(2)$, which implies that $H^2(G(K_{S\cup S_{\mathbb{R}}}(2)|K), \mathbb{Q}_2/\mathbb{Z}_2) = 0$. The exact sequence (2) of §4 applied to all finite subextensions $k'|k$ of $K|k$ yields a surjection

$$\mathop{{\bigoplus}'}_{\mathfrak{p}\in S_{\mathbb{R}}\smallsetminus S(K)} H^1(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2) \twoheadrightarrow H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2).$$

and therefore, in order to prove the proposition, it suffices to show that the group $H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2)$ is 2-divisible. This is trivial if $S \cap S_{\mathbb{R}}(K) = \varnothing$ because then $\operatorname{cd} G(K_S(2)|K) \leq 2$. Otherwise, this follows from the commutative diagram

$$\begin{array}{ccc} H^2(G(K_S(2)|K), \mathbb{Q}_2/\mathbb{Z}_2)/2 & \hookrightarrow & H^3(G(K_S(2)|K), \mathbb{Z}/2\mathbb{Z}) \\ \downarrow & & \downarrow \wr \\ \mathop{{\bigoplus}'}_{\mathfrak{p}\in S\cap S_{\mathbb{R}}(K)} H^2(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2)/2 & \hookrightarrow & \mathop{{\bigoplus}'}_{\mathfrak{p}\in S\cap S_{\mathbb{R}}(K)} H^3(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Z}/2\mathbb{Z}). \end{array}$$

The right hand vertical arrow is an isomorphism by proposition 4.1. But $H^2(T(K_{\mathfrak{p}}(2)|K_{\mathfrak{p}}), \mathbb{Q}_2/\mathbb{Z}_2) = 0$ for all $\mathfrak{p} \in S \cap S_{\mathbb{R}}(K)$ and therefore the object in the lower left corner is zero. □

5 Relation to étale cohomology

Let k be a number field and S a finite set of places of k. We think of $\operatorname{Spec}(\mathcal{O}_{k,S})$ as "{scheme-theoretic points of $\operatorname{Spec}(\mathcal{O}_{k,S})$} $\cup$ {real places of k not in S}". Essentially following Zink [17], we introduce the site $\operatorname{Spec}(\mathcal{O}_{k,S})_{et,mod}$.

Objects of the category are pairs $\bar{U} = (U, U_{real})$, where U is a scheme together with an étale structural morphism $\phi_U : U \to \operatorname{Spec}(\mathcal{O}_{k,S})$ and U_{real} is a subset of the set of real valued points $U(\mathbb{R}) = Mor_{\text{Schemes}}(\operatorname{Spec}(\mathbb{R}), U)$ of U such that $\phi_U(U_{real}) \subset S_{\mathbb{R}}(k) \smallsetminus S$.

Morphisms are scheme morphisms $f : U \to V$ over $\operatorname{Spec}(\mathcal{O}_{k,S})$ satisfying $f(U_{real}) \subset V_{real}$.

Coverings are families $\{\pi_i : \bar{U}_i \to \bar{U}\}_{i\in I}$ such that $\{\pi_i : U_i \to U\}_{i\in I}$ is an étale covering in the usual sense and $\bigcup_{i\in I} \pi_i(U_{i\,real}) = U_{real}$.

There exists an obvious morphism of sites

$$\mathrm{Spec}(\mathcal{O}_{k,S})_{et} \longrightarrow \mathrm{Spec}(\mathcal{O}_{k,S})_{et,mod}$$

and both sites coincide if S contains all real places of k. The pair $\bar{X} = (X, X_{real})$ with $X = \mathrm{Spec}(\mathcal{O}_{k,S})$ and $X_{real} = S_{\mathbb{R}}(k) \setminus S$ is the terminal object of the category and the profinite group $G_S(k)$ is nothing else but the fundamental group of $\bar{X}$ with respect to this site. Let η denote the generic point of X. For a sheaf A of abelian groups on $\mathrm{Spec}(\mathcal{O}_{k,S})_{et,mod}$ and for any point v of $\bar{X}$ we have a specialization homomorphisms $s_v : A_v \to A_\eta$ from the stalk A_v of A in v to that in η. For each point $v \in X_{real}$ we consider the local cohomology $H^i_v(\bar{X}, A)$ with support in v. There is a long exact localization sequence (see [17])

$$\cdots \to \bigoplus_{v\in X_{real}} H^i_v(\bar{X}, A) \to H^i_{et,mod}(\bar{X}, A) \to H^i_{et}(X, A) \to \cdots$$

and the local cohomology with support in real points is calculated as follows:

Lemma 5.1 *For $v \in X_{real}$ the following holds.*

$$\begin{aligned} H^0_v(\bar{X}, A) &= \ker(s_v : A_v \to A_\eta) \\ H^1_v(\bar{X}, A) &= \mathrm{coker}(s_v : A_v \to A_\eta) \\ H^i_v(\bar{X}, A) &= H^{i-1}(k_v, A_v) \qquad \textit{for } i \geq 2. \end{aligned}$$

Here the right hand side of the last isomorphism is the Galois cohomology of the field k_v.

Proof See [17], Lemma 2.3. □

Remark: Suppose that S contains all primes dividing 2 and no real primes. Let A be a locally constant constructible sheaf on $\mathrm{Spec}(\mathcal{O}_{k,S})_{et}$ which is annihilated by a power of 2. We denote the push-forward of A to $\mathrm{Spec}(\mathcal{O}_{k,S})_{et,mod}$ by the same letter. By Poitou-Tate duality, the boundary map of the long exact localization sequence

$$H^i_{et}(X, A) \longrightarrow \bigoplus_{v\in X_{real}} H^{i+1}_v(\bar{X}, A) = \bigoplus_{v \text{ arch.}} H^i(k_v, A_v)$$

is an isomorphisms for $i \geq 3$ and surjective for $i = 2$. Therefore, we obtain the vanishing of $H^i_{et,mod}(\mathrm{Spec}(\mathcal{O}_{k,S}), A)$ for $i \geq 3$. In this situation the modified étale cohomology is connected to the "positive étale cohomology" $H^*_2(\mathrm{Spec}(\mathcal{O}_{k,S}), A_+)$ defined in [3] in the following way. There exists a natural exact sequence

$$0 \to H^0_{et,mod}(\mathrm{Spec}(\mathcal{O}_{k,S}), A) \to$$
$$\bigoplus_{v \text{ arch.}} H^0(k_v, A_v) \to H^0_2(\mathrm{Spec}(\mathcal{O}_{k,S}), A_+) \to H^1_{et,mod}(\mathrm{Spec}(\mathcal{O}_{k,S}), A) \to 0.$$

and isomorphisms

$$H_2^i(\mathrm{Spec}(\mathcal{O}_{k,S}), A_+) \xrightarrow{\sim} H^{i+1}_{et,mod}(\mathrm{Spec}(\mathcal{O}_{k,S}), A)$$

for $i \geq 1$. This can be easily deduced from the long exact localization sequence, lemma 5.1 and the long exact sequence (2.4) of [3].

Now let A be a discrete $G_S(k)$-module. The module A induces locally constant sheaves on $\mathrm{Spec}(\mathcal{O}_{k,S})_{et,mod}$ and $\mathrm{Spec}(\mathcal{O}_{k,S})_{et}$, which we will denote by the same letter. According to lemma 5.1, we obtain for every $v \in X_{real}$

$$H_v^i(\bar{X}, A) = 0 \quad \text{for } i = 0, 1.$$

Let $\widetilde{X} = (\mathrm{Spec}(\mathcal{O}_{k_S,S}), S_\mathbb{R}(k_S) \smallsetminus S(k_S)$ be the universal covering of $\bar{X}$. The Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(G_S(k), H^j_{et,mod}(\widetilde{X}, A)) \Longrightarrow H^{i+j}_{et,mod}(\bar{X}, A)$$

induces natural comparison homomorphisms

$$H^i(G_S(k), A) \longrightarrow H^i_{et,mod}(\bar{X}, A)$$

for all $i \geq 0$. It follows immediately from the spectral sequence that these homomorphisms are isomorphisms if

$$H^j_{et,mod}(\widetilde{X}, A) = 0$$

for all $j \geq 1$.

Next we are going to prove theorem 6 of the introduction. Assume that S contains all primes dividing 2 and that A is 2-torsion. Both sides of the comparison homomorphism commute with direct limits, and so, in order to prove theorem 6, we may suppose that A is finite. Since A is constant on $\widetilde{X}$, we can easily reduce to the case $A = \mathbb{Z}/2\mathbb{Z}$, in order to show $H^j_{et,mod}(\widetilde{X}, A) = 0$ for $j \geq 1$. Furthermore, the assertion is trivial for $j = 1$. The theorem is well-known if S contains all real primes (see [17], prop. 3.3.1 or [7], II, 2.9) and so, passing to the limit over all finite subextensions of k in k_S, we obtain natural isomorphisms for all $j \geq 0$.

$$H^j(G_{S \cup S_\mathbb{R}}(k_S), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H^j_{et}(\widetilde{X} \smallsetminus S_\mathbb{R}(k_S), \mathbb{Z}/2\mathbb{Z}).$$

On the other hand, theorem 2 for $T = S \cup S_\mathbb{R}$, $S = S$ applied to all finite subextensions of k in k_S in conjunction with theorem 5 induces isomorphisms for all $j \geq 1$.

$$H^j(G_{S \cup S_\mathbb{R}}(k_S), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \bigoplus_{v \in S_\mathbb{R} \smallsetminus S(k_S)}{}^{'} H^j(k_v, \mathbb{Z}/2\mathbb{Z}).$$

These two isomorphisms together with the long exact localization sequence show that

$$H^j_{et,mod}(\widetilde{X}, \mathbb{Z}/2\mathbb{Z}) = 0$$

for $j \geq 1$. This completes the proof of theorem 6.

Theorem 6 is best understood in the context of étale homotopy, namely as a vanishing statement on the 2-parts of higher homotopy groups. For a scheme X we denote by X_{et} its étale homotopy type, i.e. a pro-simplicial set. The étale homotopy groups of X are by definition the homotopy groups of X_{et} and, as is well known, these pro-groups are pro-finite, whenever the scheme X is noetherian, connected and geometrically unibranch ([1] Theorem 11.1). If we consider the modified étale site $\mathrm{Spec}(\mathcal{O}_{k,S})_{et,mod}$ as above, we obtain in exactly the same manner as for the usual étale site a pro-finite simplicial set $\bar{X}_{et,mod}$. We denote the universal covering of $\bar{X}_{et,mod}$ by $\widetilde{X}_{et,mod}$. If p is a prime number and $Y.$ is a pro-simplicial set, we denote the pro-p completion of $Y.$ by $Y.^{\wedge p}$. Furthermore, we write $G(p)$ for the maximal pro-p factor group of a pro-group G.

Lemma 5.2 *Assume that $Y.$ is a simply connected (i.e. $\pi_1(Y.) = 0$) pro-simplicial set such that $\pi_i(Y.)$ is pro-finite for all $i \geq 2$. Then we have isomorphisms for all i:*

$$\pi_i(Y.)(p) \longrightarrow \pi_i(Y.^{\wedge p}).$$

Proof: See [13], prop. 13. □

For a pro-group G we denote by $K(G,1)$ the Eilenberg-MacLane pro-simplicial set associated with G (cf. [1], (2.6)). If S contains all real primes of k the following theorem was proved in [13], prop. 14.

Theorem 5.3 *Let k be a number field and S a finite set of primes of k containing all primes dividing 2. Let $\bar{X}$ be the pair (X, X_{real}) with $X = \mathrm{Spec}(\mathcal{O}_{k,S})$ and $X_{real} = S_{\mathbb{R}}(k) \smallsetminus S$ endowed with the modified étale topology. Then the higher homotopy groups of $\bar{X}_{et,mod}$ have no 2-part, i.e.*

$$\pi_i(\bar{X}_{et,mod})(2) = 0 \quad \text{for } i \geq 2.$$

Furthermore, the canonical morphism

$$(\bar{X}_{et,mod})^{\wedge 2} \longrightarrow K(G_S(k)(2), 1)$$

is a weak homotopy equivalence.

Proof: Since $G_S(k)$ is the fundamental group of $\bar{X}_{et,mod}$, theorem 6 implies that the universal covering $\widetilde{X}_{et,mod}$ of $\bar{X}_{et,mod}$ has no cohomology with values in 2-primary coefficient groups. By the Hurewicz theorem ([1], (4.5)), the pro-2 completion of $\widetilde{X}_{et,mod}$ is weakly contractible. Therefore lemma 5.2 implies

$$\pi_i(\bar{X}_{et,mod})(2) \cong \pi_i(\widetilde{X}_{et,mod})(2) \cong \pi_i\big((\widetilde{X}_{et,mod})^{\wedge 2}\big) = 0$$

for $i \geq 2$, which shows the first statement of the theorem. By theorem 5, for every finite 2-primary $G_S(k)(2)$-torsion module A the inflation homomorphism $H^i(G_S(k)(2), A) \longrightarrow H^i(G_S(k), A)$ is an isomorphism for all i. The same arguments as above show that the universal covering of $(\bar{X}_{et,mod})^{\wedge 2}$ is weakly contractible. This proves the second statement. □

6 Closing Remarks

1. *Dualizing modules*

Unfortunately, we do not have (despite semi-tautological reformulations of the definition) a good description of the p-dualizing module I of the group G_S, where S is a finite set of finite primes containing S_p. If k is totally imaginary, then I is determined by the exact sequence

$$0 \longrightarrow \mu_{p^\infty} \stackrel{diag}{\longrightarrow} \bigoplus_{\mathfrak{p} \in S(k_S)}{}' \mu_{p^\infty} \longrightarrow I \longrightarrow 0$$

(see [9], (10.2.1)) and the group G_S is a duality group at p of dimension 2 (see [13], th.4 or [9], (10.9.1)). The general case remains unsolved (also for odd p).

2. *Free profinite product decompositions*

In this paper we used free pro-p-product decompositions of Galois groups of pro-p-extensions of global fields into Galois groups of local pro-p-extensions in an essential way. One might ask whether, for sets of places $T \supset S$, the natural homomorphism

$$\phi : \mathop{\ast}_{\mathfrak{p} \in T \smallsetminus S(k_S)} T(\bar{k}_\mathfrak{p}|k) \longrightarrow G(k_T|k_S)$$

is an isomorphism, where the free product on the left hand side is the free product of *profinite* groups. More precisely, one has to ask, whether there exists a continuous section to the natural projection $T \smallsetminus S(k_T) \to T \smallsetminus S(k_S)$ such that the above map is an isomorphism (cf. the discussion in section 2). We do not know the answers to this question in general. It is 'yes' if S contains all but finitely many primes of k (see below). But it seems likely that ϕ is never an isomorphism if T and S are finite. The present level of knowledge on this question is rather low. For example, we do not know whether there are infinitely many prime numbers p such that p^∞ divides the order of G_T. The best result known in this direction is that if T contains all real places and all primes dividing one prime number p, then there exist infinitely many prime numbers ℓ dividing the order of G_T (see [14], cor.3 or [9], (10.9.4)).

In the case that S contains all but finitely many primes of k, we can deduce the above statement by applying the following slightly more general result to the complement of S:

For a finite set S of primes of k, let k^S be the maximal extension of k in which all primes in S are totally decomposed. Then there exists a continuous section to the natural projection $S(\bar{k}) \to S(k^S)$ such that the natural map

$$\mathop{*}_{\mathfrak{p}\in S(k^S)} G(\bar{k}_{\mathfrak{p}}|k) \longrightarrow G(\bar{k}|k^S)$$

is an isomorphism. This had been proved first in the special case $S = S_{\mathbb{R}}$ by Fried-Haran-Völklein [4] and then by Pop [11] for arbitrary finite S.

3. *Leopoldt's conjecture*

The Leopoldt conjecture for k and a prime number p holds if and only if the group

$$H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p)$$

is trivial for one (all) finite set(s) of primes $S \supseteq S_p$. The weak Leopoldt conjecture is true for k, p and a $\mathbb{Z}_p$-extension $k_\infty|k$ if and only if

$$H^2(G_S(k_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$$

is trivial for one (all) finite set(s) of primes $S \supseteq S_p$ (of k). This is well known for odd p and for $p = 2$ it can be easily deduced from the above results.

4. *Iwasawa theory*

Let k be a number field, $S \supseteq S_2$ a finite set of primes of k and $k_\infty|k$ the cyclotomic $\mathbb{Z}_2$-extension of k. Let $\Gamma = G(k_\infty|k) \cong \mathbb{Z}_2$ and let $\Lambda = \mathbb{Z}_2[\![\Gamma]\!] \cong \mathbb{Z}_2[[T]]$ be the Iwasawa algebra. We consider the compact Λ-module

$$X_S = G(k_S(2)|k_\infty)^{ab}.$$

Then the following holds

(i) X_S is a finitely generated Λ-module.

(ii) $\operatorname{rank}_\Lambda X_S = r_2$ (the number of complex places of k).

(iii) X_S does not contain any nontrivial finite Λ-submodule.

(iv) the μ-invariant of X_S is greater than or equal to $\# \, S \cap S_{\mathbb{R}}(k)$.

Properties (i)-(iii) follow in a purely formal way (see [9], (5.6.15)) from the facts that: (a) $\chi_2(G_S(2)) = -r_2$, (b) $H^2(G_S(k_\infty)(2), \mathbb{Q}_2/\mathbb{Z}_2) = 0$ and (c) $H^2(G_S(2), \mathbb{Q}_2/\mathbb{Z}_2)$ is 2-divisible. Assertion (iv) is trivial if S contains no real places and in the general case it follows from the exact sequence

$$0 \to (\Lambda/2)^{\# \, S\cap S_{\mathbb{R}}(k)} \longrightarrow X_S \longrightarrow X_{S\smallsetminus S_{\mathbb{R}}} \longrightarrow 0.$$

Now let k^+ be a totally real number field, $k = k^+(\mu_4)$, k_∞^+ the cyclotomic $\mathbb{Z}_2$-extension of k^+ and $k_\infty = k_\infty^+(\mu_4) = k(\mu_{2^\infty})$. Let k_n be the unique subextension of degree 2^n in k_∞ and let J be the complex conjugation. We set $A_n = \mathrm{Cl}(k_n)(2)$ and

$$A_n^- := \{a \in A_n \mid aJ(a) = 1\}.$$

Furthermore, let $A^-_\infty = \varinjlim A^-_n$, $X^+ = X_{S_2}(k^+)$, let $\vee$ denote the Pontryagin dual and (-1) the Tate-twist by -1. Then there exists a natural homomorphism

$$\phi : (A^-_\infty)^\vee \longrightarrow X^+(-1)$$

whose kernel and cokernel are annihilated by 2. If the Iwasawa μ-invariant of k is zero (this is known if $k|\mathbb{Q}$ is abelian), then ϕ is a pseudo-isomorphism, i.e. ϕ has finite kernel and cokernel. This can be seen by a slight modification of the arguments given in [5], §2:

Let M^+ be the maximal abelian 2-extension of k^+_∞ which is unramified outside S_2, in particular, M^+ is totally real. Kummer theory shows that, for an $\alpha \in k^\times_\infty$, the field $k_\infty(\sqrt[2^n]{\alpha})$ is contained in M^+k_∞ if and only if: (a) $\alpha \in k^{\times 2^n}_{\infty,\mathfrak{p}}$ for all $\mathfrak{p} \notin S(k_\infty)$ and (b) $\alpha J(\alpha) = \beta^{2^n}$ for a totally positive element $\beta \in k^+_\infty$. Let R_n be the subgroup in $k^\times_\infty / k^{\times 2^n}_\infty$ generated by elements satisfying (a) and (b) and let

$$\mathfrak{M}^- := \varinjlim_n R_n \subset k^\times_\infty \otimes \mathbb{Q}_2/\mathbb{Z}_2.$$

Then we have a perfect Kummer pairing $X^+ \times \mathfrak{M}^- \to \mu_{2^\infty}$. Since all primes dividing 2 are infinitely ramified in $k_\infty|k$, for $\alpha \otimes 2^{-n} \in \mathfrak{M}^-$ there exists a unique ideal $\mathfrak{a}$ in k_∞ with $\mathfrak{a}^{2^n} = (\alpha)$ and the class $[\,\mathfrak{a}\,]$ is contained in A^-_∞. This yields a homomorphism

$$\phi^\vee : \mathfrak{M}^- \longrightarrow A^-_\infty.$$

A straightforward computation shows that $\mathrm{im}(\phi^\vee) \supseteq (A^-_\infty)^2$ and that $\ker(\phi^\vee)$ is the image of $\mathcal{O}^\times_{k^+_\infty,\varnothing}/\mathcal{O}^{\times 2}_{k^+_\infty,S_\mathbb{R}}$ in $\mathfrak{M}^-$ (notational conventions as in §4). Thus, if the Iwasawa μ-invariant of k is zero, then the cokernel of $\phi^\vee$ is finite and it remains to show the same for its kernel. Since $\mu = 0$, the $\mathbb{F}_2$-ranks of ${}_2\mathrm{Cl}^0(k^+_n)$ (the subgroup of elements annihilated by 2 in the ideal class groups in the narrow sense) are bounded independently of n. Thus also the $\mathbb{F}_2$-ranks of the kernels of the signature maps

$$\mathcal{O}^\times_{k^+_n,S_\mathbb{R}}/\mathcal{O}^{\times 2}_{k^+_n,S_\mathbb{R}} \longrightarrow \bigoplus_{v\in S_\mathbb{R}(k^+_n)} \mathbb{R}^\times/\mathbb{R}^{\times 2}$$

are bounded independently of n. But the direct limit over n of these kernels is just the group in question. Finally, we obtain the result by taking Pontryagin duals.

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