

Tame coverings of arithmetic schemes

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The objective of this paper is to investigate tame fundamental groups of schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. More precisely, let X be a connected scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ and let \bar{X} be a compactification of X , i.e. a scheme which is proper and of finite type over $\mathrm{Spec}(\mathbb{Z})$ and which contains X as a dense open subscheme. Then the tame fundamental group of X classifies finite étale coverings of X which are tamely ramified along the boundary $\bar{X} - X$, in particular, the tame fundamental group $\pi_1^t(\bar{X}, \bar{X} - X)$ is a quotient of the étale fundamental group $\pi_1(X)$. Our interest in the tame fundamental group arises from the observation that it seems to be the maximal quotient of $\pi_1(X)$ which is ‘visible’ via class field theory by algebraic cycle theories (see [S-S1], [S] and [S-S2] for more precise statements on ‘tame class field theory’).

Coverings of a regular scheme which are tamely ramified along a normal crossing divisor have been studied in [SGA1], [G-M]. In this paper we consider tame ramification along an arbitrary Zariski-closed subset. The main reason for this is the lack of good desingularization theorems in positive and mixed characteristics. A simple imitation of the definition of tame ramification in the normal crossing case proves to be not useful in the more general situation. We give a definition of tameness in the general situation in section 1 and we show that it coincides with the previous one in the normal crossing case. For Galois coverings of normal schemes our definition of tameness coincides with that proposed by Abbes [Ab] and with the notion of “numerical tameness” defined by Chinburg/Erez [C-E].

Naturally, the question arises whether the tame fundamental group $\pi_1^t(\bar{X}, \bar{X} - X)$ is independent of the choice of the compactification \bar{X} . At the moment, we can answer this question only for the maximal pro-nilpotent quotient. In section 2, we consider discrete valuations of higher rank associated to Parshin chains. We investigate the connection between the tameness of a covering and the behaviour of the associated higher dimensional henselian fields and we show that a finite covering of a regular arithmetic scheme with nilpotent Galois group is tamely ramified if and only if all associated extensions of higher dimensional henselian fields are tame. Then we use this fact in the proof of the following

Theorem 1 *Let X be a regular and connected scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ and assume that there exists a regular scheme \bar{X} which is proper over $\mathrm{Spec}(\mathbb{Z})$ and which contains X as an open subscheme. Then $\pi_1^t(\bar{X}, \bar{X} - X)^{\mathrm{pro-nil}}$ depends only on X and not on the choice of \bar{X} .*

If X is a smooth quasiprojective variety over a finite field \mathbb{F} , then its abelianized tame fundamental group $\pi_1^t(X)^{ab} = \pi_1^t(\bar{X}, \bar{X} - X)^{ab}$ is infinite, but its degree-zero part, i.e. the kernel of the natural homomorphism

$$\pi_1^t(X)^{ab} \longrightarrow G(\bar{\mathbb{F}}|\mathbb{F}) \cong \hat{\mathbb{Z}}$$

is finite (see [S-S1]). In the arithmetic case, i.e. when X is flat over $\text{Spec}(\mathbb{Z})$, we show the following theorem in which \bar{X} is not assumed to be proper.

Theorem 2 *Let \mathcal{O} be the ring of integers in a finite extension k of \mathbb{Q} . Let \bar{X} be a flat \mathcal{O} -scheme of finite type whose geometric generic fibre $\bar{X} \otimes_{\mathcal{O}} \bar{k}$ is connected. Assume that \bar{X} is normal and that the morphism $\bar{X} \rightarrow \text{Spec}(\mathcal{O})$ is surjective. Let X be an open subscheme of \bar{X} . Then the abelianized tame fundamental group $\pi_1^t(\bar{X}, \bar{X} - X)^{ab}$ is finite.*

In the special case $X = \bar{X}$, we obtain the finiteness of the étale fundamental group $\pi_1(\bar{X})^{ab}$ which sharpens a theorem of Katz and Lang ([K-L], th.3) by weakening the assumption ‘smooth’ to ‘normal’.

1 Tame coverings

The concept of tame ramification stems from number theory: A finite extension of number fields $L|K$ is called tamely ramified at a prime \mathfrak{P} of L if the associated extension of completions $L_{\mathfrak{P}}|K_{\mathfrak{P}}$ is a tamely ramified extension of local fields. The latter means that the ramification index is prime to the characteristic of the residue field. It is a classical result that composites and towers of tamely ramified extensions are again tamely ramified. This concept generalizes to arbitrary discrete valuation rings by requiring that the associated residue field extensions are separable.

As is well known, the concept of unramified extensions has found its generalization to arbitrary schemes by the notion of étale coverings. Tame ramification along a normal crossing divisor of a regular scheme has been studied in [SGA1], [G-M]. Let us first recall the definition given there. We assume throughout this paper that all schemes under consideration are noetherian.

Let X be a regular and connected scheme and let $D = \sum D_i$ be a divisor on X . We say that D has **normal crossings**, if étale locally around every point $x \in \text{supp}(D) \subset X$, we have

$$D_i = \sum_j \text{div}(s_{ij}),$$

where $(s_{ij})_{i,j} \in \mathcal{O}_{X,x}$ is part of a regular system of parameters of the regular local ring $\mathcal{O}_{X,x}$.

Let X be regular and connected, $D \subset X$ a divisor with normal crossings and $U = X - D$. Let $P_i, i = 1, \dots, n$, be the generic points of the irreducible components D_1, \dots, D_n of D . Then the local rings \mathcal{O}_{X,P_i} are discrete valuation rings,

inducing discrete valuations v_1, \dots, v_n on the function field of X . Let $U' \rightarrow U$ be a finite étale morphism and assume for simplicity that U' is connected. Let X' be the normalization of X in the function field K' of U' .

Definition 1.1 ([G-M], 2.2.2.) *The finite étale covering $U' \rightarrow U$ is called **tamely ramified** (or *tame*, for short) along D if the extension of function fields $K'|K$ is tamely ramified at the discrete valuations associated to D_1, \dots, D_n .*

Étale locally, tame coverings are of a very simple structure by the following theorem which is known under the name Generalized Abhyankar's Lemma.

Theorem 1.2 ([SGA1], Exp. XIII, 5.3.0) *Let X be a strictly henselian local regular scheme of residue characteristic $p > 0$, $D = \sum_{i=1}^r \text{div}(f_i)$ a divisor with normal crossings on X and $U = X - D$. Then every connected finite étale covering of U which is tamely ramified along D is a quotient of a (tamely ramified) covering of the form*

$$U' = U[T_1, \dots, T_r]/(T_1^{n_1} - f_1, \dots, T_r^{n_r} - f_r),$$

where the n_i are natural numbers prime to p .

Tame coverings as defined above satisfy the axioms of a Galois theory, so (omitting the base point) there exists a profinite group $\pi_1^t(X, D)$, which is a quotient of $\pi_1(U)$ and which classifies étale coverings of U which are tame along D .

If X is strictly local and $D = \sum_{i=1}^r \text{div}(f_i)$ a normal crossing divisor, then theorem 1.2 yields a natural isomorphism

$$\pi_1^t(X, D) \cong \left(\prod_{\ell \neq p} \mathbb{Z}_\ell(1) \right)^r \stackrel{\text{notation}}{=} \left(\hat{\mathbb{Z}}^{(p)}(1) \right)^r.$$

If, more generally, $D \subset X$ is an arbitrary divisor on X , then (as already remarked in [G-M], 2.2.3.4, without further elaboration) the above definition of tame ramification is not the 'correct' one. For example, it is not stable under base change.

Example 1.3 Let $X = \text{Spec}(\mathbb{Z}[T])$ be the affine line over $\text{Spec}(\mathbb{Z})$ and consider the divisor

$$D = \text{div}(T + 4) + \text{div}(T - 4),$$

which is not a normal crossing divisor. Let $K = \mathbb{Q}(T)$ be the function field of X and $U = X - D$. Put $f = (T+4)(T-4) = T^2 - 16$, $L = K(\sqrt{f})$ and consider the normalization X_L of X in L . The ramification locus of $X_L \rightarrow X$ is either D or $D \cup X_2$, where X_2 is the unique vertical divisor on X over characteristic 2. Let us show that $X_L \rightarrow X$ is unramified at the generic point of X_2 . This is equivalent to the statement that $L|K$ is unramified at the unique discrete valuation v_2 of K which corresponds to the prime ideal $2\mathbb{Z}[T] \subset \mathbb{Z}[T]$. Therefore it suffices to show

that f is a square in the completion K_2 of K with respect to v_2 . Consider the polynomial $F(X) = X^2 - f = X^2 - T^2 + 16$. We have $F(T) \equiv 0 \pmod{16}$ and the derivative $F'(T) = 2T$ has the exact 2-valuation 1. By the usual approximation process (cf. [Se] 2.2. th.1), we see that f has a square root in K_2 . Hence the ramification locus of $X_L \rightarrow X$ is exactly D , and since D is the sum of horizontal prime divisors, the morphism $U_L \rightarrow U$ is tame along D (in the naive sense).

Now consider the closed subscheme $Y \subset X$ given by the equation $T = 0$, so $Y \cong \text{Spec}(\mathbb{Z})$. Then $D_Y = D \cap Y$ is the point on Y which corresponds to the prime number 2. Let $V = U \cap Y = Y - D_Y$. The base change $V' = U_L \times_U V \rightarrow V$ is the normalization of $V \cong \text{Spec}(\mathbb{Z}[\frac{1}{2}])$ in $\mathbb{Q}(\sqrt{-1})$. But 2 is wildly ramified in $\mathbb{Q}(\sqrt{-1})$, and so $V' \rightarrow V$ is not tame along D_Y .

Let us now give a definition of tame ramification in the general situation which will be shown in proposition 1.14 to generalize definition 1.1. Let X be a scheme, $Y \subset X$ a closed subscheme and $U = X - Y$ the open complement. For a point $y \in Y$ we write X_y^{sh} for $\text{Spec}(\mathcal{O}_{X,y}^{sh})$, where sh means strict henselization. By abuse of notation, we write U_y^{sh} for the base change $U \times_X X_y^{sh}$. The scheme U_y^{sh} is empty if $y \notin \bar{U}$.

Definition 1.4 *We say that a finite étale covering $U' \rightarrow U$ is **tamely ramified** along Y if for every point $y \in Y$ such that U_y^{sh} is nonempty, the base change*

$$U' \times_U U_y^{sh} \longrightarrow U_y^{sh}$$

can be dominated by an étale covering of the form

$$V_1 \cup \dots \cup V_r \longrightarrow U_y^{sh},$$

such that each V_i is a connected étale Galois covering of its image in U_y^{sh} and the degree of V_i over its image is prime to the characteristic of $k(y)$.

The next lemma follows in a straightforward manner from the definition of tame ramification.

Lemma 1.5 *Let X be a scheme, $U \subset X$ an open subscheme and $Y = X - U$. Let $f : X_1 \rightarrow X$ be a morphism of schemes, $U_1 = f^{-1}(U)$ and $Y_1 = f^{-1}(Y)$. If $U' \rightarrow U$ is a finite étale morphism which is tamely ramified along Y , then the base change*

$$U'_1 = U' \times_U U_1 \longrightarrow U_1$$

is tamely ramified along Y_1 .

The notion of a finite étale morphism $U' \rightarrow U$ which is tamely ramified along $Y = X - U$ is independent from X in the following sense.

Lemma 1.6 *Suppose that U is contained as an open subscheme in schemes \tilde{X} and X with closed complements \tilde{Y} and Y , respectively. Assume that there exists a finite morphism $\pi : \tilde{X} \rightarrow X$ making the diagram*

$$\begin{array}{ccc} U & \hookrightarrow & \tilde{X} \\ \parallel & & \downarrow \pi \\ U & \hookrightarrow & X \end{array}$$

commutative. Then a finite étale morphism $U' \rightarrow U$ is tamely ramified along \tilde{Y} if and only if it is tamely ramified along Y .

Proof: Let $y \in Y$ be a point and consider the cartesian diagram

$$\begin{array}{ccc} \tilde{X} \times_X X_y^{sh} & \longrightarrow & X_y^{sh} \\ \uparrow & & \uparrow \\ U \times_X X_y^{sh} & = & U_y \end{array}$$

Since $\pi : \tilde{X} \rightarrow X$ is finite, $\tilde{X} \times_X X_y^{sh}$ is strictly henselian, and

$$\tilde{X} \times_X X_y^{sh} \cong \bigcup_i X_{\tilde{y}_i}^{sh},$$

where the \tilde{y}_i are the finitely many points of \tilde{X} lying above y . Therefore, π induces a natural isomorphism

$$\bigcup_i U_{\tilde{y}_i}^{sh} \xrightarrow{\sim} U_y^{sh}.$$

Now the statement of the lemma follows easily from the definition of tame ramification. \square

Remark 1.7 Assume that X is excellent or, slightly weaker, that all local rings of X are Nagata rings. Then there exists a unique maximal scheme \tilde{X} satisfying the conditions of lemma 1.6: the normalization of X in U . This scheme is constructed in the following way. Denoting the open immersion by $j : U \hookrightarrow X$, the sheaf $j_*(\mathcal{O}_U)$ is quasicoherent and contains \mathcal{O}_X . The integral closure of \mathcal{A} of \mathcal{O}_X in $j_*(\mathcal{O}_U)$ is then a coherent sheaf of \mathcal{O}_X -algebras and the normalization of X in U is defined as $\text{Spec}(\mathcal{A})$ (cf. [Mi], I, §1, proof of th. 1.8).

If $U' \rightarrow U$ is a tame covering, then, by Zariski's main theorem, the morphism $U' \rightarrow X$ factors in the form $U' \xrightarrow{j} X' \xrightarrow{\pi} X$ with π finite and j an open embedding. In general, X' is not unique but if X'_1 and X'_2 are such schemes, we find a third scheme X'_3 dominating X'_1 and X'_2 . If X is normal and connected, then the normalization of X in the function field of U' is maximal among the schemes which are finite over X and contain U' as an open dense subscheme. The following lemma follows easily by the same considerations as in the proof of lemma 1.6.

Lemma 1.8 *Let*

$$X_1 \xrightarrow{\pi_1} X_2 \xrightarrow{\pi_2} X_3$$

be finite surjective morphisms, let $U_3 \subset X_3$ be an open subscheme and denote by U_2 and U_1 the corresponding inverse images. Suppose that $\pi_1|_{U_1} : U_1 \rightarrow U_2$ and $\pi_2|_{U_2} : U_2 \rightarrow U_3$ are étale. Furthermore, let Y_i , $1 \leq i \leq 3$, be the closed complement of U_i in X_i .

Then $(\pi_2 \circ \pi_1)|_{U_1} : U_1 \rightarrow U_3$ is tamely ramified along Y_3 if and only if $\pi_1|_{U_1} : U_1 \rightarrow U_2$ is tamely ramified along Y_2 and $\pi_2|_{U_2} : U_2 \rightarrow U_3$ is tamely ramified along Y_3 .

Consider the category $\mathbf{FEtT}/(X, Y)$ of U -schemes, finite and étale over U with tame ramification along Y . It is a full subcategory of the category \mathbf{FEt}/U of U -schemes, finite and étale over U . If $\bar{x} \rightarrow U$ is a geometric point of U , we consider the fibre functor

$$\mathbf{F} : \mathbf{FEtT}/(X, Y) \longrightarrow (\text{Sets}); \quad (V \rightarrow U) \longmapsto \text{Mor}_U(\bar{x}, V),$$

which is just the restriction of the usual fibre functor induced by \bar{x} on \mathbf{FEt}/U to $\mathbf{FEtT}/(X, Y)$. One observes that the category $\mathbf{FEtT}/(X, Y)$ together with this fibre functor satisfies the axiomatic conditions for a Galois theory (see [SGA1] Exp.V,4). Thus one obtains a profinite fundamental group

$$\pi_1^t(X, Y; \bar{x})$$

which we call the **tame fundamental group** of U with base point \bar{x} and with respect to the embedding $U \hookrightarrow X$. It is a quotient of the étale fundamental group $\pi_1(U; \bar{x})$. If $\bar{x}' \hookrightarrow U$ is another base point in the same connected component of U , then the fundamental group $\pi_1^t(X, Y; \bar{x}')$ is isomorphic to $\pi_1^t(X, Y; \bar{x})$, the isomorphism is determined up to an inner automorphism. In the following, U will always be connected and we will omit the base point from the notation. If X is a scheme of characteristic 0 (i.e. all residue fields have characteristic 0), then

$$\pi_1^t(X, Y) = \pi_1(U).$$

If X is normal and connected, then the functor $\mathbf{FEt}/X \rightarrow \mathbf{FEt}/U$ is fully faithful, and we obtain surjections

$$\pi_1(U) \twoheadrightarrow \pi_1^t(X, Y) \twoheadrightarrow \pi_1(X).$$

The next proposition follows in a straightforward manner from the theorem on the purity of the branch locus of Zariski-Nagata ([SGA2], Exp.X, th. 3.4).

Proposition 1.9 *Assume that X is regular and connected, and that Y is of codimension ≥ 2 in X . Then we have natural isomorphisms*

$$\pi_1(U) \xrightarrow{\simeq} \pi_1^t(X, Y) \xrightarrow{\simeq} \pi_1(X).$$

Let $U' \rightarrow U$ be a finite étale morphism. The **wild locus** $W_{(U' \rightarrow U)}$, i.e. the set of points $y \in Y$ such that $U' \rightarrow U$ is not tamely ramified at y , is a closed subscheme of Y . The theorem of Zariski-Nagata on the purity of the branch locus says that if X is regular, the ramification locus of a quasifinite dominant morphism to X is always pure of codimension 1 in X . The same is not true of the wild locus, since in example 1.3 we constructed a cyclic cover of a regular scheme of dimension 2 whose wild locus consists of a single closed point, i.e. is of codimension 2. But if X is equicharacteristic, we have the

Proposition 1.10 *Assume that X is regular and connected and that all points of X have the same residue characteristic. Let $U' \rightarrow U$ be a finite nilpotent étale covering. Then $W_{(U' \rightarrow U)}$ is either empty or pure of codimension 1 in X .*

Proof: We may assume that all points on X have the common residue characteristic $p > 0$, because otherwise the wild locus is empty. Let K be the function field of X and L the function field of U' . Then $L|K$ is a finite nilpotent extension and $U' = U_L$ is the normalization of U in L . Writing L as a composite $L = L_1 L_2$, where $[L_1 : K]$ is prime to p and $[L_2 : K]$ is a p -power, then $U_{L_1} \rightarrow U$ is tamely ramified along Y . The ramification locus of $X_{L_2} \rightarrow X$ is the same as the wild locus $W_{U_{L_2} \rightarrow U}$ since $G(L_2|K)$ is a p -group and all points of X have residue characteristic p . Since X is regular, the ramification locus of $X_{L_2} \rightarrow X$ is either empty or pure of codimension 1 and coincides with $W_{(U_{L_2} \rightarrow U)} = W_{(U' \rightarrow U)}$. \square

Remark 1.11 Under the assumptions of proposition 1.10, let D_1, \dots, D_r be the irreducible components of $Y = X - U$ which are of codimension 1 in X . Then a finite nilpotent covering $U' \rightarrow U$ is tamely ramified along Y if and only if the discrete valuations v_1, \dots, v_r of $K(X) = K(U)$ which are associated to D_1, \dots, D_r are tamely ramified in $K(U')|K(U)$. This justifies the ‘naive’ definition of tame ramification used in [S-S1].

From now on assume that X is a normal connected scheme and we denote the function field of X by K . Every connected finite étale covering $\tilde{X} \rightarrow X$ coincides with the normalization of X in some finite separable extension L of K . Therefore (omitting suitable chosen base points from the notation), we obtain a natural surjection $G(\tilde{K}|K) = \pi_1(\text{Spec}(K)) \twoheadrightarrow \pi_1(X)$ and an isomorphism

$$G(\tilde{K}|K) = \varprojlim_{U \subset X} \pi_1(U),$$

where U runs through the open subschemes of X .

Let us collect some facts on decomposition and inertia of integrally closed domains (see [B-Co] Ch. V, §§2,3). Let A be an integrally closed domain with quotient field K , L a Galois extension of K and B the integral closure of A in L . Let \mathfrak{P} be a prime ideal of B and $\mathfrak{p} = \mathfrak{P} \cap A$ the prime ideal in A lying under \mathfrak{P} . Let $k(\mathfrak{P}) = B_{\mathfrak{P}}/\mathfrak{P}B_{\mathfrak{P}}$ and $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ be the residue fields of \mathfrak{P} and \mathfrak{p} ,

respectively. Let $G = G(L|K)$ be the Galois group. Then G acts transitively on the set of prime ideals of B lying over \mathfrak{p} . Then one has the following subgroups in the Galois group.

- $Z = Z_{\mathfrak{p}}(L|K) = \{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$ - the decomposition group,
- $T = T_{\mathfrak{p}}(L|K) = \{\sigma \in G_{\mathfrak{p}} \mid \bar{\sigma} = id : B/\mathfrak{P} \rightarrow B/\mathfrak{P}\}$ - the inertia group.

T is a normal subgroup in Z . Let A^T and A^Z be the integral closures of A in L^T and L^Z respectively. Put $\mathfrak{P}^T = \mathfrak{P} \cap A^T$ and $\mathfrak{P}^Z = \mathfrak{P} \cap A^Z$. For a proof of the following proposition see [Ra], X, th.2.

Proposition 1.12 *If $L = \bar{K}$ is the separable closure of K , then*

- (i) \mathfrak{P} is the only prime ideal in B extending the prime ideal \mathfrak{P}^Z in A^Z ,
- (ii) $(A_{\mathfrak{P}^Z}^Z, \mathfrak{P}^Z)$ is the henselization of $(A_{\mathfrak{p}}, \mathfrak{p})$, in particular, $k(\mathfrak{P}^Z) = k(\mathfrak{p})$,
- (iii) $(A_{\mathfrak{P}^T}^T, \mathfrak{P}^T)$ is the strict henselization of $(A_{\mathfrak{p}}, \mathfrak{p})$, in particular, $k(\mathfrak{P}^T)$ is the separable closure of $k(\mathfrak{p})$.

Assume that X is normal and connected. Then the normalization of X in a finite separable extension of its function field is finite over X . As is well known, for every point x on X , the scheme X_x^{sh} is noetherian, normal and connected.

Corollary 1.13 *Let X be a normal connected scheme, $U \subset X$ an open subscheme and $Y = X - U$ the closed complement. Let L be a finite Galois extension of the function field K of X and let U_L, X_L be the normalizations of U and X in L . Then the following are equivalent*

- (i) $U_L \rightarrow U$ is étale and tamely ramified along Y ,
- (ii) For each point $\mathfrak{P} \in X_L$ the following holds, where p denotes the residue characteristic of \mathfrak{P} :
 - a) $T_{\mathfrak{P}}(L|K)$ is of order prime to p
 - b) $T_{\mathfrak{P}}(L|K) = 0$ if $\mathfrak{P} \in U_L$.

Proof: The only nontrivial point is to see that $U_L \rightarrow U$ is flat, but this is well known (cf. [Mi], I, §3, th.3.21). \square

Proposition 1.14 *Let X be a regular connected scheme and $D \subset X$ a normal crossing divisor. Then both notions of tame ramification coincide.*

Proof: Let (notation as above) $U_L \rightarrow U$ be tame in the sense of definition 1.4 and assume that v is a discrete valuation on K which corresponds to the generic point P of an irreducible component of D . Let (for any embedding to the separable closure of K) K_P^{sh} be the quotient field of the strict henselization of the discrete valuation ring $\mathcal{O}_{X,P}$. Then, by definition, LK_P^{sh} is a product of fields each of which is contained in a finite Galois extension of degree prime to

$p = \text{char } k(P)$ of K_P^{sh} . Therefore v is tamely ramified in $L|K$. On the other hand, if all these discrete valuations are tamely ramified in $L|K$, then it follows from the Generalized Abhyankar's lemma (theorem 1.2) that $U_L \rightarrow U$ is tamely ramified in the sense of definition 1.4. \square

Remark: Corollary 1.13 asserts that for coverings of normal schemes our definition of tameness coincides with that given in [Ab]. Furthermore (in the situation of corollary 1.13), if $U_L \rightarrow U$ is tame, then $X_L \rightarrow X$ is numerically tame in the sense of [C-E]. It is also not difficult to show the inverse implication.

2 Valuations and tame ramification

The aim of this section is to relate tame covers of schemes of finite type over $\text{Spec}(\mathbb{Z})$ to discrete valuations of higher rank. The quotient fields of the henselizations of such valuation rings are called higher dimensional henselian fields and are the basic constituents of Kato's and Saito's higher dimensional class field theory [K-S].

Let us recall some facts from valuation theory (see [B-Co], [Z-S], Ch. VI). Let K be a field. A subring $V \subset K$ is called valuation ring if $x \in K - V \implies x^{-1} \in V$. A valuation ring is local with maximal ideal $\mathfrak{m}_V = \{x \in V \mid x^{-1} \notin V\}$ and is integrally closed in K . The quotient $\Gamma_V = K^\times / V^\times$ is a totally ordered abelian group ($\bar{x} \leq \bar{y} \iff x^{-1}y \in V$) and is called the value group of V . Usually, one extends the natural map $v : K^\times \rightarrow \Gamma_V$, to K by setting $v(0) = \infty$. As is well known, the valuation ring V represents an equivalence class of abstractly defined valuations of K ; the valuation v is a natural representative of this class.

If $L|K$ is a finite separable field extension, then there are at least one and at most finitely many valuations w of L extending v , i.e. such that $W \cap K = V$, where $W \subset L$ is the valuation ring of w . We have an induced injective homomorphism $\Gamma_V \hookrightarrow \Gamma_W$. The index of Γ_V in Γ_W is called the ramification index and is usually denoted by $e = e_{w|v}$. We also have an associated field extension $k(v) = V/\mathfrak{m}_V \hookrightarrow k(w) = W/\mathfrak{m}_W$ whose degree is denoted by $f = f_{w|v}$. The inequality

$$(1) \quad \sum_{w|v} e_{w|v} f_{w|v} \leq [L : K],$$

(see [B-Co] Ch.6, §8.3. th.1) shows, in particular, that all these numbers are finite. v is called **defectless** in L if equality holds in (1). By [B-Co], VI, §8.5, cor.2, a discrete valuation is defectless in a finite separable extension.

Let L be a Galois extension of K , W a valuation ring of L and $V = W \cap K$. Let \mathfrak{P} be the unique maximal ideal of W . Then, besides the inertia and decomposition group, we have the **ramification group**

$$R_{\mathfrak{P}} = R_{\mathfrak{P}}(L|K) = \left\{ \sigma \in T_{\mathfrak{P}}(L|K) \mid \frac{\sigma x}{x} \equiv 1 \pmod{\mathfrak{P}} \text{ for all } 0 \neq x \in L \right\}.$$

A proof of the following proposition can be found in [End], §20.

Proposition 2.1 *If the residue characteristic of V is zero, then $R_{\mathfrak{P}} = 0$ and $T_{\mathfrak{P}}$ is abelian. If the residue characteristic of V is $p > 0$, then*

- (i) $R_{\mathfrak{P}}$ is a normal subgroup in $Z_{\mathfrak{P}}$,
- (ii) $R_{\mathfrak{P}}$ is the unique p -Sylow subgroup in $T_{\mathfrak{P}}$,
- (iii) $T_{\mathfrak{P}}/R_{\mathfrak{P}}$ is an abelian group of order prime to p .

The decomposition, inertia, ramification groups of different valuation rings W of L extending V are conjugate in $G(L|K)$. One says that $W|V$ (resp. $w|v$) is **tamely ramified** if the ramification group is trivial. V is tamely ramified in $L|K$ if $W|V$ is tamely ramified for one (every) valuation ring W of L extending V . One says that V is tamely ramified in a separable extension if it is tamely ramified in the Galois closure of this extension.

If V is tamely ramified in $L|K$, then it is defectless in $L|K$ (see [End], cor. 20.22). A valuation v on K is called **discrete valuation of rank n** if its value group is (as an ordered group) isomorphic to $\mathbb{Z} \times \cdots \times \mathbb{Z}$ (n factors) with the lexicographic order. By the unspecified term discrete valuation we always mean discrete valuation of rank 1, i.e. a discrete valuation in the usual sense.

Let V be a discrete valuation ring of rank n of K . Then V has n distinct prime ideals $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$. For $0 \leq i \leq n$, let V_i be the localization of V at \mathfrak{p}_i and let $k_i = k(\pi_i)$ be the residue field of V_i . The V_i are also discrete valuation rings (of rank $n - i$) of K and

$$V = V_0 \subset V_1 \subset \cdots \subset V_n = K.$$

For $0 \leq i < j \leq n$ the image of V_i in k_j is a discrete valuation ring of rank $j - i$. For $0 \leq i < n$, we denote the image of V_i in k_{i+1} by \bar{V}_i . Then \bar{V}_i is a usual (i.e. rank 1) discrete valuation ring with quotient field k_{i+1} and residue field k_i . By [B-Co], Ch. VI, §7, ex.6d, the ring V is henselian if and only if all \bar{V}_i , $0 \leq i < n$ are henselian (see also [Ri], ch. F, prop.9).

Let $V \subset K$ be a discrete valuation ring of rank n and let $L|K$ be a finite separable extension. Let $w|v$ be a valuation of L extending v . Then also w is a discrete valuation of rank n . Assume that the residue characteristic of v is $p > 0$.

Lemma 2.2 *Let v be a discrete valuation of rank n of a field K and let $L|K$ be a separable extension of K . Then v is tamely ramified in $L|K$ if and only if for every extension w of v to L the ramification index $e_{w|v}$ is prime to p and the successive residue extensions of $\bar{W}_i|\bar{V}_i$ are separable for $0 \leq i < n$.*

Proof: We may assume that v is strictly henselian and that $L|K$ is finite. In particular, v has a unique extension w to L . Assume that v is tamely ramified in $L|K$. By proposition 2.1, (iii), $L|K$ is an abelian extension of degree prime to p . Since v is defectless in $L|K$, the ramification index is prime to p . Furthermore, the degrees of the successive residue extensions are prime to p , and so these extensions are separable.

It remains to show the other direction. Since the successive residue extensions are separable, and since discrete valuations of rank 1 are defectless in separable extensions, one shows inductively (cf. [B-Co] Ch.VI §7 ex.5 or [Ri], G th.5) that v is defectless in $L|K$. In particular, $[L : K] = e_{w|v}$ is prime to p . By proposition 2.1 (ii),(iii), we conclude that $L|K$ is contained in a Galois extension of degree prime to p and is therefore tamely ramified. \square

We call a field k an **n -dimensional henselian field** if it is the quotient ring of a henselian discrete valuation ring of rank n with finite residue field. Equivalently, one can give an inductive definition: A 0-dimensional henselian field is just a finite field. An $(n+1)$ -dimensional henselian field is a field which is henselian under a discrete (rank 1) valuation whose residue field is an n -dimensional henselian field. So an n -dimensional henselian field κ comes along with a sequence $\kappa_{n-1}, \dots, \kappa_0$ of residue fields, where each κ_i , $i = n-1, \dots, 0$ is an i -dimensional henselian field. In particular, κ_0 is finite.

We call (by abuse of notation) an extension of n -dimensional henselian fields tamely ramified, if it is tamely ramified with respect to the discrete valuations of rank n . The next result follows easily by induction from the corresponding 1-dimensional result.

Proposition 2.3 *Let κ be a n -dimensional henselian field with last residue field κ_0 of characteristic $p > 0$ and let κ^t be the maximal tamely ramified extension of κ . Then there is a natural isomorphism*

$$G(\kappa^t|\kappa) \cong \left(\underbrace{\hat{\mathbb{Z}}^{(p')}(1) \times \dots \times \hat{\mathbb{Z}}^{(p')}(1)}_{n \text{ -times}} \right) \rtimes G(\bar{\kappa}_0|\kappa_0),$$

where $(\hat{\mathbb{Z}})^{(p')}$ denotes the prime-to- p part of $\hat{\mathbb{Z}}$ and (1) means the first Tate-twist with respect to the cyclotomic character.

From now on let X be a reduced, separated, equidimensional scheme of finite type over $\text{Spec}(\mathbb{Z})$, in particular ([Ma], Ch.13, no.34), X is excellent. Let $d = \dim(X) = \dim_{K^{\text{rull}}}(X)$. A **Parshin-chain** P on X is a sequence (P_0, P_1, \dots, P_d) of points on X such that

$$\overline{\{P_0\}} \subset \overline{\{P_1\}} \subset \dots \subset \overline{\{P_d\}} = X$$

and $\dim \overline{\{P_i\}} = i$ for $i = 0, \dots, d$.

An inductive localization-henselization procedure, which was proposed by Parshin with completion instead of henselization and which is described in [K-S], §3, provides us with a functor $(X, P) \mapsto \kappa_P^h$

$$\left(\begin{array}{l} \text{Reduced schemes } X \text{ of} \\ \text{finite type over } \text{Spec}(\mathbb{Z}) \\ \text{with a Parshin-chain } P \\ \text{of length } d = \dim X \end{array} \right) \longrightarrow \left(\begin{array}{l} \text{finite products} \\ \text{of } d\text{-dimensional} \\ \text{henselian fields} \end{array} \right).$$

Let us briefly recall this construction. First one takes the henselization \mathcal{O}_{X, P_0}^h of the local ring of X at P_0 . Then one considers the finitely many prime ideals in this ring which lie over P_1 , passes to the product of the henselizations of the localizations with respect to these prime ideals, and so on.

If in the Parshin-chain P each P_i is a regular point of $\overline{\{P_{i+1}\}}$, then κ_P^h is a d -dimensional henselian field rather than a finite product of such fields.

Assume that X is integral and let K be the function field of X . We say that a discrete valuation ring V of rank d in K **dominates** a Parshin-chain P if V_i dominates \mathcal{O}_{X, P_i} for $i = 1, \dots, d$. For a proof of the following proposition see [K-S], prop.3.3.

Proposition 2.4 *Under the above assumptions,*

$$\kappa_P^h = \prod_V \text{Quot}(V^h),$$

where V ranges over all discrete valuations of rank d which dominate P and $\text{Quot}(V^h)$ denotes the quotient field of the henselization V^h of V .

Proposition 2.5 *Let X be a d -dimensional, regular, connected scheme of finite type over $\text{Spec}(\mathbb{Z})$. Let $U \subset X$ be an open subscheme and $Y = X - U$ the complement. Let L be a finite nilpotent extension of the function field K of X . Then the following are equivalent.*

- (i) $U_L \rightarrow U$ is étale and tamely ramified along Y .
- (ii) Every discrete valuation v of rank d in K which dominates a Parshin chain on X is tamely ramified in L , and unramified if the dominated chain is contained in U .
- (iii) For every Parshin-chain P on X the extension of (finite products of) d -dimensional henselian fields $\kappa_P^h L | \kappa_P^h$ is tamely ramified, and unramified if P is contained in U .

Proof: The implication (i) \Rightarrow (ii) is obvious and the equivalence (ii) \Leftrightarrow (iii) follows from proposition 2.4. Suppose that (ii) holds. Let P_{d-1} be a point of codimension 1 on U . We extend P_{d-1} to a Parshin-chain $(P_0, \dots, P_{d-1}, P_d)$ on U . Let V be a discrete valuation ring in K dominating P . Then (notation as above), V_{d-1} dominates and hence is equal to $\mathcal{O}_{U, P_{d-1}}$. Since V is unramified in $L|K$, so is V_{d-1} . By the theorem of Zariski-Nagata on the purity of the branch locus, we conclude that $U_L \rightarrow U$ is étale.

It remains to show that $U_L \rightarrow U$ is tamely ramified along Y . Since $L|K$ is nilpotent, we can easily reduce to the case that the Galois group is a finite p -group for some prime number p . Let x_1, \dots, x_n be the finitely many codimension 1 points which ramify in $X_L \rightarrow X$. Since the set of points in X where $U_L \rightarrow U$ is not tame is closed, it suffices to show that it is tamely ramified at every closed point $y \in Y$.

$U_L \rightarrow U$ is tamely ramified at every point of residue characteristic different from p and at every point which is not contained in $\cup_{i=1}^n \overline{\{x_i\}}$.

We show that a closed point y of residue characteristic p and with $y \in \overline{\{x_i\}}$ for some $1 \leq i \leq n$ does not exist. Assume that y is such a point. Choose a Parshin-chain $P = (P_0, \dots, P_d)$ on X with $P_0 = y$ and $P_{d-1} = x_i$ (we find such a chain because X is catenary). Choose a discrete valuation ring V of rank d in K dominating P . Then (for appropriately chosen embeddings to \bar{K}) the quotient field $K_{x_i}^{sh}$ of the strict henselization of \mathcal{O}_{X, x_i} contains $\text{Quot}(V^{sh})$. By assumption $L\text{Quot}(V^{sh})|\text{Quot}(V^{sh})$ is a field extension of degree prime to p , hence trivial. But, since x_i is ramified in the p -extension $L|K$, $LK_{x_i}^{sh}|K_{x_i}^{sh}$ is nontrivial. This yields the required contradiction. \square

From the proof of the last proposition, we obtain the

Corollary 2.6 *Assume X is connected but only normal. Let $L|K$ be a finite Galois extension of p -power degree. If $U_L \rightarrow U$ is étale and tamely ramified along Y , then $L|K$ is unramified at every discrete valuation associated to a codimension 1 point of X which either lies on U or whose closure in X contains a point of residue characteristic p . If X is regular, the inverse implication is also true.*

Proposition 2.7 *Suppose that U is regular and let X_1 and X_2 be regular schemes which are proper over $\text{Spec}(\mathbb{Z})$ and which contain U as an open subscheme. Let $Y_i = X_i - U$, $i = 1, 2$, be the closed complements. Let $U' \rightarrow U$ be a finite nilpotent étale covering. Then $U' \rightarrow U$ is tamely ramified along Y_1 if and only if it is tamely ramified along Y_2 .*

Proof: We may suppose that the degree of $U' \rightarrow U$ is a power of some prime number p . Let X_3 be a normal scheme which is proper over $\text{Spec}(\mathbb{Z})$, contains U as an open subscheme and has proper surjective morphisms $\pi_i : X_3 \rightarrow X_i$, $i = 1, 2$ making the diagrams

$$\begin{array}{ccc} U & \hookrightarrow & X_3 \\ \parallel & & \downarrow \pi_i \\ U & \hookrightarrow & X_i \end{array}$$

commutative. (The existence of such a X_3 is well known, for instance, take X_3 as the normalization of the closure of the image of U in $X_1 \times_{\mathbb{Z}} X_2$.) If $U' \rightarrow U$ is tamely ramified along Y_1 , then it is tamely ramified along $Y_3 = X_3 - U$ (see lemma 1.5). Let L be the function field of U' . Then, by corollary 2.6, $L|K$ is tamely ramified at every discrete valuation on L which is defined by a codimension 1 point of X_3 which lies on Y_3 and whose closure contains a point of residue characteristic p . This set of valuation contains the set of discrete valuation on L which are defined by a codimension 1 point of X_2 which lies on Y_2 and whose closure contains a point of residue characteristic p . Applying corollary 2.6 again, we conclude that $U' \rightarrow U$ is tamely ramified along Y_2 . \square

Theorem 1 of the introduction follows immediately from proposition 2.7. In particular, the abelianized tame fundamental group of a regular scheme of finite type over $\text{Spec}(\mathbb{Z})$ does not depend on the choice of a compactification. So it is justified to use the notation $\pi_1^t(X)^{ab}$ for $\pi_1^t(\bar{X}, \bar{X} - X)^{ab}$ for this group.

3 A finiteness result

The aim of this section is to prove the following theorem 3.1 (=th.2 from the introduction). In the proof we essentially use a finiteness result of N. Katz and S. Lang on relative étale fundamental groups ([K-L], th.1).

Theorem 3.1 *Let \mathcal{O} be the ring of integers in a finite extension k of \mathbb{Q} . Let X be a flat \mathcal{O} -scheme of finite type whose geometric generic fibre $X \otimes_{\mathcal{O}} \bar{k}$ is connected. Assume that X is normal and that the morphism $X \rightarrow \text{Spec}(\mathcal{O})$ is surjective. Let U be an open subscheme of X . Then the abelianized tame fundamental group $\pi_1^t(X, X - U)^{ab}$ is finite.*

For the proof of theorem 3.1 we need the following

Lemma 3.2 *Let A be a strictly henselian discrete valuation ring with perfect (hence algebraically closed) residue field and with quotient field k . Let $k_{\infty}|k$ be a \mathbb{Z}_p -extension. Let $K|k$ be a regular field extension and let $B \subset K$ be a discrete valuation ring dominating A . Then B is ramified in $Kk_{\infty}|K$.*

Proof: For each $n \geq 0$, let $k_n|k$ be the unique subextension of degree p^n in $k_{\infty}|k$. Let A_n (resp. B_n) denote the normalization of A (resp. B) in k_n (resp. Kk_n). A_n is again a strictly henselian discrete valuation ring. B_n is a semi-local Dedekind domain.

Suppose that B is unramified in Kk_{∞} . Fix an n . Let \mathfrak{p} be the prime ideal of B and let $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ be the prime ideals of B_n . Since $B_n|B$ is Galois, all \mathfrak{p}_i have a common inertia index $f = f_n$, so $N_{B_n|B}(\mathfrak{p}_i) = \mathfrak{p}^f$ for $i = 1, \dots, g$. Since $B_n|B$ is étale, we have $p^n = [k_n : k] = gf$.

Let π_n be a uniformizer of A_n . Since $A_n|A$ is totally ramified, we have

$$v_A(N_{k_n|k}(\pi_n)) = 1.$$

Considered as an element of B_n , the \mathfrak{p}_i -valuation of π_n is positive and independent of i , $1 \leq i \leq g$. Denoting this positive number by a , we have the following equality of ideals in B_n

$$(\pi_n) = (\mathfrak{p}_1 \cdots \mathfrak{p}_g)^a.$$

Hence $N_{B_n|B}((\pi_n)) = \mathfrak{p}^{afg}$ and therefore

$$v_B(N_{k_n|k}(\pi_n)) = ap^n \geq p^n.$$

On the other hand, we have

$$v_B(N_{k_n|k}(\pi_n)) = v_B(\pi_0) \cdot v_A(N_{k_n|k}(\pi_n)) = v_B(\pi_0),$$

where π_0 is any uniformizer in $A = A_0$. Since n was arbitrary, the inequality $v_B(\pi_0) \geq p^n$ yields a contradiction. \square

Proof of theorem 3.1: Since X is normal, for any open subscheme V of U the natural homomorphism $\pi_1(V) \rightarrow \pi_1(U)$ is surjective. Therefore also the homomorphism

$$\pi_1^t(X, X - V)^{ab} \longrightarrow \pi_1^t(X, X - U)^{ab}$$

is surjective and so we may replace U by a suitable open subscheme and assume that U is smooth over $\bar{S} = \text{Spec}(\mathcal{O})$. Let $S \subset \bar{S}$ be the image of U . Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(U/S) & \longrightarrow & \pi_1(U)^{ab} & \longrightarrow & \pi_1(S)^{ab} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker}^t(U/S) & \longrightarrow & \pi_1^t(X, X - U)^{ab} & \longrightarrow & \pi_1^t(\bar{S}, \bar{S} - S)^{ab} \end{array}$$

where the groups $\text{Ker}(U/S)$ and $\text{Ker}^t(U/S)$ are defined by the exactness of the corresponding rows, and the two right vertical homomorphisms are surjective. By a theorem of N. Katz and S. Lang ([K-L], th.1), the group $\text{Ker}(U/S)$ is finite. By classical one-dimensional class field theory, the group $\pi_1^t(\bar{S}, \bar{S} - S)^{ab}$ is finite (it is the Galois group of the ray class field of k with modulus $\prod_{\mathfrak{p} \notin S} \mathfrak{p}$). The kernel of $\pi_1(S)^{ab} \rightarrow \pi_1^t(\bar{S}, \bar{S} - S)^{ab}$ is generated by the ramification groups of the primes of \bar{S} which are not in S . Denoting the product of the residue characteristics of these primes by N , we see that $\pi_1(S)^{ab}$ is the product of a finite group and a topologically finitely generated pro- N group. Therefore the same is also true for $\pi_1(U)^{ab}$ and for $\pi_1^t(X, X - U)^{ab}$. By the snake lemma, it will suffice to show that the cokernel C of the induced map $\text{Ker}(U/S) \rightarrow \text{Ker}^t(U/S)$ is a torsion group.

Let K be the function field of X and let k_1 be the maximal abelian extension of k such that the normalization U_{Kk_1} of U in the composite Kk_1 is étale over U and tamely ramified along Y . By [K-L], lemma 2, (2), the normalization of S in k_1 is ind-étale over S . Let $k_2|k$ be the maximal subextension of $k_1|k$ such that the normalization S_{k_2} of S in k_2 is étale over S and tamely ramified along $\bar{S} - S$. Then $G(k_2|k) = \pi_1^t(\bar{S}, \bar{S} - S)^{ab}$ and, by the snake lemma, $C \cong G(k_1|k_2)$.

In order to show that C is a torsion group, we therefore have to show that $k_1|k_2$ does not contain a \mathbb{Z}_p -extension of k_2 for any prime number p . Since $k_2|k$ is a finite extension and $k_1|k$ is abelian, this is equivalent to the assertion that $k_1|k$ contains no \mathbb{Z}_p -extension of k for any prime number p . Let p be a prime number and suppose that $k_\infty|k$ is a \mathbb{Z}_p -extension such that the normalization U_{Kk_∞} is ind-étale over U and ind-tamely ramified along Y . A \mathbb{Z}_p -extension is unramified outside p and at least ramified at one prime dividing p , see e.g. [NSW], (10.3.20)(ii). Since the normalization of S in k_1 is ind-étale, we may suppose that $p|N$. Let k' be the maximal unramified subextension of $k_\infty|k$ and let \bar{S}' be the normalization of \bar{S} in k' . Then the base change $X' = X \times_{\bar{S}} \bar{S}' \rightarrow X$ is étale. Hence X' is normal and the pre-image U' of U is smooth and

geometrically connected over k' . So, after replacing k by k' , we may suppose that $k_\infty|k$ is totally ramified at a prime $\mathfrak{p}|p$, $\mathfrak{p} \in \bar{S} - S$.

Let $\mathcal{O}_{\mathfrak{p}}$ be the local ring of \bar{S} at \mathfrak{p} . After a base change to the strict henselization A of $\mathcal{O}_{\mathfrak{p}}$, we arrive at the following situation (several letters get a new meaning):

- (1) a strictly henselian discrete valuation ring A with perfect residue field of characteristic p and with quotient field k of characteristic 0
- (2) a flat connected and normal A -scheme of finite type X which projects surjectively to $\text{Spec}(A)$
- (3) the function field K of X is a regular extension of k (i.e. k is algebraically closed in K)
- (4) an open subscheme $U \subset X$ which is smooth over K
- (5) a \mathbb{Z}_p -extension $k_\infty|k$ (in which A is totally ramified)
- (6) The normalization of U in Kk_∞ is ind-étale over U and ind-tamely ramified along $Y = X - U$.

Let us show that such a situation cannot occur. Let P be the generic point of an irreducible component of the special fibre of X over A . Then $B = \mathcal{O}_{X_P}$ is a discrete valuation ring dominating A . Now lemma 3.2 shows that B ramifies in Kk_∞ . Therefore the order of the inertia group of P in Kk_∞ is divisible by $p = \text{char } k(P)$. By corollary 1.13, this contradicts assumption (6). \square

In the special case $U = X$ we obtain the following corollary.

Corollary 3.3 *Let \mathcal{O} be the ring of integers in a finite extension k of \mathbb{Q} . Let X be a flat \mathcal{O} -scheme of finite type whose geometric generic fibre $X \otimes_{\mathcal{O}} \bar{k}$ is connected. Assume that X is normal and that the morphism $X \rightarrow \text{Spec}(\mathcal{O})$ is surjective. Then the abelianized étale fundamental group $\pi_1(X)^{ab}$ is finite.*

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