On Poitou's duality theorem

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Dedicated to the memory of Jürgen Neukirch

The Galois cohomology of the S-idele class group

$$H^i(G(k_S|k), C_S(k_S))$$

plays an important role in class field theory. Here S is a set of primes of a number field k, k_S denotes the maximal extension of k which is unramified outside S and, though carrying a canonical topology, $C_S = C_S(k_S)$ is considered as a discrete $G_S = G(k_S|k)$-module. The natural locally compact topology on C_S, however, becomes essential in the proof of the global duality theorem of Tate-Poitou. The crucial point is the following theorem of Poitou which is applied to the submodule $C_S^0 \subseteq C_S$ of idele classes of norm 1.

Theorem 1 (Poitou) *Let G be a profinite group and let C be a class formation for G such that the group of universal norms*

$$N_U C = \bigcap_{V \subseteq U} N_{U/V} C^V \subseteq C^U$$

is divisible for all open subgroups $U \subseteq G$. Furthermore, suppose that for every open subgroup $U \subseteq G$ the group of invariants C^U carries a natural compact topology. Then the cup product

$$\hat{H}^i(G, \mathrm{Hom}(A, C)) \times \hat{H}^{2-i}(G, A) \xrightarrow{\cup} H^2(G, C) \xrightarrow[\sim]{inv} \tfrac{1}{\#G}\mathbb{Z}/\mathbb{Z}$$

induces isomorphisms

$$\hat{H}^i(G, \mathrm{Hom}(A, C)) \xrightarrow{\sim} \hat{H}^{2-i}(G, A)^\vee$$

for all $i \leq 0$ and every discrete G-module A which is finitely generated as a $\mathbb{Z}$-module.

Here $\vee$ denotes the Pontryagin dual, $\hat{H}^i(G, \mathrm{Hom}(A, C))$ is the usual cohomology of profinite groups in positive dimensions and for $i \leq 0$ it denotes the Tate cohomology of profinite groups, which is defined by

$$\hat{H}^i(G, M) = \varprojlim_U \hat{H}^i(G/U, M^U), \quad i \leq 0, \tag{1}$$

where U runs through the open normal subgroups of G. For $V \subseteq U$ the transition map is given by the deflation $\mathit{def}: \hat{H}^i(G/V, M^V) \to \hat{H}^i(G/U, M^U)$, whose definition we will recall below.

The aim of this paper is twofold. First, answering a question of J. Neukirch, we give a complete proof of theorem 1, i.e. we fill a gap in the original proof of Poitou. Secondly, in the arithmetic case, we extend Poitou's theorem to positive dimensions i, proving theorems 2 and 3 below. We denote the set of archimedean places of a number field k by S_∞ and, for a prime number p, the set of primes dividing p by S_p. If A is a G-module and $i > 0$, we write $\hat{H}^i(G_S, A)(p)$ for the p-torsion subgroup of the torsion group $\hat{H}^i(G_S, A)$. If $i \leq 0$ and $\hat{H}^i(G_S, A)$ is profinite, then we use the notation $\hat{H}^i(G_S, A)(p)$ for the maximal pro-p-factor group.

Theorem 2 *Let p be a prime number and let S be a finite set of primes of the number field k such that $S \supseteq S_p \cup S_\infty$. Then the cup product*

$$\hat{H}^i(G_S, \mathrm{Hom}(A, C_S)) \times \hat{H}^{2-i}(G_S, A) \xrightarrow{\cup} H^2(G_S, C_S) \xrightarrow[\sim]{inv} \tfrac{1}{\#G_S}\mathbb{Z}/\mathbb{Z}$$

induces isomorphisms

$$\hat{H}^i(G_S, \mathrm{Hom}(A, C_S))(p) \xrightarrow{\sim} \hat{H}^{2-i}(G_S, A)(p)^\vee$$

for all $i \in \mathbb{Z}$ and every discrete G-module A which is finitely generated as a $\mathbb{Z}$-module.

So far, the pairing $H^1(G_S, \mathrm{Hom}(A, C_S)) \times H^1(G_S, A) \xrightarrow{\cup} H^2(G_S, C_S)(p) \cong \mathbb{Q}_p/\mathbb{Z}_p$ has been known to be non-degenerate only for finite p-primary A (cf. [S] thm.4). The extension of this duality to finitely generated modules comes somewhat unexpected because we do not know whether the strict cohomological p-dimension of G_S actually equals 2. (A proof of this would require to show the validity of Leopoldt's conjecture for the prime number p and for every finite subextension of k in k_S). Furthermore, note that we did not assume that k is totally imaginary, if $p = 2$. The essential input in the proof of theorem 2 is theorem 6 of [S] which asserts that C_S is p-divisible under the given assumptions.

As an application of the principal ideal theorem, the p-divisibility of C_S can also be verified for prime numbers p such that none of the primes dividing p is in S. Recall that one says that p^∞ divides the order $\#G$ of a profinite group G, if we find open subgroups in G of index divisible by arbitrary high p-powers.

Theorem 3 *Let p be a prime number and let S be a finite set of primes of the number field k such that $S \supseteq S_\infty$ and $S \cap S_p = \emptyset$. If $p^\infty | \#G_S$, then the cup product*

$$\hat{H}^i(G_S, \mathrm{Hom}(A, C_S)) \times \hat{H}^{2-i}(G_S, A) \xrightarrow{\cup} H^2(G_S, C_S) \xrightarrow[\sim]{inv} \tfrac{1}{\#G_S}\mathbb{Z}/\mathbb{Z}$$

induces isomorphisms

$$\hat{H}^i(G_S, \mathrm{Hom}(A, C_S))(p) \xrightarrow{\sim} \hat{H}^{2-i}(G_S, A)(p)^\vee$$

for all $i \in \mathbb{Z}$ and every discrete G-module A which is finitely generated as a $\mathbb{Z}$-module.

We do not know anything about the cohomological p-dimension of G_S in the situation of theorem 3. It applies, for example, to the case $S = S_\infty$, if the p-class field tower of k is infinite.

In order to explain the above mentioned problem in Poitou's proof of theorem 1, let us recall the definition of the deflation map. Consider for $q \geq 0$ the composition

$$H_q(G/V, M^V) \xrightarrow{edge} H_q(G/U, (M^V)_U) \xrightarrow{N_*} H_q(G/U, M^U),$$

where *edge* is the edge morphism (coinflation) of the homological Hochschild-Serre spectral sequence associated to the group extension $1 \to U/V \to G/V \to G/U \to 1$ and N_* is the map which is induced on homology by the norm $N_{U/V} : (M^V)_U \to M^U$. Via the identification of homology with Tate cohomology in negative dimensions, this defines the deflation in dimension $i < -1$. In dimensions $i = 0, -1$, the map *def* is induced via the identifications $\hat{H}^0(G/U, A^U) \cong A^G/N_{G/U}A^U$ and $\hat{H}^{-1}(G/U, A^U) \cong {}_{N_{G/U}}A^U/I_{G/U}A^U$ by the identity and the norm map, respectively.

It is clear from its definition that the Tate cohomology of profinite groups does not satisfy all properties of a usual cohomology functor. Theorem 1 follows by a limit process from the following well known theorem of Tate and Nakayama (see e.g. [NSW] Thm. 3.1.5).

Theorem 4 (Tate-Nakayama) *Let G be a finite group, let C be a class module for G and let $\gamma \in H^2(G, C)$ be a fundamental class. Then for all integers $i \in \mathbb{Z}$ the cup product*

$$\hat{H}^i(G, \mathrm{Hom}(A, C)) \times \hat{H}^{2-i}(G, A) \xrightarrow{\cup} H^2(G, C) \cong \tfrac{1}{\#G}\mathbb{Z}/\mathbb{Z},$$

where $H^2(G, C) \cong \frac{1}{\#G}\mathbb{Z}/\mathbb{Z}$ is given by $\gamma \mapsto \frac{1}{\#G} \bmod \mathbb{Z}$, induces an isomorphism

$$\hat{H}^i(G, \mathrm{Hom}(A, C)) \cong \hat{H}^{2-i}(G, A)^\vee$$

of finite abelian groups, provided that A is finitely generated and $\mathbb{Z}$-free.

Already in order to define the duality map in theorem 1, one has to show that the cup product is compatible with deflation. Explicitly, given a finite group G, a normal subgroup $U \subseteq G$ and G-modules A, B, we have to show that for $i \leq 0$ and $q \geq 1$ the diagram

$$(*) \qquad \begin{array}{ccccc} \hat{H}^i(G,A) & \times & \hat{H}^{q-i}(G,B) & \xrightarrow{\ \cup\ } & \hat{H}^q(G, A\otimes B) \\ \big\downarrow{\scriptstyle def} & & \big\uparrow{\scriptstyle inf} & & \big\uparrow{\scriptstyle inf} \\ \hat{H}^i(G/U, A^U) & \times & \hat{H}^{q-i}(G/U, B^U) & \xrightarrow{\cup} & \hat{H}^q(G/U, (A\otimes B)^U) \end{array}$$

commutes (*inf* denotes the inflation map). In 1995 J. Neukirch pointed out that there is no proof for the commutativity of $(*)$ in the literature. Indeed, G. Poitou in his original paper [P] incorrectly claims that the required commutativity follows easily from the case $i = 0$ by dimension shifting. L. V. Kuz'min, cf. [K], noticed that this is not true and gave a proof in a special case. All other authors (at least to the restricted knowledge of the author of the present paper) ignored the problem.

In dimensions $i = -1, 0$, however, the commutativity of $(*)$ can be verified directly, which justifies theorem 1 for $i = 0$. Therefore Neukirch proposed to give a proof of the duality theorem of Tate-Poitou using theorem 1 only in dimension $i = 0$. Actually, this can be done, cf. [NSW], Ch.VIII. However, lacking duality for $\hat{H}^{-1}$, the proof of Tate's 9-term sequence for finite modules becomes rather intricate and one has to combine its proof with that of the global Euler-Poincaré characteristic formula (loc. cit.). The flow of arguments would become more streamlined if we were allowed to use Poitou's theorem in arbitrary dimensions. Furthermore, we need the correctness of the definition of the duality homomorphism also for theorems 2 and 3. Therefore the first sections of this paper are devoted to the rather technical verification of the commutativity of diagram $(*)$ for all $i \leq 0$. Then we recall the proof of theorem 1 and show theorems 2 and 3.

0. Some facts about Tate cohomology

Let G be a finite group. Recall (see e.g. [B] Ch.IV) that Tate cohomology is constructed using a complete resolution $P_\bullet$ (of $\mathbb{Z}$)

$$\begin{tikzcd} \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[rr, "d_0"] \arrow[dr, "\epsilon"'] & & P_{-1} \arrow[r, "d_{-1}"] & P_{-2} \arrow[r, "d_{-2}"] & P_{-3} \arrow[r] & \cdots \\ & & & \mathbb{Z} \arrow[ur, "\mu"'] \arrow[dr] & & & & \\ & & 0 \arrow[ur] & & 0 & & & \end{tikzcd}$$

consisting of finitely generated projective $\mathbb{Z}[G]$-modules P_n, $n \in \mathbb{Z}$, by the rule

$$\hat{H}^i(G,A) := H^i(\mathrm{Hom}(P_\bullet, A)^G), \tag{2}$$

where $\mathrm{Hom}(P_\bullet, A)^i := \mathrm{Hom}(P_i, A)$. If P is finitely generated and projective, then the same holds for $P^+ = \mathrm{Hom}(P, \mathbb{Z})$ [1] and for an arbitrary G-module A the module $\mathrm{Hom}(P, A)$ is cohomologically trivial. Therefore the norm induces an isomorphism

$$(P^+ \otimes A)_G \xrightarrow{\sim} \mathrm{Hom}(P, A)_G \underset{\sim}{\xrightarrow{N_G}} \mathrm{Hom}(P, A)^G, \tag{3}$$

which, applied to the negative part of the complete resolution, implies isomorphisms

$$H_i(G, A) \underset{\sim}{\xrightarrow{N_G}} \hat{H}^{-i-1}(G, A) \quad \text{for } i \geq 1. \tag{4}$$

Recall the definition of the cup product (cf. [B] chap. IV §5) $\hat{H}^p \times \hat{H}^q \longrightarrow \hat{H}^{p+q}$. It is induced by the tensor product in dimension 0 and is uniquely characterized by its functorial properties. In order to construct it, one uses an (essentially unique) diagonal approximation

$$\Delta : P_\bullet \longrightarrow P_\bullet \otimes P_\bullet, \tag{5}$$

i.e. a family of G-module homomorphisms $\varphi_{p,q} : P_{p+q} \to P_p \otimes P_q$, $p, q \in \mathbb{Z}$ satisfying a certain list of axioms (loc. cit.).

The abstract definition of the cup product, however, is not sufficient for our purposes. We will have to fix a particular complete resolution $P_\bullet$ and an explicitly given diagonal approximation $\varphi_{p,q}$, $p, q \in \mathbb{Z}$. Fortunately, such an explicit diagonal approximation is given in [AW] for the homogeneous standard resolution. The modules in this resolution are given by

$$P_i = P_{-1-i} = \mathbb{Z}[G^{i+1}], \quad i \geq 0 \tag{6}$$

and the differentials are defined for $i > 0$ by

$$d_i(g_0, \ldots, g_i) = \sum_{j=0}^{i} (-1)^j (g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_i), \tag{7}$$

$$d_{-i}(g_1, \ldots, g_i) = \sum_{\tau \in G} \sum_{j=0}^{i} (-1)^j (g_1, \ldots, g_j, \tau, g_{j+1}, \ldots, g_i), \tag{8}$$

while $d_0 : P_0 \longrightarrow P_{-1}$ is given by

$$d_0(g_0) = \sum_{\tau \in G} \tau g_0 \quad (= \sum_{\tau \in G} \tau). \tag{9}$$

An explicit diagonal approximation for this complex is given by (cf. [AW]):
If $p \geq 0$ and $q \geq 0$,

$$\varphi_{p,q}(\sigma_0, \ldots, \sigma_{p+q}) = (\sigma_0, \ldots, \sigma_p) \otimes (\sigma_p, \ldots, \sigma_{p+q}) \,. \tag{10}$$

[1] Hom always means $\mathrm{Hom}_{\mathbb{Z}}$

If $p \geq 1$ and $q \geq 1$,

$$\varphi_{-p,-q}(\sigma_1, \dots, \sigma_{p+q}) = (\sigma_1, \dots, \sigma_p) \otimes (\sigma_{p+1}, \dots, \sigma_{p+q}) \,. \tag{11}$$

If $p \geq 0$ and $q \geq 1$,

$$\begin{aligned}
\varphi_{p,-p-q}(\sigma_1, \dots, \sigma_q) &= \sum (\sigma_1, \tau_1, \dots, \tau_p) \otimes (\tau_p, \dots, \tau_1, \sigma_1, \dots, \sigma_q) \,, & (12)\\
\varphi_{-p-q,p}(\sigma_1, \dots, \sigma_q) &= \sum (\sigma_1, \dots, \sigma_q, \tau_1, \dots, \tau_p) \otimes (\tau_p, \dots, \tau_1, \sigma_q) \,, & (13)\\
\varphi_{p+q,-q}(\sigma_0, \dots, \sigma_p) &= \sum (\sigma_0, \dots, \sigma_p, \tau_1, \dots, \tau_q) \otimes (\tau_q, \dots, \tau_1) \,, & (14)\\
\varphi_{-q,p+q}(\sigma_0, \dots, \sigma_p) &= \sum (\tau_1, \dots, \tau_q) \otimes (\tau_q, \dots, \tau_1, \sigma_0, \dots, \sigma_p) \,, & (15)
\end{aligned}$$

where the τ_i on the right-hand side run independently through G.

1. The diagram $(*)$ commutes

Let $P_\bullet^{(G)}$, resp. $P_\bullet^{(G/U)}$ be the complete standard resolution for G, resp. G/U as defined in the last section.

Lemma 1 *Let G be a finite group and let $U \subseteq G$ be a normal subgroup. Let in negative dimension the map $\alpha_{-i} :\ P_{-i}^{(G/U)} \longrightarrow P_{-i}^{(G)}$ be given by*

$$(\sigma_1 U, \dots, \sigma_i U) \longmapsto \sum_{\tau_1, \dots, \tau_i \in U} (\sigma_1 \tau_1, \dots, \sigma_i \tau_i), \tag{16}$$

where the τ_i on the right-hand side run independently through U. Then in dimension ≤ -2 the deflation map is induced by a cocycle map

$$\mathit{def} \colon \mathrm{Hom}_G(P_\bullet^{(G)}, A) \longrightarrow \mathrm{Hom}_{G/U}(P_\bullet^{(G/U)}, A^U),$$

which is uniquely defined by the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(P_{-i}^{(G)}, A)_G & \xrightarrow[\sim]{N_G} & \mathrm{Hom}_G(P_{-i}^{(G)}, A) \\
\Big\downarrow{\scriptstyle (\alpha_{-i}^*, (N_U)_*)} & & \Big\downarrow{\scriptstyle \mathit{def}} \\
\mathrm{Hom}(P_{-i}^{(G/U)}, A^U)_{G/U} & \xrightarrow[\sim]{N_{G/U}} & \mathrm{Hom}_{G/U}(P_{-i}^{(G/U)}, A^U).
\end{array}$$

Proof: First note that the map α_{-i} commutes with the differential on the negative part of the standard complex. Let $(g_1, \dots, g_i)^*_{g_1, \dots, g_i \in G}$ denote the dual basis of $(P_{-i}^{(G)})^+ = \mathrm{Hom}(P_{-i}^{(G)}, \mathbb{Z})$, i.e. $(g_1, \dots, g_i)^*$ maps $(g_1, \dots, g_i)$ to 1 and all other basis elements of $P_{-i}^{(G)}$ to zero. The dual basis of $P_{-i}^{(G/U)+}$ is denoted by

$((g_1U),\ldots,(g_iU))^*_{g_1U,\ldots,g_iU\in G/U}$. Consider the diagram

$$\begin{array}{ccccc}
((P_{-i}^{(G)})^+\otimes A)_G & \xrightarrow[\sim]{D} & \mathrm{Hom}(P_{-i}^{(G)},A)_G & \xrightarrow[\sim]{N_G} & \mathrm{Hom}_G(P_{-i}^{(G)},A) \\
\big\downarrow{\scriptstyle coinf} & & \big\downarrow{\scriptstyle (\alpha_{-i}^*,pr_*)} & & \big\downarrow{\scriptstyle \beta} \\
((P_{-i}^{(G/U)})^+\otimes A_U)_{G/U} & \xrightarrow[\sim]{D} & \mathrm{Hom}(P_{-i}^{(G/U)},A_U)_{G/U} & \xrightarrow[\sim]{N_{G/U}} & \mathrm{Hom}_{G/U}(P_{-i}^{(G/U)},A_U) \\
\big\downarrow{\scriptstyle id\otimes N_U} & & \big\downarrow{\scriptstyle (id^*,(N_U)_*)} & & \big\downarrow{\scriptstyle (id^*,(N_U)_*)} \\
((P_{-i}^{(G/U)})^+\otimes A^U)_{G/U} & \xrightarrow[\sim]{D} & \mathrm{Hom}(P_{-i}^{(G/U)},A^U)_{G/U} & \xrightarrow[\sim]{N_{G/U}} & \mathrm{Hom}_{G/U}(P_{-i}^{(G/U)},A^U).
\end{array}$$

Explanations: $pr: A \to A_U$ is the natural projection, *coinf* is the chain map which induces coinflation on homology, it is induced by pr and by the map $P_{-i}^{(G)+} \to P_{-i}^{(G/U)+}$, which sends $(g_1^*,\ldots,g_i^*)$ to $((g_1U)^*,\ldots,(g_iU)^*)$. The maps D are induced by the canonical duality isomorphisms $P^+\otimes A \cong \mathrm{Hom}(P,A)$ (the P_{-i} are free of finite rank). A simple calculation shows that the upper left square commutes and we define β in order to let the upper right square commute. The lower squares commute obviously. By definition, *def* is induced by the composition of the two vertical arrows on the right. This completes the proof of the lemma. $\square$

Now assume that A and B are G-modules. The problem in extending the commutativity of $(*)$ from $i=0$ to the case of arbitrary dimensions is that we cannot make the shift for A and A^U simultaneously (and, clearly, the same problem occurs for B). That is why we are forced to make these explicit chain calculations below.

Suppose that $i \geq 2$ and $q \geq 1$. Let $\bar{x} \in \hat{H}^{-i}(G,A)$, $\bar{y} \in H^{q+i}(G/U,B^U)$ be cohomology classes represented by cocycles $x \in \mathrm{Hom}_G(P_{-i}^{(G)},A)$ and $y \in \mathrm{Hom}_{G/U}(P_{q+i}^{(G/U)},A)$, respectively.

Proposition 2

$$\bar{x}\cup \mathit{inf}\,\bar{y} = \mathit{inf}(\mathit{def}\,\bar{x}\cup\bar{y}) \in H^q(G,A\otimes B).$$

Proof: We calculate both sides on the level of cochains. Let us start with the left side. For $(\sigma_0,\ldots,\sigma_q)\in P_q^{(G)}$ we obtain by (15)

$$(x\cup \mathit{inf} y)(\sigma_0,\ldots,\sigma_q) = \sum_{\tau_1,\ldots,\tau_i\in G} x(\tau_1,\ldots,\tau_i)\otimes y(\tau_iU,\ldots,\tau_1U,\sigma_0U,\ldots,\sigma_qU).$$

In order to compute the right side, choose a $z\in \mathrm{Hom}(P_{-i}^{(G)},A)$ with $x = N_Gz$. Then by (15) and by lemma 1 we have

$$inf(def x \cup y)(\sigma_0, \ldots, \sigma_q) = \Big((N_{G/U}(\alpha_{-i}^*, (N_U)_*)z) \cup y\Big)(\sigma_0 U, \ldots, \sigma_q U)$$

$$= \sum_{\tau_1 U, \ldots, \tau_i U \in G/U} (N_{G/U}(\alpha_{-i}^*, (N_U)_*)z)(\tau_1 U, \ldots, \tau_i U) \otimes y(\tau_i U, \ldots, \tau_1 U, \sigma_0 U, \ldots, \sigma_q U).$$

Using the definition of α_{-i} and of N_U this transforms to

$$\sum_{\tau U \in G/U} \sum_{u \in U} \sum_{\tau_1 U, \ldots, \tau_i U \in G/U} \sum_{u_1, \ldots, u_i \in U}$$

$$\tau u\, z(\tau^{-1}\tau_1 u_1, \ldots, \tau^{-1}\tau_i u_i) \otimes y(\tau_i U, \ldots, \tau_1 U, \sigma_0 U, \ldots, \sigma_q U),$$

which coincides with

$$\sum_{\tau_1, \ldots, \tau_i \in G} (N_G z)(\tau_1, \ldots, \tau_i) \otimes y(\tau_i U, \ldots, \tau_1 U, \sigma_0 U, \ldots, \sigma_q U).$$

This shows the proposition. $\square$

Proposition 3 *For $i \leq 0$ and $q \geq 1$ the diagram*

$$(*) \qquad \begin{array}{ccccc} \hat{H}^i(G, A) & \times & \hat{H}^{q-i}(G, B) & \xrightarrow{\ \cup\ } & \hat{H}^q(G, A \otimes B) \\ \big\downarrow{\scriptstyle def} & & \big\uparrow{\scriptstyle inf} & & \big\uparrow{\scriptstyle inf} \\ \hat{H}^i(G/U, A^U) & \times & \hat{H}^{q-i}(G/U, B^U) & \xrightarrow{\cup} & \hat{H}^q(G/U, (A \otimes B)^U), \end{array}$$

commutes.

Proof: For $i \leq -2$ this follows from the last proposition. Via the isomorphism

$$\begin{array}{rcl} {}_{N_G}A/I_G A & \xrightarrow{\sim} & H^{-1}(G, A) \\ a & \longmapsto & x_a : \mathbb{Z}[G] \to A,\ \sigma \mapsto \sigma(a) \end{array}$$

the cup product is given on the chain level by

$$x_a \cup y(\sigma_0, \ldots, \sigma_q) = \sum_{\sigma \in G} \sigma a \otimes y(\sigma, \sigma_0, \ldots, \sigma_q).$$

The commutativity of $(*)$ for $i = -1$ follows immediately, since *def* is induced by the norm

$$N_U : {}_{N_G}A/I_G A \longrightarrow {}_{N_{G/U}}A^U/I_{G/U}A^U.$$

The case $i = 0$ is obvious. $\square$

2. Proof of the theorems

From now on let G be a profinite group. Having verified the commutativity of the diagram $(*)$, we can now deduce the following proposition from the theorem of Tate-Nakayama for finite groups.

Proposition 4 *Under the assumption of theorem 1 suppose that A is $\mathbb{Z}$-free. Then the cup product*

$$\hat{H}^i(G, \operatorname{Hom}(A, C)) \times \hat{H}^{2-i}(G, A) \xrightarrow{\cup} H^2(G, C) \xrightarrow[\sim]{inv} \tfrac{1}{\#G}\mathbb{Z}/\mathbb{Z}$$

induces isomorphisms

$$\hat{H}^i(G, \operatorname{Hom}(A, C)) \xrightarrow{\sim} \hat{H}^{2-i}(G, A)^\vee$$

for all $i \in \mathbb{Z}$.

Proof: For $i \neq 1$ this follows easily from the theorem 4 by passing to the limit over G/U, where U runs through the open normal subgroups in G. For $i = 1$, let $U \subseteq G$ be an open normal subgroup which acts trivially on A. Since A is $\mathbb{Z}$-free, $H^1(U/V, A^V) = 0$ for every open normal subgroup $V \subseteq U$ and therefore

$$H^1(G/U, A^U) \xrightarrow{\sim} H^1(G/V, A^V).$$

Using the fact that C is a class formation, we obtain

$$H^1(U/V, \operatorname{Hom}(A, C)^V) = H^1(U/V, \operatorname{Hom}(A, C^V)) \cong H^1(U/V, C^V)^{\operatorname{rank}_{\mathbb{Z}} A} = 0.$$

This implies that

$$H^1(G/U, \operatorname{Hom}(A, C)^U) \xrightarrow{\sim} H^1(G/V, \operatorname{Hom}(A, C)^V).$$

A stationary limit process shows that theorem 4 implies proposition 4 also in dimension 1. $\square$

For a discrete G-module M we denote by

$$N_G M = \bigcap_{U \subseteq G} N_{G/U} M^U \subseteq M^G$$

the module of universal norms of M. In the following definition we systematize the assumption on the compactness of C^U in theorem 1.

Definition 1 *Let G be a profinite group. A level-compact G-module is a discrete G-module M (i.e. $M = \bigcup_U M^U$, where U runs through the open subgroups of G) which is endowed with an additional topology, such that the action*

$$G \times M \longrightarrow M$$

is continuous and M^U is compact for every open subgroup $U \subseteq G$.

If M is level-compact, then for $i \leq 0$ the groups $\hat{H}^i(G,M)$ are abelian profinite groups in a natural way: For every open normal $U \subseteq G$ the group $\hat{H}^i(G/U, M^U)$ inherits a natural compact topology from M^U via the standard complex. Furthermore, this group is annihilated by $(G:U)$, hence $\hat{H}^i(G/U, M^U)$ and thus also $\hat{H}^i(G,M)$ is profinite. For $i > 0$ we give $H^i(G,M)$ the discrete topology.

Lemma 5 *Let M be a level-compact G-module. Then*

$$\hat{H}^0(G,M) = M^G/N_G M.$$

Proof: For every open normal subgroup $U \subseteq G$ we have a short exact sequence

$$N_{G/U}M^U \longrightarrow M^G \longrightarrow \hat{H}^0(G/U, M^U) \longrightarrow 0.$$

Since all groups are compact by assumption, the sequence remains exact after passing to the projective limit over all U. □

Lemma 6 *Let M be a level-compact G-module. Suppose that for every open subgroup $U \subseteq G$ a closed subgroup*

$$M(U) \subseteq M^U$$

is given in such a way that the following conditions hold

(i) $N_U M \subseteq M(U)$ *for every open* $U \subseteq G$,

(ii) *if V is normal in U, then $N_{U/V} : M^V \to M^U$ maps $M(V)$ to $M(U)$.*

Then

$$\hat{H}^i(G,M) \cong \varprojlim_U \hat{H}^i(G/U, M(U))$$

for all $i \leq -1$.

Proof: We have seen in lemma 1 that the deflation maps are in negative dimensions given by a map on the chain level. As projective limits are exact on compact groups, we see that for $i \leq -2$ the group $\hat{H}^i(G,M)$ also can be calculated as the quotient of the inverse limit of the cocycles modulo the inverse limit of the coboundaries. These, however take values in the groups of universal norms on the corresponding levels, i.e. we may take the limit over the groups $M(U)$ instead of M^U as well. This shows the lemma for $i \leq -2$. For $i = -1$ deflation is explicitly given by the norm and a straightforward computation, similarly exploiting the level-compactness, shows the statement also in this case (cf. [NSW] 3.1.7). □

Proposition 7 *Let*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{j} M''$$

be an exact sequence of level-compact G-modules, such that the induced map $N_U M \xrightarrow{N_U j} N_U M''$ is surjective for all open normal subgroups U of G. Then there is an associated long exact cohomology sequence

$$\cdots \longrightarrow \hat{H}^{-n}(G, M') \longrightarrow \hat{H}^{-n}(G, M) \longrightarrow \hat{H}^{-n}(G, M'') \longrightarrow \cdots$$

ending with $\cdots \to \hat{H}^0(G, M') \to \hat{H}^0(G, M) \to \hat{H}^0(G, M'')$. *If, moreover, j is surjective, we obtain the long exact cohomology sequence unbounded in both directions (i.e. from $-\infty$ to $+\infty$).*

Proof: For every open normal subgroup U in G we consider the kernel $M'(U) := \ker(N_U M \xrightarrow{N_U j} N_U M'')$. We have inclusions $N_U M' \subseteq M'(U) \subseteq M'^U$ and obtain the exact and commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M'(U) \arrow[r] \arrow[d, "N_{G/U}"] & N_U M \arrow[r] \arrow[d, "N_{G/U}"] & N_U M'' \arrow[r] \arrow[d, "N_{G/U}"] & 0 \\
0 \arrow[r] & M'^G \arrow[r] & M^G \arrow[r] & M''^G.
\end{tikzcd}$$

Consider the long exact cohomology sequence

$$\cdots \to \hat{H}^i(G/U, M'(U)) \to \hat{H}^i(G/U, N_U M) \to \hat{H}^i(G/U, N_U M'') \to \cdots \tag{17}$$

associated to the upper line. It consists of compact abelian groups, is clearly exact and all homomorphisms including the boundary maps are continuous (use, e.g., the snake lemma in the abelian category of compact abelian groups). Passing to the inverse limit over U, we obtain using lemma 6 the asserted long exact sequence up to dimension -1.
By compactness, the image of $N_{G/U} : N_U M \longrightarrow M^G$ is $N_G M$ and hence, by lemma 5, the cokernel of this map is $\hat{H}^0(G, M)$. We denote its kernel by

$$X(M, U) = {}_{N_{G/U}} N_U M,$$

which contains $Y(M, U) := I_{G/U} N_U M$, and the same holds for M''. The snake lemma implies an exact commutative diagram

$$\begin{tikzcd}
Y(M,U) \arrow[r, "j"] \arrow[d] & Y(M'',U) \arrow[d] \\
X(M,U) \arrow[r] & X(M'',U) \arrow[r, "\delta"] & M'^G/N_{G/U}M'(U) \arrow[r] & \hat{H}^0(G,M) \arrow[r] & \hat{H}^0(G,M'').
\end{tikzcd}$$

Observe that

$$\varprojlim_{U} (M'^G / N_{G/U} M'(U)) = \hat{H}^0(G, M').$$

Furthermore, the upper map $j : I_{G/U}N_UM \to I_{G/U}N_UM''$ is obviously surjective and $N_{G/U} : N_UM \to M^G$ maps $I_{G/U}N_UM$ to zero. Therefore $\delta : X(M'', U) \to M'^G/N_{G/U}M'(U)$ maps $Y(M'', U)$ to zero by the definition of δ. This means that we may replace in the last diagram the group $X(M, U)$ by $\hat{H}^{-1}(G/U, N_UM) = X(M, U)/Y(M, U)$ and $X(M'', U)$ by $\hat{H}^{-1}(G/U, N_UM'') = X(M'', U)/Y(M'', U)$ and obtain an exact sequence of compact groups and continuous homomorphisms. Taking projective limits over U, we obtain the exact sequence

$$\hat{H}^{-1}(G, M) \to \hat{H}^{-1}(G, M'') \xrightarrow{\delta} \hat{H}^0(G, A) \to \hat{H}^0(G, B) \to \hat{H}^0(G, C).$$

Now suppose that j is surjective. Consider the commutative and exact diagram

$$\begin{tikzcd}
0 & 0 & & \\
\hat{H}^0(G, M) \arrow[u] \arrow[r] & \hat{H}^0(G, M'') \arrow[u] \arrow[dr, dashed] & & \\
M^G \arrow[u] \arrow[r] & M''^G \arrow[u] \arrow[r] & H^1(G, M') \arrow[r] & H^1(G, M) \\
N_GM \arrow[u] \arrow[r, two heads, "N_Gj"] & N_GM'' \arrow[u] \arrow[ur, "0"'] & & \\
0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The vertical sequences are exact by lemma 5. We conclude the existence of the dotted arrow, which glues the already proven negative part long exact sequence with the long exact sequence

$$H^1(G, M') \longrightarrow H^1(G, M) \longrightarrow H^1(G, M'') \longrightarrow \cdots . \qquad \square$$

Remark: The homomorphisms in the negative part of the long exact sequence are easily seen to be continuous. The maps in dimensions greater or equal to one are continuous because the groups are discrete. Only the boundary map $\delta^0 : \hat{H}^0(G, M'') \to H^1(G, M')$ might cause a problem. It can be seen to be continuous in our application, however, under the general assumptions of proposition 7 this is not necessarily true.

Corollary 8 *If p is a prime number, then the long exact sequence of proposition 7 induces a long exact sequence of the p-parts, too.*

Proof: All occurring cohomology groups are either abelian profinite groups or abelian discrete torsion groups and therefore they naturally decompose into the direct sum of their p-parts and their prime-to-p-parts. In order to prove the corollary, it suffices to show that also the differentials decompose into a direct sum of homomorphisms. This is trivially true for continuous differentials, hence it remains to consider the homomorphism $\delta^0 : \hat{H}^0(G, M'') \to H^1(G, M')$. Let $x \in \hat{H}^0(G, M'')(p)$. Choose a pre-image $m'' \in M''^G$ of x and let $m \in M$ be a pre-image of m''. Furthermore, let $U \subseteq G$ be an open normal subgroup in G such that $m \in M^U$. The closed subgroup generated by m in M^U maps onto the closed subgroup generated by x which coincides with $\mathbb{Z}_p \cdot x \subseteq \hat{H}^0(G, M'')(p)$. We conclude that $\delta^0(\lambda x) \in H^1(G/U, M'^U) \subseteq H^1(G, M')$ for all $\lambda \in \mathbb{Z}_p$. Writing $\#(G/U) = Np^k$ with $(N, p) = 1$, we obtain that $\delta^0(x) = N\delta^0(N^{-1}x) \in N \cdot H^1(G/U, M'^U) = H^1(G/U, M'^U)(p) \subseteq H^1(G, M')(p)$. A similar argument shows that δ^0 sends the prime-to-p-part of $\hat{H}^0(G, M'')$ to the prime-to-p-part of $H^1(G, M')$. $\square$

We use the first part of proposition 7 in order to prove Poitou's theorem.

Proof of theorem 1: If A is $\mathbb{Z}$-free, the assertion is contained in proposition 4. Let A be an arbitrary finitely generated G-module. There exists an exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

of finitely generated G-modules, where R and F are $\mathbb{Z}$-free. Applying the functor $\mathrm{Hom}(-, C)$, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, C) \longrightarrow \mathrm{Hom}(F, C) \longrightarrow \mathrm{Hom}(R, C)\,.$$

Let U run through the open normal subgroups such that $F^U = F$, hence $A^U = A$, $R^U = R$. Then $N_U\mathrm{Hom}(F, C) = \mathrm{Hom}(F, N_UC)$, $N_U\mathrm{Hom}(R, C) = \mathrm{Hom}(R, N_UC)$ and the map $N_U\mathrm{Hom}(F, C) \longrightarrow N_U\mathrm{Hom}(R, C)$ is surjective since N_UC is divisible. For $i \leq 0$ proposition 7 induces an exact commutative diagram in which we write $\tilde{}$ for $\mathrm{Hom}(-, C)$:

$$\begin{array}{ccccccccc}
\hat{H}^{i-1}(G, \tilde{F}) & \longrightarrow & \hat{H}^{i-1}(G, \tilde{R}) & \longrightarrow & \hat{H}^{i}(G, \tilde{A}) & \longrightarrow & \hat{H}^{i}(G, \tilde{F}) & \longrightarrow & \hat{H}^{i}(G, \tilde{R}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{3-i}(G, F)^\vee & \to & H^{3-i}(G, R)^\vee & \to & H^{2-i}(G, A)^\vee & \to & H^{2-i}(G, F)^\vee & \to & H^{2-i}(G, R)^\vee.
\end{array}$$

The vertical arrows except the middle one are isomorphisms by proposition 4. Hence, by the five lemma the middle one is also an isomorphism. This proves theorem 1. $\square$

In the proofs of theorems 2 and 3 we will use the second statement of proposition 7 as well as the proposition 10 below. First we need the

Lemma 9 *Suppose that A is finitely generated and that $p^\infty | \#G$. Then $\hat{H}^0(G, A)$ is profinite and we have a canonical isomorphism*

$$\hat{H}^0(G, A)(p) \cong A^G \otimes \mathbb{Z}_p.$$

Proof: By definition, $\hat{H}^0(G,A)$ is the inverse limit of the groups $A^G/N_{G/U}A^U$. Since A is finitely generated, all norm groups are of finite index in A^G and it suffices to show that for arbitrarily given $n \in \mathbb{N}$ the subgroup $N_{G/U}A^U$ is contained in $p^n A^G$ for sufficiently small U. Let U be an open normal subgroup such that $A^U = A$. Then for every normal open subgroups $V \subseteq U$ with $p^n|(U:V)$ and for every $a \in A^V = A^U = A$ we have

$$N_{G/V}a = N_{G/U}N_{U/V}a = (U:V)N_{G/U}a \in p^n A^G.$$

This shows the lemma. □

Proposition 10 *Suppose that $p^\infty|\#G$ and let*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

be a short exact sequence of finitely generated G-modules. Then we obtain a long exact sequence

$$\cdots \longrightarrow \hat{H}^i(G,A')(p) \longrightarrow \hat{H}^i(G,A)(p) \longrightarrow \hat{H}^i(G,A'')(p) \longrightarrow \cdots$$

which is unbounded in both directions. The groups are compact for $i \leq 0$, discrete for $i > 0$ and all homomorphisms are continuous.

Proof: Since A is finitely generated, $\hat{H}^i(G/U,A^U)$ is finite for all U. Furthermore, $A^U = A$ for small U, hence we obtain for small U a long exact sequence

$$\cdots \longrightarrow \hat{H}^0(G/U,A'^U)(p) \longrightarrow \hat{H}^0(G/U,A^U)(p) \longrightarrow \hat{H}^0(G/U,A''^U)(p)$$

of finite abelian p-torsion groups. Passing to the projective limit, we obtain the negative part of our long exact sequence. Now consider for small U the long exact sequence

$$0 \longrightarrow A'^G \longrightarrow A^G \longrightarrow A''^G \longrightarrow \hat{H}^1(G/U,A^U) \longrightarrow \cdots .$$

Tensoring by $\mathbb{Z}_p$ and passing to the direct limit over U, we obtain the right part of our long exact sequence. Both fit together by lemma 9. The continuity of the maps is clear from their definitions. □

The following is the abstract form of theorems 2 and 3.

Theorem 5 *Let G be a profinite group and let p be a prime number with $p^\infty|\#G$. Suppose that C is a level-compact class formation for G such that the group of universal norms*

$$N_U C = \bigcap_{V \subseteq U} N_{U/V} C^V \subseteq C^U$$

is p-divisible for all open subgroups $U \subseteq G$. If, moreover, C is p-divisible, then the cup product

$$\hat{H}^i(G, \mathrm{Hom}(A,C)) \times \hat{H}^{2-i}(G,A) \xrightarrow{\cup} H^2(G,C) \xrightarrow[\sim]{inv} \tfrac{1}{\#G}\mathbb{Z}/\mathbb{Z}$$

induces isomorphisms

$$\hat{H}^i(G, \mathrm{Hom}(A,C))(p) \xrightarrow{\sim} \hat{H}^{2-i}(G,A)(p)^\vee$$

for all $i \in \mathbb{Z}$ and every discrete G-module A which is finitely generated as a $\mathbb{Z}$-module.

Proof: If A is $\mathbb{Z}$-free, the assertion follows immediately from proposition 4. Let A be an arbitrary finitely generated G-module, whose torsion part consists, without loss of generality, only of p-torsion. There exists an exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

of finitely generated G-modules, where R and F are $\mathbb{Z}$-free. Applying the functor $\mathrm{Hom}(-,C)$, since C is p-divisible, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A,C) \longrightarrow \mathrm{Hom}(F,C) \longrightarrow \mathrm{Hom}(R,C) \longrightarrow 0.$$

Let U run through the open normal subgroups such that $F^U = F$, hence $A^U = A$, $R^U = R$. Then $N_U\mathrm{Hom}(F,C) = \mathrm{Hom}(F, N_UC)$, $N_U\mathrm{Hom}(R,C) = \mathrm{Hom}(R, N_UC)$ and the map $N_U\mathrm{Hom}(F,C) \longrightarrow N_U\mathrm{Hom}(R,C)$ is surjective since N_UC is p-divisible. By corollary 8 we obtain a long exact cohomology sequence which is unbounded in both directions

$$\cdots \longrightarrow \hat{H}^i(G,\tilde{A})(p) \longrightarrow \hat{H}^i(G,\tilde{F})(p) \longrightarrow \hat{H}^i(G,\tilde{R})(p) \longrightarrow \cdots$$

and in which we wrote $\sim$ for $\mathrm{Hom}(-,C)$. Furthermore, proposition 10 gives us a corresponding long exact cohomology sequence associated to the short exact sequence $0 \to R \to F \to A \to 0$ in which all groups are compact or discrete and all maps are continuous. Hence this sequence remains exact after taking Pontryagin duals. Thus the duality map of theorem 5 defines a map between two long exact sequences. Therefore the statement of theorem 5 for A follows easily from that for F and R and from the five lemma. $\square$

Finally, we apply theorem 5 to the arithmetic situation. Let k be a number field, p a prime number and let S be any finite set of primes of k such that $S \supseteq S_\infty$. For every finite subextension K of k in k_S consider the subgroup $C_S^0(K)$ of ideles of norm 1 in $C_S(K)$ and let $C_S^0 = \varinjlim_{K \subset k_S} C_S^0(K)$. The exact sequences

$$0 \longrightarrow C_S^0(K) \longrightarrow C_S(K) \xrightarrow{|\ |} \mathbb{R}_+^\times \longrightarrow 0$$

show isomorphisms

$$\hat{H}^i(G_S, C_S^0) \cong \hat{H}^i(G_S, C_S)$$

for all $i \in \mathbb{Z}$. Since $C_S^0(K)$ is compact for all finite subextensions K of k in k_S, we conclude that C_S^0 is a level-compact class formation for G_S. One easily obtains that the groups of universal norms of C_S^0 are divisible, because this is true for C_S (see e.g. [NSW] 8.4.10, 8.5.2).

Proof of theorem 2: If $S \supseteq S_p$, then $p^\infty | \#G_S$, for instance because k_S contains the cyclotomic $\mathbb{Z}_p$-extension of k. By theorem 6 of [S] (see also 10.9.5 in [NSW]) C_S and hence also C_S^0 is p-divisible. Therefore theorem 2 follows from theorem 5. □

Proof of theorem 3: Under the given assumptions, for every finite subextension K of k in k_S the group $G_S(K)^{ab}(p)$ is finitely generated (since S is finite) and torsion (a $\mathbb{Z}_p$-extension is ramified at least at one prime dividing p and $S \cap S_p = \emptyset$), hence finite. By the group theoretic form of the principal ideal theorem (see [N] chap. VI thm. 7.6), we conclude that

$$\varinjlim_{K \subseteq k_S} G_S(K)^{ab}(p) = 0.$$

Since $D_S(K)$ is divisible, the exact sequences

$$D_S(K) \longrightarrow C_S(K) \longrightarrow G_S(K)^{ab} \longrightarrow 0$$

induce an isomorphism

$$C_S/p \xrightarrow{\sim} \varinjlim_{K \subseteq k_S} G_S(K)^{ab}/p = 0$$

in the limit. Hence C_S and thus also C_S^0 is p-divisible and theorem 3 follows from theorem 5. □

Closing remarks: 1. If $S \supseteq S_p \cup S_\infty$, theorem 2 can be used to calculate the group of universal norms of the G_S-module $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, C_S) = {}_pC_S$ as

$$N_{G_S}({}_pC_S) = \prod_{v \in S_\mathbb{C}} \mu_p,$$

where $S_\mathbb{C}$ denotes the set of complex places of k. One can show that the validity of the Leopoldt conjecture for k and p is equivalent to the fact, that these are all the universal norms in $C_S(k)$ which are annihilated by p (cf. [NSW] 10.3.7). However, we can *not* derive the latter statement from the duality theorems above.

2. Theorem 5 also applies to the absolute Galois group of a finite extension $k|\mathbb{Q}_l$, where l is any prime number. The level-compact class module

$$\mathcal{A} = \varinjlim_{K|k} (\varprojlim_{n} K^\times/n),$$

is easily seen to have trivial universal norms. Theorem 5 applies to every prime number p and we obtain the well known duality theorem for local Galois modules (cf. [NSW] 7.2.8).

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