The $K(\pi, 1)$-property for marked curves over finite fields

Philippe Lebacque and Alexander Schmidt

January 13, 2015

Abstract

We investigate the $K(\pi, 1)$-property for p of smooth, marked curves (X, T) defined over finite fields of characteristic p. We prove that (X, T) has the $K(\pi, 1)$-property if X is affine and give positive and negative examples in the proper case. We also consider the unmarked proper case over a finite field of characteristic different to p.
2010 Math. Subj. Class. 11R34, 11R37, 14F20
Key words: Galois cohomology, étale cohomology, restricted ramification

1 Introduction

In [1],[2],[3], the second author investigated the $K(\pi, 1)$-property for p of arithmetic curves whose function field is of characteristic different to p. As a result, the Galois group of the maximal unramified outside S and T-split pro-p-extension of a global field of characteristic different to p is often of cohomological dimension less or equal to two. In this paper we consider the case of a smooth curve over a finite field of characteristic p. We prove that (X, T) has the $K(\pi, 1)$-property if X is affine and give positive and negative examples in the proper case. We also consider the unmarked proper case over a finite field of characteristic different to p, which was left out in the earlier papers.

The authors would like to thank the referee for his valuable suggestions.

1.1 The marked étale site and the $K(\pi, 1)$-property

Let X be a regular one-dimensional noetherian scheme defined over $\mathbb{F}_q$ (with $q = p^f$) and let T be a finite set of closed points. In [3], the second author defined the marked site (X, T) of X at T considering finite étale morphisms $Y \to X$ inducing isomorphisms $k(y) \cong k(x)$ on the residue fields for any closed point $y \in Y$ mapping to $x \in T$. Let M be a p-torsion sheaf. The resulting cohomology groups are denoted by $H^i(X, T, M)$ and they satisfy the usual properties we expect from étale cohomology groups. He also proved (see [3] for more details) that these finite marked étale morphisms fit into a Galois theory and (after choosing a base geometric point $\bar{x} \notin T$) we denote by $\pi_1(X, T)$ the profinite group classifying étale coverings of X in which the points of T split completely. We denote by $\widetilde{(X,T)}(p)$ the universal pro-p-covering of (X,T). The projection $\widetilde{(X,T)}(p) \to X$ is Galois with Galois group the maximal pro-p-quotient $\pi_1(X,T)(p)$ of $\pi_1(X,T)$.

Let M be a discrete p-torsion $\pi_1(X,T)(p)$-module. Consider the Hochschild-Serre spectral sequence:

$$E_2^{ij} = H^i(\pi_1(X,T)(p), H^j(\widetilde{(X,T)}(p), T, M)) \Rightarrow H^{i+j}(X,T,M).$$

The edge morphisms provide homomorphisms

$$\phi_{i,M} : H^i(\pi_1(X,T)(p), M) \to H^i(X,T,M).$$

We say that (X,T) has the $K(\pi,1)$-property for p if $\phi_{i,M}$ is an isomorphism for all M and all $i \geqslant 0$. The following Lemma 1.1 implies in particular, that (X,T) has the $K(\pi,1)$-property for p if $\phi_{i,\mathbb{F}_p}$ is an isomorphism for $i \geqslant 2$.

Lemma 1.1. *(cf. [3] Lemma 2.2) $\phi_{i,M}$ is an isomorphism for $i = 0,1$ and is a monomorphism for $i = 2$. Moreover, $\phi_{i,M}$ is an isomorphism for all $i \geqslant 0$ if and only if*

$$\varinjlim_{(Y,T')} H^i(Y,T',M) = 0 \text{ for all } i \geqslant 1,$$

where the direct limit is taken over all finite intermediate coverings (Y,T') of the universal pro-p-covering $\widetilde{(X,T)}(p) \to (X,T)$.

1.2 Notation

Unless otherwise stated, we use the following notation:

- p denotes a prime number.
- $\mathbb{F}$ is a finite field, $\bar{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$, $\tilde{\bar{\mathbb{F}}}$ its maximal pro-p-extension inside $\bar{\mathbb{F}}$ and $G_{\mathbb{F}}$ the Galois group of $\bar{\mathbb{F}}/\mathbb{F}$.
- X is a smooth projective absolutely irreducible curve defined over $\mathbb{F}$.
- $k = \mathbb{F}(X)$ the function field of X.
- g_X the genus of X.
- $\bar{X} = X \times_{\mathbb{F}} \bar{\mathbb{F}}$, $\tilde{\bar{X}} = X \times_{\mathbb{F}} \tilde{\bar{\mathbb{F}}}$.
- S, T are two disjoint sets (possibly empty) of closed points of X.
- if x is a closed point of a X, X_x denotes the henselization of X at x and $T_x = \{x\}$ if $x \in T$ and $\varnothing$ otherwise.
- k_S^T denotes the maximal pro-p-extension of k which is unramified outside S and in which all places of T split completely. If empty, we omit S (or T) from the notation.
- $G_S^T(k) = \mathrm{Gal}(k_S^T/k) = \pi_1(X-S,T)(p)$.
- $H^i(X-S,T)$ denotes the i-th étale cohomology group $H^i_{et}(X-S,T,\mathbb{F}_p)$ of the marked curve $(X-S,T)$.
- for a pro-p-group G we set $H^i(G) = H^i(G,\mathbb{F}_p)$.
- for an abelian group A and an integer m we write $A[m] = \ker(A \xrightarrow{\cdot m} A)$

1.3 New results

Let X be a smooth projective absolutely irreducible curve defined over the finite field $\mathbb{F}$ and let $k = \mathbb{F}(X)$ be the function field of X. Let S and T be finite disjoint sets of closed points of X. In this paper, we prove the following result:

Theorem 1.2. *Assume that $p = \mathrm{char}(\mathbb{F})$.*

(i) *If $S \neq \varnothing$, then $(X - S, T)$ has the $K(\pi, 1)$-property for p and $cd\, G_S^T(k) = 1$.*

(ii) *If $T = \varnothing$, then $(X - S)$ has the $K(\pi, 1)$-property for p and $cd\, G_S(k) \leqslant 2$.*

In the remaining cases, we have the following results.

Theorem 1.3. *Assume that $p = \mathrm{char}(\mathbb{F})$, $S = \varnothing$ and $T \neq \varnothing$.*

(i) *If $\mathrm{Pic}(X)[p] = 0$, then (X, T) has the $K(\pi, 1)$-property for p if and only if $T = \{x\}$ consists of a single point with $p \nmid \deg x$. In this case $\pi_1(X, T)(p) = 1$.*

(ii) *If $\mathrm{Pic}(X)[p] \neq 0$ and*

$$\sum_{x \in T} \frac{\deg(x)}{(\#\mathbb{F})^{\deg(x)/2} - 1} > g_X - 1,$$

then $\pi_1(X, T)(p)$ is finite and (X, T) has not the $K(\pi, 1)$-property for p.

Finally, we consider the unmarked proper case over a finite field of characteristic different to p, which was left out in the earlier papers.

Theorem 1.4. *Assume that $p \neq \mathrm{char}(\mathbb{F})$. Then X has the $K(\pi, 1)$-property for p if and only if $\mu_p(\mathbb{F}) = 1$ or $\mathrm{Pic}(X)[p] \neq 0$.*

In the remaining case $\mu_p \subset \mathbb{F}$ and $\mathrm{Pic}(X)[p] = 0$ we have

$$\pi_1^{et}(X)(p) \cong \pi_1^{et}(\mathbb{F})(p) \cong \mathbb{Z}_p.$$

In particular, $H^i(\pi_1^{et}(X)(p))$ is always finite and vanishes for $i > 3$.

2 Computation of étale cohomology groups

Proposition 2.1 (Local computation)**.** *Let K be a nonarchimedean local (or henselian) field of characteristic p. Let $Y = Spec\, \mathcal{O}_K$, $y \in Y$ the closed point and let T be $\varnothing$ or $\{y\}$. Then the local cohomology groups $H_y^i(Y, T)$ vanish for $i \neq 2$ and*

$$H_y^2(Y, T) = \begin{cases} H^1_{/nr}(K) & \text{if } T = \varnothing \\ H^1(K) & \text{if } T = \{y\}, \end{cases}$$

where $H^1_{/nr} = H^1(K)/H^1_{nr}(K)$.

Proof: We use the excision sequence:

$$\cdots \to H_y^i(Y, T) \to H^i(Y, T) \to H^i(Y - \{y\}) \to H_y^{i+1}(Y, T) \to \cdots.$$

Since Y is henselian, $H^i(Y) \cong H^i(y) = H^i_{nr}(K)$, hence $H^i(Y) = 0$ for $i \geqslant 2$. Since Y is normal, $H^1(Y, T) \to H^1(Y - \{y\})$ is injective, hence $H_y^1(Y, T) = 0$. Furthermore,

$H^i(Y-\{y\}) = H^i(K)$ and this group vanishes for $i \geqslant 2$ since $cd_pK = 1$ (see [4], Cor. 6.1.3). It follows that $H^i_y(Y,T) \cong H^i(Y,T)$ for $i \geqslant 3$.

For $T = \varnothing$ we obtain $H^i_y(Y) = 0$ for $i \geqslant 3$ and the short exact sequence

$$0 \to H^1(Y) \to H^1(Y-\{y\}) \to H^2_y(Y) \to 0$$

implies the result for $H^2_y(Y)$.

If $T = \{y\}$, the identity of (Y,T) is cofinal among the covering families of (Y,T), hence $H^i(Y,\{y\}) = 0$ for $i \geqslant 1$. We obtain $H^2_y(Y,\{y\}) \cong H^1(K)$ and $H^i_y(Y,\{y\}) = 0$ for $i \geqslant 3$. □

Proposition 2.2. *(Global computation) Let X be a smooth projective and geometrically irreducible curve over $\mathbb{F}$, $k = \mathbb{F}(X)$ and S and T finite, disjoint sets of closed points of X.*

Then $H^i(X-S,T) = 0$ for $i \geqslant 3$ and $H^2(X-S,T) = 0$ if $S \neq \varnothing$. We have an exact sequence

$$0 \to H^1(X-S,T) \to H^1(X-S) \to \bigoplus_{x\in T} H^1_{nr}(k_x) \to H^2(X-S,T) \to H^2(X-S) \to 0.$$

Proof: In the case $T = \varnothing$ we have $H^i(X-S) = 0$ for $i \geqslant 3$ and $H^2(X-S) = 0$ if $S \neq \varnothing$ by [5] exp. 10, Thm. 5.1 and Cor. 5.2. Moreover, the sequence is exact for trivial reasons.

Now assume $T \neq \varnothing$. Consider the excision sequence for $(X-S,T)$ and $(X-(S\cup T))$:

$$\cdots \to \bigoplus_{x\in T} H^i_x((X-S)_x, T_x) \to H^i(X-S,T) \to H^i(X-(S\cup T)) \to \cdots.$$

Proposition 2.1 shows that $H^i(X-S,T) \cong H^i(X-(S\cup T)) = 0$ for $i \geqslant 3$ and the exactness of the sequence

$$0 \to H^1(X-S,T) \to H^1(X-(S\cup T)) \to \bigoplus_{x\in T} H^1(k_x) \to H^2(X-S,T) \to 0. \quad (*)$$

Comparing this with the excision sequence for $(X-S)$ and $(X-(S\cup T))$

$$0 \to H^1(X-S) \to H^1(X-(S\cup T)) \to \bigoplus_{x\in T} H^1_{/nr}(k_x) \to H^2(X-S) \to 0,$$

we obtain the exact sequence of the proposition.

If $S \neq \varnothing$, the Strong Approximation Theorem implies that

$$H^1(X-(S\cup T)) \longrightarrow \bigoplus_{x\in T} H^1(k_x)$$

is surjective (see [4] Thm. 9.2.5). Using $(*)$ this shows that $H^2(X-S,T) = 0$ in this case. □

Corollary 2.3. *If $G^T_S(k)$ is finite and nontrivial, then $(X-S,T)$ does not have the $K(\pi,1)$-property for p.*

Proof. In this case we have $cd\, G_S^T(k) = \infty$ but $H^i(X - S, T) = 0$ for $i \geqslant 3$. ☐

Corollary 2.4. *We have the Euler-Poincaré characteristic formula*

$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_p} H^i(X, T) = \#T.$$

Proof. If $S = \varnothing$, all groups in the exact sequence of Proposition 2.2 are finite and we obtain

$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_p} H^i(X, T) = \#T + \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_p} H^i(X).$$

Recall that $H^1(\bar{X}) = \mathrm{Hom}(\mathrm{Pic}(\bar{X})[p], \mathbb{F}_p)$ (every connected étale covering of $\bar{X}$ comes by base change from an isogeny of the Jacobian of $\bar{X}$). Hence

$$H^2(X) = H^1(\mathbb{F}, H^1(\bar{X})) = H^1(\mathbb{F}, \mathrm{Hom}(\mathrm{Pic}(\bar{X})[p], \mathbb{F}_p)) = \mathrm{Hom}(\mathrm{Pic}(\bar{X})[p], \mathbb{F}_p)_{G_{\mathbb{F}}}.$$

Furthermore, we have an exact sequence

$$0 \to H^1(\mathbb{F}) \to H^1(X) \to \mathrm{Hom}(\mathrm{Pic}(\bar{X})[p], \mathbb{F}_p)^{G_{\mathbb{F}}} \to 0.$$

Thus Lemma 2.5 below shows

$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_p} H^i(X) = 1 - \dim_{\mathbb{F}_p} H^1(\mathbb{F}) = 0.$$

☐

Lemma 2.5. *We have* $\mathrm{Pic}(\bar{X})[p]^{G_{\mathbb{F}}} = \mathrm{Pic}(X)[p]$ *and*

$$\dim_{\mathbb{F}_p} \mathrm{Pic}(\bar{X})[p]_{G_{\mathbb{F}}} = \dim_{\mathbb{F}_p} \mathrm{Pic}(X)[p].$$

Proof. The first equality follows from the Leray spectral sequence

$$E_2^{ij} = H^i(\mathbb{F}, H^j(\bar{X}, \mathbb{G}_m)) \Rightarrow H^{i+j}(X, \mathbb{G}_m)$$

and the vanishing of the Brauer group of a finite field:

$$H^2(\mathbb{F}, H^0(\bar{X}, \mathbb{G}_m)) = H^2(\mathbb{F}, \bar{\mathbb{F}}^\times) = 0.$$

The equality of dimensions follows from the exact sequence of finite-dimensional $\mathbb{F}_p$-vector spaces

$$0 \to \mathrm{Pic}(\bar{X})[p]^{G_{\mathbb{F}}} \to \mathrm{Pic}(\bar{X})[p] \xrightarrow{1-\mathrm{Frob}} \mathrm{Pic}(\bar{X})[p] \to \mathrm{Pic}(\bar{X})[p]_{G_{\mathbb{F}}} \to 0.$$

☐

3 Proof of Theorem 1.2

Assume $S \neq \varnothing$. From the computations in the last section, we know that $H^i(X-S,T) = 0$ for $i \geqslant 2$. By Lemma 1.1, $(X-S,T)$ has the $K(\pi,1)$-property for p and $cd\, G_S^T(k) \leqslant 1$. But $G_S^T(k)$ is nontrivial, which follows from the exact sequence

$$0 \to H^1(X-S,T) \to H^1(X-S) \to \bigoplus_{x\in T} H^1_{nr}(k_x) \to 0$$

together with the fact that $\bigoplus_{x\in T} H^1_{nr}(k_x)$ has finite $\mathbb{F}_p$-dimension whereas $H^1(X-S)$ is infinite dimensional.

Now assume that $S = \varnothing$ and $T = \varnothing$. Let $\widetilde{\mathbb{F}}$ be the maximal p-extension of $\mathbb{F}$ in $\bar{\mathbb{F}}$. Then $H^2(X_{\widetilde{\mathbb{F}}}) = H^2(X_{\bar{\mathbb{F}}})^{\mathrm{Gal}(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})} = 0$. Hence $X_{\widetilde{\mathbb{F}}}$ is a $K(\pi,1)$ for p and the Hochschild-Serre spectral sequence for $X_{\widetilde{\mathbb{F}}}/X$ shows the same for X. This finishes the proof of Theorem 1.2.

4 Proof of Theorem 1.3

Proposition 4.1. *Assume that* $\mathrm{Pic}(X)[p] = 0$ *and* $T \neq \varnothing$ *and let* p^r *be the maximal* p*-power dividing* $gcd(\deg x,\ x \in T)$. *Then*

$$G^T(k) = \pi_1(X,T)(p) \cong \mathrm{Gal}(\mathbb{F}'/\mathbb{F}),$$

where $\mathbb{F}'$ *is the unique extension of* $\mathbb{F}$ *of degree* p^r.

Proof. Let $\widetilde{\mathbb{F}}$ be the maximal p-extension of $\mathbb{F}$ in $\bar{\mathbb{F}}$. Using Lemma 2.5, we have

$$H^2(X) = H^1(\mathbb{F}, H^1(\bar{X})) \cong \mathrm{Hom}(\mathrm{Pic}(X)[p], \mathbb{F}_p) = 0$$

and Corollary 2.4 shows that $H^1(X)$ is 1-dimensional. Hence $\pi_1(X)(p)$ is free of rank 1 and therefore the surjection

$$\pi_1(X)(p) \twoheadrightarrow \mathrm{Gal}(\widetilde{\mathbb{F}}/\mathbb{F})$$

is an isomorphism (cf. [4], Prop. 1.6.15). The maximal subextension $\mathbb{F}'/\mathbb{F}$ of $\widetilde{\mathbb{F}}/\mathbb{F}$ such that all points in T split completely in the base change $X \otimes_{\mathbb{F}} \mathbb{F}' \to X$ is exactly the unique extension of degree p^r of $\mathbb{F}$. □

Corollary 4.2. *Assume that* $\mathrm{Pic}(X)[p] = 0$ *and* $T \neq \varnothing$. *Then* (X,T) *is a* $K(\pi,1)$ *for* p *if and only if* $T = \{x\}$ *consists of a single point with* $p \nmid \deg x$. *In this case the fundamental group* $\pi_1(X,T)(p)$ *is trivial.*

Proof. By Proposition 4.1, $\pi_1(X,T)(p)$ is finite cyclic. If $p \mid gcd(\deg x,\ x \in T)$, then $\pi_1(X,T)(p)$ is nontrivial and (X,T) is not a $K(\pi,1)$ for p by Corollary 2.3.

Assume $p \nmid gcd(\deg x \mid x \in T)$. Then $\pi_1(X,T)(p)$ is the trivial group, $H^1(X,T) = 0$ and (X,T) is a $K(\pi,1)$ if and only if $H^2(X,T) = 0$. By Corollary 2.4 this is equivalent to $\#T = 1$. □

Lemma 4.3. *Assume that $\pi_1(X,T)(p)$ is finite and* $\mathrm{Pic}(X)[p] \neq 0$. *Then (X,T) is not a $K(\pi,1)$ for p.*

Proof. By Corollary 2.3, (X,T) is not a $K(\pi,1)$ for p if $\pi_1(X,T)(p)$ is nontrivial. Assume that $\pi_1(X,T)(p) = 1$. Then (X,T) is a $K(\pi,1)$ for p if and only if $H^2(X,T) = 0$. But by Proposition 2.2, $H^2(X) \cong \mathrm{Hom}(\mathrm{Pic}(X)[p], \mathbb{F}_p) \neq 0$ is a quotient of $H^2(X,T)$. □

The following theorem is due to Ihara, see [6], Thm. 1 (FF).

Theorem 4.4. *Assume that $T \neq \varnothing$ and let $q = \#\mathbb{F}$. If*

$$\sum_{x\in T} \frac{\deg(x)}{q^{\deg(x)/2}-1} > \max(g_X - 1, 0),$$

then $\pi_1(X,T)$ is finite. In particular, $\pi_1(X,T)(p)$ is finite.

Summing up, we obtain Theorem 1.3.

5 Proof of Theorem 1.4

Let $\widetilde{\mathbb{F}}$ be the maximal p-extension of $\mathbb{F}$ in $\bar{\mathbb{F}}$ and $\widetilde{X} = X \times_{\mathbb{F}} \widetilde{\mathbb{F}}$. Then

$$X \text{ is a } K(\pi,1) \text{ for } p \Longleftrightarrow \tilde{X} \text{ is a } K(\pi,1) \text{ for } p$$

and we have

$$H^i_{et}(\widetilde{X}) \cong H^i_{et}(\bar{X})^{G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})}$$

for all i. Hence $H^i_{et}(\widetilde{X})$ vanishes for $i \geqslant 3$ and $H^2_{et}(\widetilde{X}) = \mu_p(\widetilde{\mathbb{F}})^* = \mu_p(\mathbb{F})^*$. We conclude that X has the $K(\pi,1)$-property for p if $\mu_p(\mathbb{F}) = 1$. In the following we assume that $\mathbb{F}$ contains all p-th roots of unity. For every tower of finite connected étale p-coverings $Z \to Y \to X$ the natural map

$$\mathbb{Z}/p\mathbb{Z} = H^2_{et}(\widetilde{Y}, \mu_p) \longrightarrow H^2_{et}(\widetilde{Z}, \mu_p) = \mathbb{Z}/p\mathbb{Z}$$

is multiplication by the degree $[\widetilde{Z} : \widetilde{Y}]$. Hence, by Lemma 1.1,

$$\widetilde{X} \text{ is a } K(\pi,1) \text{ for } p \Longleftrightarrow \#\big(\pi_1^{et}(\widetilde{X})(p)\big) = \infty.$$

Note that

$$\begin{aligned} \pi_1^{ab}(\widetilde{X})/p &\cong H^1_{et}(\widetilde{X})^* \\ &\cong \big(H^1_{et}(\bar{X})^{G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})}\big)^* \\ &\cong \mathrm{Pic}(\bar{X})[p]_{G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})} \end{aligned}$$

and by Lemma 2.5

$$\mathrm{Pic}(\bar{X})[p]_{G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})} = 0 \Longleftrightarrow \mathrm{Pic}(\widetilde{X})[p] = 0.$$

Furthermore, since $G(\widetilde{\mathbb{F}}/\mathbb{F})$ is a pro-p-group:

$$\mathrm{Pic}(\widetilde{X})[p] = 0 \Longleftrightarrow \mathrm{Pic}(X)[p] = \mathrm{Pic}(\widetilde{X})[p]^{G(\widetilde{\mathbb{F}}/\mathbb{F})} = 0,$$

and therefore, $\mathrm{Pic}(X)[p] \neq 0 \iff \pi_1^{ab}(\widetilde{X})/p \neq 0$. Hence it suffices to show the equivalences

$$\#\big(\pi_1^{et}(\widetilde{X})(p)\big) = \infty \Leftrightarrow \#\big(\pi_1^{ab}(\widetilde{X})(p)\big) = \infty \Leftrightarrow \#\pi_1^{ab}(\widetilde{X})/p \neq 0.$$

Elementary theory of pro-p-groups shows that it remains to show the implication

$$\pi_1^{ab}(\tilde{X})/p \neq 0 \Longrightarrow \#\big(\pi_1^{ab}(\tilde{X})(p)\big) = \infty.$$

Setting $T := T_p(\bar{X}) = \pi_1^{ab}(\bar{X})(p)$, we can write this implication in the form

$$(T_{G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})})/p \neq 0 \Longrightarrow \#(T_{G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})}) = \infty.$$

The group $G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})$ is pro-cyclic of supernatural order prime to p. Furthermore, $T \cong \mathbb{Z}_p^{2g}$ and the kernel of the reduction map $\mathrm{Gl}_{2g}(\mathbb{Z}_p) \to \mathrm{Gl}_{2g}(\mathbb{F}_p)$ is a pro-p-group. Hence the action of $G(\bar{\mathbb{F}}/\widetilde{\mathbb{F}})$ on T factors through a finite cyclic group of order prime to p. We conclude that Theorem 1.4 follows from Lemma 5.1 below. □

The following Lemma 5.1 and its application in the proof of Theorem 1.4 were proposed to us by J. Stix. We thank the referee for suggesting the short proof given below.

Lemma 5.1. *Let G be a finite group of order n, p a prime number with $p \nmid n$ and T a finitely generated free $\mathbb{Z}_p$-module with a G-action. Then*

$$\#T_G = \infty \iff (T/p)_G \neq 0.$$

Proof. Since the Tate cohomology of $\mathbb{Z}_p[G]$-modules vanishes, we obtain the split exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \longrightarrow \ker(N) \longrightarrow T \xrightarrow{N} T^G \longrightarrow 0,$$

where $N = \sum_{g\in G} g$. For $B = \ker(N)$, $\hat{H}^{-1}(G,B) = 0$ implies $B_G = 0$. We obtain $T_G \cong T^G$ and $(T/p)_G \cong (T^G)/p$. Hence both assertions of the lemma are equivalent to $T^G \neq 0$. □

References

[1] A. Schmidt, "Rings of integers of type $K(\pi, 1)$", *Doc. Math.* **12** (2007), 441–471.

[2] A. Schmidt, "On the $K(\pi, 1)$-property for rings of integers in the mixed case", *Algebraic number theory and related topics*, 2007, 91–100, RIMS Kôkyûroku Bessatsu, B12, Res. Inst. Math. Sci. (RIMS), Kyoto, 2009.

[3] A. Schmidt, Über Pro-p-Fundamentalgruppen markierter arithmetischer Kurven, *J. Reine Angew. Math.* **640** (2010), 203–235.

[4] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of Number Fields*, 2nd ed., 2nd corr. print., Grundlehren der math. Wiss. **323**, Springer 2013.

[5] M. Artin, A. Grothendieck et J. L. Verdier, "Théorie des topos et cohomologie étale des schémas", Tome 3", *Lecture Notes in Mathematics*, Vol. **305**, Springer-Verlag 1973.

[6] Y. Ihara, "How many primes decompose completely in an infinite unramified Galois extension of a global field?", *J. Math. Soc. Japan* **35** (1983), no. 4, 693–709.

Laboratoire de Mathématiques de Besançon, 16 route de Gray, 25030 Besançon, France
email: philippe.lebacque@univ-fcomte.fr

Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 288, D-69120 Heidelberg, Deutschland
email: schmidt@mathi.uni-heidelberg.de