Circular sets of prime numbers and p-extensions of the rationals

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Abstract: Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. We prove that the group $G_S(\mathbb{Q})(p)$ has cohomological dimension 2 if the linking diagram attached to S and p satisfies a certain technical condition, and we show that $G_S(\mathbb{Q})(p)$ is a duality group in these cases. Furthermore, we investigate the decomposition behaviour of primes in the extension $\mathbb{Q}_S(p)/\mathbb{Q}$ and we relate the cohomology of $G_S(\mathbb{Q})(p)$ to the étale cohomology of the scheme $Spec(\mathbb{Z}) - S$. Finally, we calculate the dualizing module.

1 Introduction

Let k be a number field, p a prime number and S a finite set of places of k. The pro-p-group $G_S(k)(p) = G(k_S(p)/k)$, i.e. the Galois group of the maximal p-extension of k which is unramified outside S, contains valuable information on the arithmetic of the number field k. If all places dividing p are in S, then we have some structural knowledge on $G_S(k)(p)$, in particular, it is of cohomological dimension less or equal to 2 (if $p = 2$ one has to require that S contains no real place, [Sc3]), and it is often a so-called duality group, see [NSW], X, §7. Furthermore, the cohomology of $G_S(k)(p)$ coincides with the étale cohomology of the arithmetic curve $Spec(\mathcal{O}_k) - S$ in this case.

In the opposite case, when S contains no prime dividing p, only little is known. By a famous theorem of Golod and Šafarevič, $G_S(k)(p)$ may be infinite. A conjecture due to Fontaine and Mazur [FM] asserts that $G_S(k)(p)$ has no infinite quotient which is an analytic pro-p-group. So far, nothing was known on the cohomological dimension of $G_S(k)(p)$ and on the relation between its cohomology and the étale cohomology of the scheme $Spec(\mathcal{O}_k) - S$.

Recently, J. Labute [La] showed that pro-p-groups with a certain kind of relation structure have cohomological dimension 2. By a result of H. Koch [Ko], $G_S(\mathbb{Q})(p)$ has such a relation structure if the set of prime numbers S satisfies a certain technical condition. In this way, Labute obtained first examples of pairs (p, S) with $p \notin S$ and $cd\, G_S(\mathbb{Q})(p) = 2$, e.g. $p = 3$, $S = \{7, 19, 61, 163\}$.

The objective of this paper is to use arithmetic methods in order to extend Labute's result. First of all, we weaken the condition on S which implies cohomological dimension 2 (and strict cohomological dimension 3!) and we show that $G_S(\mathbb{Q})(p)$ is a duality group in these cases. Furthermore, we investigate the decomposition behaviour of primes in the extension $\mathbb{Q}_S(p)/\mathbb{Q}$ and we relate the cohomology of $G_S(\mathbb{Q})(p)$ to the étale cohomology of the scheme $Spec(\mathbb{Z}) - S$. Finally, we calculate the dualizing module.

2 Statement of results

Let p be an odd prime number, S a finite set of prime numbers not containing p and $G_S(p) = G_S(\mathbb{Q})(p)$ the Galois group of the maximal p-extension $\mathbb{Q}_S(p)$ of $\mathbb{Q}$ which is unramified outside S. Besides p, only prime numbers congruent to 1 modulo p can ramify in a p-extension of $\mathbb{Q}$, and we assume that all primes in S have this property. Then $G_S(p)$ is a pro-p-group with n generators and n relations, where $n = \#S$ (see lemma 3.1).

Inspired by some analogies between knots and prime numbers (cf. [Mo]), J. Labute [La] introduced the notion of the linking diagram $\Gamma(S)(p)$ attached to p and S and showed that $cd\, G_S(p) = 2$ if $\Gamma(S)(p)$ is a 'non-singular circuit'. Roughly speaking, this means that there is an ordering $S = \{q_1, q_2, \dots, q_n\}$ such that $q_1 q_2 \cdots q_n q_1$ is a circuit in $\Gamma(S)(p)$ (plus two technical conditions, see section 7 for the definition).

We generalize Labute's result by showing

Theorem 2.1. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Assume there exists a subset $T \subset S$ such that the following conditions are satisfied.*

(i) $\Gamma(T)(p)$ *is a non-singular circuit.*

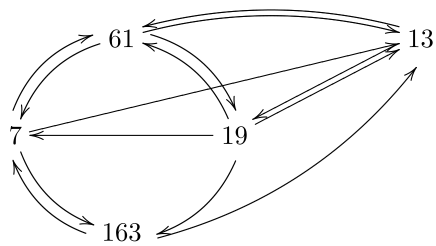
(ii) *For each $q \in S \backslash T$ there exists a directed path in $\Gamma(S)(p)$ starting in q and ending with a prime in T.*

Then $cd\, G_S(p) = 2$.

Remarks. 1. Condition (ii) of Theorem 2.1 can be weakened, see section 7.

2. Given p, one can construct examples of sets S of arbitrary cardinality $\#S \geq 4$ with $cd\, G_S(p) = 2$.

Example. For $p = 3$ and $S = \{7, 13, 19, 61, 163\}$, the linking diagram has the following shape



The linking diagram associated to the subset $T = \{7, 19, 61, 163\}$ is a non-singular circuit, and we obtain $cd\, G_S(3) = 2$ in this case.

The proof of Theorem 2.1 uses arithmetic properties of $G_S(p)$ in order to enlarge the set of prime numbers S without changing the cohomological dimension of $G_S(p)$. In particular, we show

Theorem 2.2. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Assume that $G_S(p) \neq 1$ and $cd\, G_S(p) \leq 2$. Then the following holds.*

(i) *$cd\, G_S(p) = 2$ and $scd\, G_S(p) = 3$.*

(ii) *$G_S(p)$ is a pro-p duality group (of dimension 2).*

(iii) *For all $\ell \in S$, $\mathbb{Q}_S(p)$ realizes the maximal p-extension of $\mathbb{Q}_\ell$, i.e. (after choosing a prime above ℓ in $\bar{\mathbb{Q}}$), the image of the natural inclusion $\mathbb{Q}_S(p) \hookrightarrow \mathbb{Q}_\ell(p)$ is dense.*

(iv) *The scheme $X = Spec(\mathbb{Z}) - S$ is a $K(\pi, 1)$ for p and the étale topology, i.e. for any p-primary $G_S(p)$-module M, considered as a locally constant étale sheaf on X, the natural homomorphism*

$$H^i(G_S(p), M) \to H^i_{et}(X, M)$$

is an isomorphism for all i.

Remarks. 1. If S consists of a single prime number, then $G_S(p)$ is finite, hence $\#S \geq 2$ is necessary for the theorem. At the moment, we do not know examples of cardinality 2 or 3.
2. The property asserted in Theorem 2.2 (iv) implies that the natural morphism of pro-spaces

$$X_{et}(p) \longrightarrow K(G_S(p), 1)$$

from the pro-p-completion of the étale homotopy type X_{et} of X (see [AM]) to the $K(\pi, 1)$-pro-space attached to the pro-p-group $G_S(p)$ is a weak equivalence. Since $G_S(p)$ is the fundamental group of $X_{et}(p)$, this justifies the notion '$K(\pi, 1)$ for p and the étale topology'. If S contains the prime number p, this property always holds (cf. [Sc2]).

We can enlarge the set of prime numbers S by the following

Theorem 2.3. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Assume that $cd\, G_S(p) = 2$. Let $\ell \notin S$ be another prime number congruent to 1 modulo p which does not split completely in the extension $\mathbb{Q}_S(p)/\mathbb{Q}$. Then $cd\, G_{S \cup \{\ell\}}(p) = 2$.*

3 Comparison with étale cohomology

In this section we show that cohomological dimension 2 implies the $K(\pi, 1)$-property.

Lemma 3.1. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Then*

$$dim_{\mathbb{F}_p} H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) = \begin{cases} 1 & \text{if } i = 0 \\ \#S & \text{if } i = 1 \\ \#S & \text{if } i = 2\,. \end{cases}$$

Proof. The statement for H^0 is obvious. [NSW], Theorem 8.7.11 implies the statement on H^1 and yields the inequality

$$\dim_{\mathbb{F}_p} H^2(G_S(p), \mathbb{Z}/p\mathbb{Z}) \leq \#S\,.$$

The abelian pro-p-group $G_S(p)^{ab}$ has $\#S$ generators. There is only one $\mathbb{Z}_p$-extension of $\mathbb{Q}$, namely the cyclotomic $\mathbb{Z}_p$-extension, which is ramified at p. Since p is not in S, $G_S(p)^{ab}$ is finite, which implies that $G_S(p)$ must have at least as many relations as generators. By [NSW], Corollary 3.9.5, the relation rank of $G_S(p)$ is $\dim_{\mathbb{F}_p} H^2(G_S(p), \mathbb{Z}/p\mathbb{Z})$, which yields the remaining inequality for H^2. □

Proposition 3.2. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. If $cd\, G_S(p) \leq 2$, then the scheme $X = Spec(\mathbb{Z}) - S$ is a $K(\pi, 1)$ for p and the étale topology, i.e. for any discrete p-primary $G_S(p)$-module M, considered as locally constant étale sheaf on X, the natural homomorphism*

$$H^i(G_S(p), M) \to H^i_{et}(X, M)$$

is an isomorphism for all i.

Proof. Let L/k be a finite subextension of k in $k_S(p)$. We denote the normalization of X in L by X_L. Then $H^i_{et}(X_L, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > 3$ ([Ma], §3, Proposition C). Since flat and étale cohomology coincide for finite étale group schemes ([Mi1], III, Theorem 3.9), the flat duality theorem of Artin-Mazur ([Mi2], III Theorem 3.1) implies

$$H^3_{et}(X_L, \mathbb{Z}/p\mathbb{Z}) = H^3_{fl}(X_L, \mathbb{Z}/p\mathbb{Z}) \cong H^0_{fl,c}(X_L, \mu_p)^\vee = 0,$$

since a p-extension of $\mathbb{Q}$ cannot contain a primitive p-th root of unity. Let $\tilde{X}$ be the universal (pro-)p-covering of X. We consider the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G_S(p), H^q_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})) \Rightarrow H^{p+q}_{et}(X, \mathbb{Z}/p\mathbb{Z}).$$

Étale cohomology commutes with inverse limits of schemes if the transition maps are affine (see [AGV], VII, 5.8). Therefore we have $H^i_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 3$, and for $i = 1$ by definition. Hence $E_2^{ij} = 0$ unless $i = 0, 2$. Using the assumption $cd\, G_S(p) \leq 2$, the spectral sequence implies isomorphisms $H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^i_{et}(X, \mathbb{Z}/p\mathbb{Z})$ for $i = 0, 1$ and a short exact sequence

$$0 \to H^2(G_S(p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\phi} H^2_{et}(X, \mathbb{Z}/p\mathbb{Z}) \to H^2_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})^{G_S(p)} \to 0.$$

Let $\bar{X} = Spec(\mathbb{Z})$. By the flat duality theorem of Artin-Mazur, we have an isomorphism $H^2_{et}(\bar{X}, \mathbb{Z}/p\mathbb{Z}) \cong H^1_{fl}(\bar{X}, \mu_p)^\vee$. The flat Kummer sequence $0 \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to 0$, together with $H^0_{fl}(\bar{X}, \mathbb{G}_m)/p = 0 = {}_pH^1_{fl}(\bar{X}, \mathbb{G}_m)$ implies $H^2_{et}(\bar{X}, \mathbb{Z}/p\mathbb{Z}) = 0$. Furthermore, $H^3_{et}(\bar{X}, \mathbb{Z}/p\mathbb{Z}) \cong H^0_{fl}(\bar{X}, \mu_p)^\vee = 0$. Considering the étale excision sequence for the pair $(\bar{X}, X)$, we obtain an isomorphism

$$H^2_{et}(X, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} \bigoplus_{\ell \in S} H^3_\ell(Spec(\mathbb{Z}_\ell), \mathbb{Z}/p\mathbb{Z}).$$

The local duality theorem ([Mi2], II, Theorem 1.8) implies

$$H^3_\ell(Spec(\mathbb{Z}_\ell), \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_\ell)}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)^\vee.$$

All primes $\ell \in S$ are congruent to 1 modulo p by assumption, hence $\mathbb{Z}_\ell$ contains a primitive p-th root of unity for $\ell \in S$, and we obtain $\dim_{\mathbb{F}_p} H^2_{et}(X, \mathbb{Z}/p\mathbb{Z}) = \#S$. Now Lemma 3.1 implies that ϕ is an isomorphism. We therefore obtain

$$H^2_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})^{G_S(p)} = 0.$$

Since $G_S(p)$ is a pro-p-group, this implies ([NSW], Corollary 1.7.4) that

$$H^2_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z}) = 0.$$

We conclude that the Hochschild-Serre spectral sequence degenerates to a series of isomorphisms

$$H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^i_{et}(X, \mathbb{Z}/p\mathbb{Z}), \qquad i \geq 0.$$

If M is a finite p-primary $G_S(p)$-module, it has a composition series with graded pieces isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with trivial $G_S(p)$-action ([NSW], Corollary 1.7.4), and the statement of the proposition for M follows from that for $\mathbb{Z}/p\mathbb{Z}$ and from the five-lemma. An arbitrary discrete p-primary $G_S(p)$-module is the filtered inductive limit of finite p-primary $G_S(p)$-modules, and the statement of the proposition follows since group cohomology ([NSW], Proposition 1.5.1) and étale cohomology ([AGV], VII, 3.3) commute with filtered inductive limits. □

4 Proof of Theorem 2.2

In this section we prove Theorem 2.2. Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Assume that $G_S(p) \neq 1$ and $cd\, G_S(p) \leq 2$.

Let $U \subset G_S(p)$ be an open subgroup. The abelianization U^{ab} of U is a finitely generated abelian pro-p-group. If U^{ab} were infinite, it would have a quotient isomorphic to $\mathbb{Z}_p$, which corresponds to a $\mathbb{Z}_p$-extension K_∞ of the number field $K = \mathbb{Q}_S(p)^U$ inside $\mathbb{Q}_S(p)$. By [NSW], Theorem 10.3.20 (ii), a $\mathbb{Z}_p$-extension of a number field is ramified at at least one prime dividing p. This contradicts $K_\infty \subset \mathbb{Q}_S(p)$ and we conclude that U^{ab} is finite.

In particular, $G_S(p)^{ab}$ is finite. Hence $G_S(p)$ is not free, and we obtain $cd\, G_S(p) = 2$. This shows the first part of assertion (i) of Theorem 2.2 and assertion (iv) follows from Proposition 3.2.

By Lemma 3.1, we know that for each prime number $\ell \in S$, the group $G_{S\setminus\{\ell\}}(p)$ is a proper quotient of $G_S(p)$, hence each $\ell \in S$ is ramified in the extension $\mathbb{Q}_S(p)/\mathbb{Q}$. Let $G_\ell(\mathbb{Q}_S(p)/\mathbb{Q})$ denote the decomposition group of ℓ in $G_S(p)$ with respect to some prolongation of ℓ to $\mathbb{Q}_S(p)$. As a subgroup of $G_S(p)$, $G_\ell(\mathbb{Q}_S(p)/\mathbb{Q})$ has cohomological dimension less or equal to 2. We have a natural surjection $G(\mathbb{Q}_\ell(p)/\mathbb{Q}_\ell) \twoheadrightarrow G_\ell(\mathbb{Q}_S(p)/\mathbb{Q})$. By [NSW], Theorem 7.5.2, $G(\mathbb{Q}_\ell(p)/\mathbb{Q}_\ell)$ is the pro-p-group on two generators σ, τ subject to the relation $\sigma\tau\sigma^{-1} = \tau^\ell$. τ is a generator of the inertia group and σ is a Frobenius lift.

Therefore, $G(\mathbb{Q}_\ell(p)/\mathbb{Q}_\ell)$ has only three quotients of cohomological dimension less or equal to 2: itself, the trivial group and the Galois group of the maximal unramified p-extension of $\mathbb{Q}_\ell$. Since ℓ is ramified in the extension $\mathbb{Q}_S(p)/\mathbb{Q}$, the map $G(\mathbb{Q}_\ell(p)/\mathbb{Q}_\ell) \twoheadrightarrow G_\ell(\mathbb{Q}_S(p)/\mathbb{Q})$ is an isomorphism, and hence $\mathbb{Q}_S(p)$ realizes the maximal p-extension of $\mathbb{Q}_\ell$. This shows statement (iii) of Theorem 2.2.

Next we show the second part of statement (i). By [NSW], Proposition 3.3.3, we have $scd\, G_S(p) \in \{2,3\}$. Assume that $scd\, G = 2$. We consider the $G_S(p)$-module

$$D_2(\mathbb{Z}) = \varinjlim_U U^{ab},$$

where the limit runs over all open normal subgroups $U \lhd G_S(p)$ and for $V \subset U$ the transition map is the transfer $\mathrm{Ver}\colon U^{ab} \to V^{ab}$, i.e. the dual of the corestriction map $\mathrm{cor}\colon H^2(V,\mathbb{Z}) \to H^2(U,\mathbb{Z})$ (see [NSW], I, §5). By [NSW], Theorem 3.6.4 (iv), we obtain $G_S(p)^{ab} = D_2(\mathbb{Z})^{G_S(p)}$. On the other hand, U^{ab} is finite for all U and the group theoretical version of the Principal Ideal Theorem (see [Ne], VI, Theorem 7.6) implies $D_2(\mathbb{Z}) = 0$. Hence $G_S(p)^{ab} = 0$ which implies $G_S(p) = 1$ producing a contradiction. Hence $scd\, G_S(p) = 3$ showing the remaining assertion of Theorem 2.2, (i).

It remains to show that $G_S(p)$ is a duality group. By [NSW], Theorem 3.4.6, it suffices to show that the terms

$$D_i(G_S(p), \mathbb{Z}/p\mathbb{Z}) = \varinjlim_U H^i(U, \mathbb{Z}/p\mathbb{Z})^\vee$$

are trivial for $i = 0, 1$. Here U runs through the open subgroups of $G_S(p)$, $^\vee$ denotes the Pontryagin dual and the transition maps are the duals of the corestriction maps. For $i = 0$, and $V \subsetneqq U$, the transition map

$$\mathrm{cor}^\vee : \mathbb{Z}/p\mathbb{Z} = H^0(V, \mathbb{Z}/p\mathbb{Z})^\vee \to H^0(U, \mathbb{Z}/p\mathbb{Z})^\vee = \mathbb{Z}/pZ$$

is multiplication by $(U : V)$, hence zero. Since $G_S(p)$ is infinite, we obtain $D_0(G_S(p), \mathbb{Z}/p\mathbb{Z}) = 0$. Furthermore,

$$D_1(G_S(p), \mathbb{Z}/p\mathbb{Z}) = \varinjlim_U U^{ab}/p = 0$$

by the Principal Ideal Theorem. This finishes the proof of Theorem 2.2.

5 The dualizing module

Having seen that $G_S(p)$ is a duality group under certain conditions, it is interesting to calculate its dualizing module. The aim of this section is to prove

Theorem 5.1. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Assume that $cd\, G_S(p) = 2$. Then we have a natural isomorphism*

$$D \cong \mathrm{tor}_p\big(C_S(\mathbb{Q}_S(p))\big)$$

between the dualizing module D of $G_S(p)$ and the p-torsion submodule of the S-idèle class group of $\mathbb{Q}_S(p)$. There is a natural short exact sequence

$$0 \to \bigoplus_{\ell \in S} \mathrm{Ind}_{G_S(p)}^{G_\ell}\ \mu_{p^\infty}(\mathbb{Q}_\ell(p)) \to D \to E_S(\mathbb{Q}_S(p)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$

in which G_ℓ is the decomposition group of ℓ in $G_S(p)$ and $E_S(\mathbb{Q}_S(p))$ is the group of S-units of the field $\mathbb{Q}_S(p)$.

Working in a more general situation, let S be a non-empty set of primes of a number field k. We recall some well-known facts from class field theory and we give some modifications for which we do not know a good reference.

By k_S we denote the maximal extension of k which is unramified outside S and we denote $G(k_S/k)$ by $G_S(k)$. For an intermediate field $k \subset K \subset k_S$, let $C_S(K)$ denote the S-idèle class group of K. If S contains the set S_∞ of archimedean primes of k, then the pair $(G_S(k), C_S(k_S))$ is a class formation, see [NSW], Proposition 8.3.8. This remains true for arbitrary non-empty S, as can be seen as follows: We have the class formation

$$(G_S(k), C_{S\cup S_\infty}(k_S)).$$

Since k_S is closed under unramified extensions, the Principal Ideal Theorem implies $Cl_S(k_S) = 0$. Therefore we obtain the exact sequence

$$0 \to \bigoplus_{v\in S_\infty\setminus S(k)} \mathrm{Ind}_{G_S(k)} k_v^\times \to C_{S\cup S_\infty}(k_S) \to C_S(k_S) \to 0.$$

Since the left term is a cohomologically trivial $G_S(k)$-module, we obtain that $(G_S(k), C_S(k_S))$ is a class formation. Finally, if p is a prime number, then also $(G_S(k)(p), C_S(k_S(p))$ is a class formation.

Remark: An advantage of considering the class formation $(G_S(k)(p), C_S(k_S(p))$ for sets S of primes which do not contain S_∞ is that we get rid of 'redundancy at infinity'. A technical disadvantage is the absence of a reasonable Hausdorff topology on the groups $C_S(K)$ for finite subextensions K of k in $k_S(p)$.

Next we calculate the module

$$D_2(\mathbb{Z}_p) = \varinjlim_{U,n} H^2(U, \mathbb{Z}/p^n\mathbb{Z})^\vee,$$

where n runs through all natural numbers, U runs through all open subgroups of $G_S(k)(p)$ and $^\vee$ is the Pontryagin dual. If $cd\, G_S(p) = 2$, then $D_2(\mathbb{Z}_p)$ is the dualizing module D of $G_S(k)(p)$.

Theorem 5.2. *Let k be a number field, p an odd prime number and S a finite non-empty set of non-archimedean primes of k such that the norm $N(\mathfrak{p})$ of $\mathfrak{p}$ is congruent to 1 modulo p for all $\mathfrak{p} \in S$. Assume that the scheme $X = Spec(\mathcal{O}_k) - S$ is a $K(\pi, 1)$ for p and the étale topology and that $k_S(p)$ realizes the maximal p-extension $k_\mathfrak{p}(p)$ of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$. Then $G_S(p)$ is a pro-p-duality group of dimension 2 with dualizing module*

$$D \cong \mathrm{tor}_p\big(C_S(k_S(p))\big).$$

Remarks. 1. In view of Theorem 2.2, Theorem 5.2 shows Theorem 5.1.
2. In the case when S contains all primes dividing p, a similar result has been proven in [NSW], X, §5.

Proof of Theorem 5.2. We consider the schemes $\bar{X} = Spec(\mathcal{O}_k)$ and $X = \bar{X} - S$ and we denote the natural embedding by $j : X \to \bar{X}$. As in the proof of Proposition 3.2, the flat duality theorem of Artin-Mazur implies

$$H^3_{et}(X, \mathbb{Z}/p\mathbb{Z}) \cong H^0_{fl,c}(X, \mu_p)^\vee,$$

and the group on the right vanishes since $k_\mathfrak{p}$ contains a primitive p-th root of unity for all $\mathfrak{p} \in S$. The $K(\pi, 1)$-property yields $cd\, G_S(k)(p) \le 2$. Since $k_S(p)$ realizes the maximal p-extension $k_\mathfrak{p}(p)$ of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$, the inertia groups of these primes are of cohomological dimension 2 and we obtain $cd\, G_S(p) = 2$.

Next we consider, for some $n \in \mathbb{N}$, the constant sheaf $\mathbb{Z}/p^n\mathbb{Z}$ on X. The duality theorem of Artin-Verdier shows an isomorphism

$$H^i_{et}(\bar{X}, j_!(\mathbb{Z}/p^n\mathbb{Z})) = H^i_c(X, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathrm{Ext}^{3-i}_X(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m)^\vee.$$

For $\mathfrak{p} \in S$, a standard calculation (see, e.g., [Mi2], II, Proposition 1.1) shows

$$H^i_\mathfrak{p}(\bar{X}, j_!(\mathbb{Z}/p^n\mathbb{Z}) \cong H^{i-1}(k_\mathfrak{p}, \mathbb{Z}/p^n\mathbb{Z}),$$

where $k_\mathfrak{p}$ is (depending on the readers preference) the henselization or the completion of k at $\mathfrak{p}$. The excision sequence for the pair $(\bar{X}, X)$ and the sheaf $j_!(\mathbb{Z}/p^n\mathbb{Z})$ therefore implies a long exact sequence

$$(*)\quad \cdots \to H^i_{et}(X, \mathbb{Z}/p^n\mathbb{Z}) \to \bigoplus_{\mathfrak{p}\in S} H^i(k_\mathfrak{p}, \mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{Ext}^{2-i}_X(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m)^\vee \to \cdots$$

The local duality theorem ([NSW], Theorem 7.2.6) yields isomorphisms

$$H^i(k_\mathfrak{p}, \mathbb{Z}/p^n\mathbb{Z})^\vee \cong H^{2-i}(k_\mathfrak{p}, \mu_{p^n})$$

for all $i \in \mathbb{Z}$. Furthermore,

$$\mathrm{Ext}^0_X(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m) = H^0(k, \mu_{p^n}).$$

We denote by $E_S(k)$ and $Cl_S(k)$ the group of S-units and the S-ideal class group of k, respectively. By $Br(X)$, we denote the Brauer group of X. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ together with

$$\mathrm{Ext}^i_X(\mathbb{Z}, \mathbb{G}_m) = H^i_{et}(X, \mathbb{G}_m) = \begin{cases} E_S(k) & \text{for } i = 0 \\ Cl_S(k) & \text{for } i = 1 \\ Br(X) & \text{for } i = 2 \end{cases}$$

and the Hasse principle for the Brauer group implies exact sequences

$$0 \to E_S(k)/p^n \to \mathrm{Ext}^1_X(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m) \to {}_{p^n}Cl_S(k) \to 0$$

and

$$0 \to Cl_S(k)/p^n \to \mathrm{Ext}^2_X(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{G}_m) \to \bigoplus_{\mathfrak{p}\in S} {}_{p^n}Br(k_\mathfrak{p}).$$

The same holds, if we replace X by its normalization X_K in a finite extension K of k in $k_S(p)$. Now we go to the limit over all such K. Since $k_S(p)$ realizes the maximal p-extension of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$, we have

$$\varinjlim_K \bigoplus_{\mathfrak{p}\in S(K)} H^i(K_\mathfrak{p}, \mathbb{Z}/p^n\mathbb{Z})^\vee = \varinjlim_K \bigoplus_{\mathfrak{p}\in S(K)} H^i(K_\mathfrak{p}, \mu_{p^n}) = 0$$

for $i \geq 1$ and

$$\varinjlim_{K} \bigoplus_{\mathfrak{p} \in S(K)} {}_{p^n}Br(K_{\mathfrak{p}}) = 0.$$

The Principal Ideal Theorem implies $Cl_S(k_S(p))/p = 0$ and since this group is a torsion group, its p-torsion part is trivial. Going to the limit over the exact sequences $(*)$ for all X_K, we obtain $D_i(\mathbb{Z}/p\mathbb{Z}) = 0$ for $i = 0, 1$, hence $G_S(k)(p)$ is a duality group of dimension 2. Furthermore, we obtain the exact sequence

$$0 \to \mathrm{tor}_p\big(E_S(k_S(p))\big) \to \bigoplus_{\mathfrak{p} \in S} \mathrm{Ind}\,_{G_S(k)(p)}^{G_{\mathfrak{p}}} \mathrm{tor}_p\big(k_{\mathfrak{p}}(p)^{\times}\big) \to$$
$$D \to E_S(k_S(p)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Let $U \subset G_S(k)(p)$ be an open subgroup and put $K = k_S(p)^U$. The invariant map

$$\mathrm{inv}_K \colon H^2(U, C_S(k_S(p))) \to \mathbb{Q}/\mathbb{Z}$$

induces a pairing

$$\mathrm{Hom}_U(\mathbb{Z}/p^n\mathbb{Z}, C_S(k_S(p))) \times H^2(U, \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\cup} H^2(U, C_S(K)) \xrightarrow{\mathrm{inv}_K} \mathbb{Q}/\mathbb{Z},$$

and therefore a compatible system of maps

$${}_{p^n}C_S(K) \to H^2(U, \mathbb{Z}/p^n\mathbb{Z})^{\vee}$$

for all U and n. In the limit, we obtain a natural map

$$\phi \colon \mathrm{tor}_p\big(C_S(k_S(p)\big) \longrightarrow D.$$

By our assumptions, the idèle group $J_S(k_S(p))$ is p-divisible. We therefore obtain the exact sequence

$$0 \to \mathrm{tor}_p\big(E_S(k_S(p))\big) \to \bigoplus_{\mathfrak{p} \in S} \mathrm{Ind}\,_{G_S(k)(p)}^{G_{\mathfrak{p}}} \mathrm{tor}_p\big(k_{\mathfrak{p}}(p)^{\times}\big) \to$$
$$\mathrm{tor}_p\big(C_S(k_S(p))\big) \to E_S(k_S(p)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

which, via the just constructed map ϕ, compares to the similar sequence with D above. Hence ϕ is an isomorphism by the five lemma. □

Finally, without any assumptions on $G_S(k)(p)$, we calculate the $G_S(k)(p)$-module $D_2(\mathbb{Z}_p)$ as a quotient of $\mathrm{tor}_p\big(C_S(k_S(p))\big)$ by a subgroup of universal norms. We therefore can interpret Theorem 5.2 as a vanishing statement on universal norms.

Let us fix some notation. If G is a profinite group and if M is a G-module, we denote by ${}_{p^n}M$ the submodule of elements annihilated by p^n. By $N_G(M) \subset M^G$ we denote the subgroup of universal norms, i.e.

$$N_G(M) = \bigcap_U N_{G/U}(M^U),$$

where U runs through the open normal subgroups of G and $N_{G/U}(M^U) \subset M^G$ is the image of the norm map

$$N \colon M^U \to M^G,\ m \mapsto \sum_{\sigma \in G/U} \sigma m.$$

Proposition 5.3. *Let S be a non-empty finite set of non-archimedean primes of k and let p be an odd prime number such that S contains no prime dividing p. Then*

$$D_2(G_S(k)(p), \mathbb{Z}_p) \cong \varinjlim_{K,n} {}_{p^n}C_S(K)/N_{G(k_S(p)/K)}({}_{p^n}C_S(K)),$$

where n runs through all natural numbers and K runs through all finite subextension of k in $k_S(p)$.

Proof. We want to use Poitou's duality theorem ([Sc2], Theorem 1). But the class module $C_S(k_S(p))$ is not level-compact and we cannot apply the theorem directly. Instead, we consider the level-compact class formation

$$\big(G_S(k)(p), C^0_{S\cup S_\infty}(k_S(p))\big),$$

where $C^0_{S\cup S_\infty}(k_S(p)) \subset C_{S\cup S_\infty}(k_S(p))$ is the subgroup of idèle classes of norm 1. By [Sc2], Theorem 1, we have for all natural numbers n and all finite subextensions K of k in $k_S(p)$ a natural isomorphism

$$H^2(G_S(K)(p), \mathbb{Z}/p^n\mathbb{Z})^\vee \cong \hat{H}^0(G_S(K)(p),\ {}_{p^n}C^0_{S\cup S_\infty}(k_S(p))),$$

where $\hat{H}^0$ is Tate-cohomology in dimension 0 (cf. [Sc2]). The exact sequence

$$0 \to \bigoplus_{v\in S_\infty(K)} K_v^\times \to C_{S\cup S_\infty}(K) \to C_S(K) \to 0$$

and the fact that $K_v^\times$ is p-divisible for archimedean v, implies for all n and all finite subextensions K of k in $k_S(p)$ an exact sequence of finite abelian groups

$$0 \to \bigoplus_{v\in S_\infty(K)} \mu_{p^n}(K_v) \to {}_{p^n}C_{S\cup S_\infty}(K) \to\ {}_{p^n}C_S(K) \to 0.$$

[Sc2], Proposition 7 therefore implies isomorphisms

$$\hat{H}^0(G_S(K)(p), {}_{p^n}C_{S\cup S_\infty}(k_S(p))) \cong \hat{H}^0(G_S(K)(p), {}_{p^n}C_S(k_S(p)))$$

for all n and K. Furthermore, the exact sequence

$$0 \to C^0_{S\cup S_\infty}(K) \to C_{S\cup S_\infty}(K) \xrightarrow{|\ |} \mathbb{R}_+^\times \to 0$$

shows ${}_{p^n}C^0_{S\cup S_\infty}(K)) = {}_{p^n}C_{S\cup S_\infty}(K))$ for all n and all finite subextensions K of k in $k_S(p)$. Finally, [Sc2], Lemma 5 yields isomorphisms

$$\hat{H}^0(G_S(K)(p), {}_{p^n}C_S(k_S(p))) \cong\ {}_{p^n}C_S(K)/N_{G(k_S(p)/K)}({}_{p^n}C_S(K)).$$

Going to the limit over all n and K, we obtain the statement of the Proposition. □

6 Going up

The aim of this section is to prove Theorem 2.3. We start with the following lemma.

Lemma 6.1. *Let $\ell \neq p$ be prime numbers. Let $\mathbb{Q}_\ell^h$ be the henselization of $\mathbb{Q}$ at ℓ and let K be an algebraic extension of $\mathbb{Q}_\ell^h$ containing the maximal unramified p-extension $(\mathbb{Q}_\ell^h)^{nr,p}$ of $\mathbb{Q}_\ell^h$. Let $Y = Spec(\mathcal{O}_K)$, and denote the closed point of Y by y. Then the local étale cohomology group $H_y^i(Y, \mathbb{Z}/p\mathbb{Z})$ vanishes for $i \neq 2$ and we have a natural isomorphism*

$$H_y^2(Y, \mathbb{Z}/p\mathbb{Z}) \cong H^1(G(K(p)/K), \mathbb{Z}/p\mathbb{Z}).$$

Proof. Since K contains $(\mathbb{Q}_\ell^h)^{nr,p}$, we have $H_{et}^i(Y, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > 0$. The excision sequence shows $H_y^i(Y, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i = 0, 1$ and $H_y^i(Y, \mathbb{Z}/p\mathbb{Z}) \cong H^{i-1}(G(\bar{K}/K), \mathbb{Z}/p\mathbb{Z})$ for $i \geq 2$. By [NSW], Proposition 7.5.7, we have

$$H^{i-1}(G(\bar{K}/K), \mathbb{Z}/p\mathbb{Z}) = H^{i-1}(G(K(p)/K), \mathbb{Z}/p\mathbb{Z})$$

But $G(K(p)/K)$ is a free pro-p-group (either trivial or isomorphic to $\mathbb{Z}_p$). This concludes the proof. □

Let k be a number field and let S be finite set of primes of k. For a (possibly infinite) algebraic extension K of k we denote by $S(K)$ the set of prolongations of primes in S to K. Now assume that $M/K/k$ is a tower of pro-p Galois extensions. We denote the inertia group of a prime $\mathfrak{p} \in S(K)$ in the extension M/K by $T_\mathfrak{p}(M/K)$. For $i \geq 0$ we write

$$\bigoplus_{\mathfrak{p} \in S(K)}{}' H^i(T_\mathfrak{p}(M/K), \mathbb{Z}/p\mathbb{Z}) \stackrel{df}{=} \varinjlim_{k' \subset K} \bigoplus_{\mathfrak{p} \in S(k')} H^i(T_\mathfrak{p}(M/k'), \mathbb{Z}/p\mathbb{Z}),$$

where the limit on the right hand side runs through all finite subextensions k' of k in K. The $G(K/k)$-module $\bigoplus'_{\mathfrak{p} \in S(K)} H^i(T_\mathfrak{p}(M/K), \mathbb{Z}/p\mathbb{Z})$ is the maximal discrete submodule of the product $\prod_{\mathfrak{p} \in S(K)} H^i(T_\mathfrak{p}(M/K), \mathbb{Z}/p\mathbb{Z})$.

Proposition 6.2. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p such that $cd\, G_S(p) = 2$. Let $\ell \notin S$ be another prime number congruent to 1 modulo p which does not split completely in the extension $\mathbb{Q}_S(p)/\mathbb{Q}$. Then, for any prime $\mathfrak{p}$ dividing ℓ in $\mathbb{Q}_S(p)$, the inertia group of $\mathfrak{p}$ in the extension $\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)$ is infinite cyclic. Furthermore,*

$$H^i(G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}) = 0$$

for $i \geq 2$. For $i = 1$ we have a natural isomorphism

$$H^1(G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{\mathfrak{p} \in S_\ell(\mathbb{Q}_S(p))}{}' H^1(T_\mathfrak{p}(\mathbb{Q}_{S\cup\{\ell\}}(p))/\mathbb{Q}_S(p), \mathbb{Z}/p\mathbb{Z}),$$

where $S_\ell(\mathbb{Q}_S(p))$ denotes the set of primes of $\mathbb{Q}_S(p)$ dividing ℓ. In particular, $G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p))$ is a free pro-p-group.

Proof. Since ℓ does not split completely in $\mathbb{Q}_S(p)/\mathbb{Q}$ and since $cdG_S(p) = 2$, the decomposition group of ℓ in $\mathbb{Q}_S(p)/\mathbb{Q}$ is a non-trivial and torsion-free quotient of $\mathbb{Z}_p \cong G(\mathbb{Q}_\ell^{nr,p}/\mathbb{Q}_\ell)$. Therefore $\mathbb{Q}_S(p)$ realizes the maximal unramified p-extension of $\mathbb{Q}_\ell$. We consider the scheme $X = Spec(\mathbb{Z}) - S$ and its universal pro-p covering $\tilde{X}$ whose field of functions is $\mathbb{Q}_S(p)$. Let Y be the subscheme of $\tilde{X}$ obtained by removing all primes of residue characteristic ℓ. We consider the étale excision sequence for the pair $(\tilde{X}, Y)$. By Theorem 3.2, we have $H^i_{et}(\tilde{X}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > 0$, which implies isomorphisms

$$H^i_{et}(Y, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} {\bigoplus_{\mathfrak{p}|\ell}}' H^{i+1}_{\mathfrak{p}}(Y^h_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$$

for $i \geq 1$. By Lemma 6.1, we obtain $H^i_{et}(Y, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 2$. The universal p-covering $\tilde{Y}$ of Y has $\mathbb{Q}_{S\cup\{\ell\}}(p)$ as its function field, and the Hochschild-Serre spectral sequence for $\tilde{Y}/Y$ yields an inclusion

$$H^2(G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}) \hookrightarrow H^2_{et}(Y, \mathbb{Z}/p\mathbb{Z}) = 0.$$

Hence $G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p))$ is a free pro-p-group and for H^1 we obtain

$$\begin{aligned} H^1(G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}) &\xrightarrow{\sim} H^1_{et}(Y, \mathbb{Z}/p\mathbb{Z}) \\ &\cong {\bigoplus_{\mathfrak{p}\in S_\ell(\mathbb{Q}_S(p))}}' H^1(G(\mathbb{Q}_S(p)_{\mathfrak{p}}(p)/\mathbb{Q}_S(p)_{\mathfrak{p}}), \mathbb{Z}/p\mathbb{Z}). \end{aligned}$$

This shows that each $\mathfrak{p} \mid \ell$ ramifies in $\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)$, and since the Galois group is free, $\mathbb{Q}_{S\cup\{\ell\}}(p)$ realizes the maximal p-extension of $\mathbb{Q}_S(p)_{\mathfrak{p}}$. In particular,

$$H^1(G(\mathbb{Q}_S(p)_{\mathfrak{p}}(p)/\mathbb{Q}_S(p)_{\mathfrak{p}}), \mathbb{Z}/p\mathbb{Z}) \cong H^1(T_{\mathfrak{p}}(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z})$$

for all $\mathfrak{p} \mid \ell$, which finishes the proof. □

Let us mention in passing that the above calculations imply the validity of the following arithmetic form of Riemann's existence theorem.

Theorem 6.3. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p such that $\operatorname{cd} G_S(p) = 2$. Let $T \supset S$ be another set of prime numbers congruent to 1 modulo p. Assume that all $\ell \in T\backslash S$ do not split completely in the extension $\mathbb{Q}_S(p)/\mathbb{Q}$. Then the inertia groups in $\mathbb{Q}_T(p)/\mathbb{Q}_S(p)$ of all primes $\mathfrak{p} \in T\backslash S(\mathbb{Q}_S(p))$ are infinite cyclic and the natural homomorphism*

$$\phi : \mathop{\ast}_{\mathfrak{p}\in T\backslash S(\mathbb{Q}_S(p))} T_{\mathfrak{p}}(\mathbb{Q}_T(p)/\mathbb{Q}_S(p)) \longrightarrow G(\mathbb{Q}_T(p)/\mathbb{Q}_S(p))$$

is an isomorphism.

Remark: A similar theorem holds in the case that S contains p, see [NSW], Theorem 10.5.1.

Proof. By Proposition 6.2 and by the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4), ϕ is a homomorphism between free pro-p-groups which induces an isomorphism on mod p cohomology. Therefore ϕ is an isomorphism. □

Proof of theorem 2.3. We consider the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(G_S(p), H^j(G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}) \Rightarrow H^{i+j}(G_{S\cup\{\ell\}}(p), \mathbb{Z}/p\mathbb{Z}).$$

By Proposition 6.2, we have $E_2^{ij} = 0$ for $j \geq 2$ and

$$\begin{aligned} H^1(G(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}) &\cong \bigoplus_{\mathfrak{p}|\ell}{}' H^1(T_{\mathfrak{p}}(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}) \\ &\cong \mathrm{Ind}_{G_S(p)}^{G_\ell} H^1(T_\ell(\mathbb{Q}_{S\cup\{\ell\}}(p)/\mathbb{Q}_S(p)), \mathbb{Z}/p\mathbb{Z}), \end{aligned}$$

where $G_\ell \cong \mathbb{Z}_p$ is the decomposition group of ℓ in $G_S(p)$. We obtain $E_2^{i,1} = 0$ for $i \geq 2$. By assumption, $cd\, G_S(p) = 2$, hence $E_2^{0,j} = 0$ for $j \geq 3$. This implies $H^3(G_{S\cup\{\ell\}}(p), \mathbb{Z}/p\mathbb{Z}) = 0$, and hence $cd\, G_{S\cup\{\ell\}}(p) \leq 2$. Finally, the decomposition group of ℓ in $G_{S\cup\{\ell\}}(p)$ is full, i.e. of cohomological dimension 2. Therefore, $cd\, G_{S\cup\{\ell\}}(p) = 2$. □

We obtain the following

Corollary 6.4. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Let $\ell \notin S$ be a another prime number congruent to 1 modulo p. Assume that there exists a prime number $q \in S$ such that the order of ℓ in $(\mathbb{Z}/q\mathbb{Z})^\times$ is divisible by p (e.g. ℓ is not a p-th power modulo q). Then $cd\, G_S(p) = 2$ implies $cd\, G_{S\cup\{\ell\}}(p) = 2$.*

Proof. Let K_q be the maximal subextension of p-power degree in $\mathbb{Q}(\mu_q)/\mathbb{Q}$. Then K_q is a non-trivial finite subextension of $\mathbb{Q}$ in $\mathbb{Q}_S(p)$ and ℓ does not split completely in $K_q/\mathbb{Q}$. Hence the result follows from Theorem 2.3. □

Remark. One can sharpen Corollary 6.4 by finding weaker conditions on a prime ℓ not to split completely in $\mathbb{Q}_S(p)$.

7 Proof of Theorem 2.1

In this section we prove Theorem 2.1. We start by recalling the notion of the linking diagram attached to S and p from [La]. Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Let $\Gamma(S)(p)$ be the directed graph with vertices the primes of S and edges the pairs $(r, s) \in S \times S$ with r not a p-th power modulo s. We now define a function ℓ on the set of pairs of distinct primes of S with values in $\mathbb{Z}/p\mathbb{Z}$ by first choosing a primitive root g_s modulo s for each $s \in S$. Let $\ell_{rs} = \ell(r, s)$ be the image in $\mathbb{Z}/p\mathbb{Z}$ of any integer c satisfying

$$r \equiv g_s^{-c} \bmod s\ .$$

The residue class ℓ_{rs} is well-defined since c is unique modulo $s-1$ and $p \mid s-1$. Note that (r, s) is an edge of $\Gamma(S)(p)$ if and only if $\ell_{rs} \neq 0$. We call ℓ_{rs} the *linking number* of the pair (r, s). This number depends on the choice of primitive roots, if g is another primitive root modulo s and $g_s \equiv g^a \bmod s$, then the linking number attached to (r, s) would be multiplied by a if g were used instead of g_s. The directed graph $\Gamma(S)(p)$ together with ℓ is called the *linking diagram* attached to S and p.

Definition 7.1. We call a finite set S of prime numbers congruent to 1 modulo p *strictly circular with respect to* p (and $\Gamma(S)(p)$ a *non-singular circuit*), if there exists an ordering $S = \{q_1, \dots, q_n\}$ of the primes in S such that the following conditions hold.

(a) The vertices $q_1, \dots, q_n$ of $\Gamma(S)(p)$ form a circuit $q_1 q_2 \cdots q_n q_1$.

(b) If i, j are both odd, then $q_i q_j$ is not an edge of $\Gamma(S)(p)$.

(c) If we put $\ell_{ij} = \ell(q_i, q_j)$, then

$$\ell_{12}\ell_{23} \cdots \ell_{n-1,n}\ell_{n1} \neq \ell_{1n}\ell_{21} \cdots \ell_{n,n-1}.$$

Note that condition (b) implies that n is even ≥ 4 and that (c) is satisfied if there is an edge $q_i q_j$ of the circuit $q_1 q_2 \cdots q_n q_1$ such that $q_j q_i$ is not an edge of $\Gamma(S)(p)$. Condition (c) is independent of the choice of primitive roots since the condition can be written in the form

$$\frac{\ell_{1n}}{\ell_{n-1,n}} \frac{\ell_{21}}{\ell_{n1}} \frac{\ell_{32}}{\ell_{12}} \cdots \frac{\ell_{n,n-1}}{\ell_{n-2,n-1}} \neq 1,$$

where each ratio in the product is independent of the choice of primitive roots.

If p is an odd prime number and if $S = \{q_1, \dots, q_n\}$ is a finite set of prime numbers congruent to 1 modulo p, then, by a result of Koch [Ko], the group $G_S(p)$ has a minimal presentation $G_S(p) = F/R$, where F is a free pro-p-group on generators $x_1, \dots, x_n$ and R is the minimal normal subgroup in F on generators $r_1, \dots, r_n$, where

$$r_i \equiv x_i^{q_i - 1} \prod_{j \neq i} [x_i, x_j]^{\ell_{ij}} \bmod F_3.$$

Here F_3 is the third step of the lower p-central series of F and the $\ell_{ij} = \ell(q_i, q_j)$ are the linking numbers for some choice of primitive roots. If S is strictly circular, Labute ([La], Theorem 1.6) shows that $G_S(p)$ is a so-called 'mild' pro-p-group, and, in particular, is of cohomological dimension 2 ([La], Theorem 1.2).

Proof of Theorem 2.1. By [La], Theorem 1.6, we have $cd\, G_T(p) = 2$. By assumption, we find a series of subsets

$$T = T_0 \subset T_1 \subset \cdots \subset T_r = S,$$

such that for all $i \geq 1$, the set $T_i \backslash T_{i-1}$ consists of a single prime number q congruent to 1 modulo p and there exists a prime number $q' \in T_{i-1}$ with q not a p-th power modulo q'. An inductive application of Corollary 6.4 yields the result. □

Remark. Labute also proved some variants of his group theoretic result [La], Theorem 1.6. The same proof as above shows corresponding variants of Theorem 2.1, by replacing condition (i) by other conditions on the subset T as they are described in [La], §3.

A straightforward applications of Čebotarev's density theorem shows that, given $\Gamma(S)(p)$, a prime number q congruent to 1 modulo p can be found with the additional edges of $\Gamma(S \cup \{q\})(p)$ arbitrarily prescribed (cf. [La], Proposition 6.1). We therefore obtain the following corollaries.

Corollary 7.2. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p, containing a strictly circular subset $T \subset S$. Then there exists a prime number q congruent to 1 modulo p with*

$$cd\, G_{S\cup\{q\}}(p) = 2.$$

Corollary 7.3. *Let p be an odd prime number and let S be a finite set of prime numbers congruent to 1 modulo p. Then we find a finite set T of prime numbers congruent to 1 modulo p such that*

$$cd\, G_{S\cup T}(p) = 2.$$

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