Tame Class Field Theory for Singular Varieties over Algebraically Closed Fields

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1 Introduction

Let X be a (possibly singular) separated scheme of finite type over an algebraically closed field k of characteristic $p \geq 0$ and let m be a natural number. We construct a pairing between the first mod m algebraic singular homology $H_1^S(X, \mathbb{Z}/m\mathbb{Z})$ and the first mod m tame étale cohomology group $H_t^1(X, \mathbb{Z}/m\mathbb{Z})$. For $\pi_1^{t,ab}(X) = H_t^1(X, \mathbb{Q}/\mathbb{Z})^*$ we prove the following analogue of Hurewicz's theorem in algebraic topology:

Theorem 1.1. *The induced homomorphism*

$$rec_X : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \pi_1^{t,ab}(X)/m$$

is surjective. It is an isomorphism of finite abelian groups if $(m, p) = 1$*, and for general* m *if resolution of singularities holds for schemes of dimension* $\leq \dim X + 1$ *over* k*.*

For $p \nmid m$, the groups $H_1^S(X, \mathbb{Z}/m\mathbb{Z})$ and $\pi_1^{t,ab}(X)/m$ are known to be isomorphic by the work of Suslin and Voevodsky [SV1]. Theorem 1.1 above provides an explicit isomorphism which extends to the case $p \mid m$ (under resolution of singularities). Moreover, in the last section we show that for $p \nmid m$ our isomorphism coincides with the one constructed in [SV1].

The motivation for constructing our pairing between the groups $H_1^S(X, \mathbb{Z}/m\mathbb{Z})$ and $H_t^1(X, \mathbb{Z}/m\mathbb{Z})$ comes from topology: For a locally contractible Hausdorff space X and a natural number m, the canonical duality pairing

$$\langle \cdot, \cdot \rangle : H_1^{\mathrm{sing}}(X, \mathbb{Z}/m\mathbb{Z}) \times H^1(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z},$$

between singular homology and sheaf cohomology with mod m coefficients can be given explicitly in the following way: represent $b \in H^1(X, \mathbb{Z}/m\mathbb{Z})$ by a $\mathbb{Z}/m\mathbb{Z}$-torsor $\mathcal{T} \to X$ and $a \in H_1^{\mathrm{sing}}(X, \mathbb{Z}/m\mathbb{Z})$ by a 1-cycle α in the singular complex of X. Then

$$\langle a, b \rangle = \Phi_{par}^{-1} \circ \Phi_{taut} \in \mathbb{Z}/m\mathbb{Z}, \quad \text{where } \Phi_{taut}, \Phi_{par} : \alpha^*(\mathcal{T})|_0 \xrightarrow{\sim} \alpha^*(\mathcal{T})|_1$$

are the isomorphisms between the fibres over 0 and 1 of the pull-back torsor $\alpha^*(\mathcal{T}) \to \Delta^1 = [0, 1]$ given tautologically ($0^*\alpha = 1^*\alpha$) and by parallel transport (every $\mathbb{Z}/m\mathbb{Z}$-torsor on $[0, 1]$ is trivial).

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For a variety X, the pairing between $H_1^S(X,\mathbb{Z}/m\mathbb{Z})$ and $H_t^1(X,\mathbb{Z}/m\mathbb{Z})$ inducing the homomorphism rec_X of our Main Theorem 1.1 will be constructed in the same way. However, 1-cycles in the algebraic singular complex are not linear combinations of morphisms but finite correspondences from Δ^1 to X. In order to mimic the above construction, we thus have to define the pull-back of a torsor along a finite correspondence, which requires the construction of the push-forward torsor along a finite surjective morphism.

To prove Theorem 1.1, we first consider the case of a smooth curve C. If $\mathcal{A}$ is the Albanese variety of C, then we have isomorphisms

$$H_1^S(C,\mathbb{Z}/m\mathbb{Z}) \xrightarrow[\sim]{\delta} {}_mH_0^S(C,\mathbb{Z}) \cong {}_m\mathcal{A}(k). \tag{1}$$

The first isomorphism follows from the coefficient sequence together with the divisibility of $H_1^S(C,\mathbb{Z})$, and the second from the Abel-Jacobi theorem. On the other hand,

$$\mathrm{Hom}({}_mA(k),\mathbb{Z}/m\mathbb{Z}) \xrightarrow[\sim]{\tau} H_t^1(C,\mathbb{Z}/m\mathbb{Z}). \tag{2}$$

This follows because the maximal étale subcovering $\widetilde{\mathcal{A}} \to \mathcal{A}$ of the m-multiplication map $\mathcal{A} \overset{m}{\to} \mathcal{A}$ is the quotient of $\mathcal{A}$ by the connected component of the finite group scheme ${}_m\mathcal{A}$, and the maximal abelian tame étale covering of C with Galois group annihilated by m is $\widetilde{C} := C \times_{\mathcal{A}} \widetilde{\mathcal{A}}$. The heart of the proof of Theorem 1.1 for smooth curves is to show that under the above identifications, our pairing agrees with the evaluation map.

We then show surjectivity of rec_X for general X by reducing to the case of smooth curves. Finally, we use duality theorems to show that both sides of rec_X have the same order: For the p-primary part, we use resolution of singularities to reduce to the smooth projective case considered in [Ge3]. For $(m,\mathrm{char}(k)) = 1$, Suslin and Voevodsky [SV1] construct an isomorphism

$$\alpha_X : H^1_{\mathrm{et}}(X,\mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} H^1_S(X,\mathbb{Z}/m\mathbb{Z}).$$

Hence the source and the target of rec_X have the same order and therefore rec_X is an isomorphism. In Section 7 we show that rec_X is dual to the map α_X. Thus, for $\mathrm{char}(k) \nmid m$, our construction gives an explicit description of the Suslin-Voevodsky isomorphism α_X, which zig-zags through Ext-groups in various categories and is difficult to understand.

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2 Torsors and finite correspondences

All occurring schemes in this section are separated schemes of finite type over a field k. For any abelian group A and a finite surjective morphism $\pi : Z \to X$ with Z integral and X normal, connected, we have transfer maps

$$\pi_* : H^i_{\mathrm{et}}(Z,A) \to H^i_{\mathrm{et}}(X,A)$$

for all $i \geq 0$ (see [MVW], 6.11, 6.21). The group $H^1_{\mathrm{et}}(Z,A)$ classifies isomorphism classes of étale A-torsors (i.e., principal homogeneous spaces) over the scheme Z. We are going to construct a functor

$$\pi_* : \mathcal{PHS}(Z,A) \longrightarrow \mathcal{PHS}(X,A)$$

from the category of étale A-torsors on Z to the category of étale A-torsors on X, which induces the transfer map $\pi_* : H^1_{\mathrm{et}}(Z,A) \to H^1_{\mathrm{et}}(X,A)$ above on isomorphism classes.

We recall how to add and subtract torsors. For an abelian group A and A-torsors $\mathcal{T}_1$, $\mathcal{T}_2$ on a scheme Y, define

$$\mathcal{T}_1 + \mathcal{T}_2$$

to be the quotient scheme of $\mathcal{T}_1 \times_Y \mathcal{T}_2$ by the action of A given by $(t_1, t_2) + a = (t_1 + a, t_2 - a)$. It carries the structure of an A-torsor by setting

$$\overline{(t_1, t_2)} + a := \overline{(t_1 + a, t_2)} \quad (= \overline{(t_1, t_2 + a)}).$$

The functor

$$+ : \mathcal{PHS}(Y, A) \times \mathcal{PHS}(Y, A) \longrightarrow \mathcal{PHS}(Y, A), \quad (\mathcal{T}_1, \mathcal{T}_2) \longmapsto \mathcal{T}_1 + \mathcal{T}_2,$$

lifts the addition in $H^1_{\text{et}}(Y, A)$ to torsors (cf. [Mi], III, Rem. 4.8 (b)). Note that "+" is associative and commutative up to natural functor isomorphisms. In particular, we can multiply a torsor by any natural number m, putting $m \cdot \mathcal{T} = \mathcal{T} + \cdots + \mathcal{T}$ (m times). If $mA = 0$, then we have a natural isomorphism of torsors

$$m \cdot \mathcal{T} \xrightarrow{\sim} Y \times A, \quad \overline{(t_1, \ldots, t_m)} \mapsto (t_2 - t_1) + \cdots + (t_m - t_1) \in A, \tag{3}$$

where $Y \times A$ is the trivial A-torsor on Y representing the constant sheaf $\underline{A}$ over Y. Here $t_i - t_j$ denotes the unique element $a \in A$ with $t_i = t_j + a$.

Furthermore, given a torsor $\mathcal{T}$, define $(-\mathcal{T})$ to be the torsor which is isomorphic to $\mathcal{T}$ as a scheme and on which $a \in A$ acts as $-a$. This yields a functor

$$(-1) : \mathcal{PHS}(Y, A) \longrightarrow \mathcal{PHS}(Y, A), \quad \mathcal{T} \longmapsto (-\mathcal{T}),$$

which lifts multiplication by (-1) from $H^1_{\text{et}}(Y, A)$ to an endofunctor of $\mathcal{PHS}(Y, A)$. We have a natural isomorphism of torsors

$$\mathcal{T} + (-\mathcal{T}) \xrightarrow{\sim} Y \times A, \quad \overline{(t_1, t_2)} \mapsto t_1 - t_2 \in A. \tag{4}$$

Now let $\pi : Z \to X$ be finite and surjective, Z integral, X normal, connected, and let $\mathcal{T}$ be an A-torsor on Z. For every point $x \in X$, the base change $Z \times_X X_x^{sh}$ is a product of strictly henselian local schemes. Therefore we find an étale cover $(U_i \to X)_{i \in I}$ of X such that $\mathcal{T}$ trivializes over the pull-back étale cover $(\pi^{-1}(U_i) \to Z)_{i \in I}$ of Z.

Next choose a pseudo-Galois covering $\widetilde{\pi} : \widetilde{Z} \to X$ dominating $Z \to X$. Recall that this means that $k(\widetilde{Z})|k(X)$ is a normal field extension and that the natural map $\mathrm{Aut}_X(\widetilde{Z}) \to \mathrm{Aut}_{k(X)}(k(\widetilde{Z}))$ is bijective (cf. [SV1], Lemma 5.6). Let $\pi_{in} : X_{in} \to X$ be the quotient scheme $\widetilde{Z}/G$, where $G = \mathrm{Aut}_X(\widetilde{Z})$. Then X_{in} is the normalization of X in the maximal purely inseparable subextension $k(X)^{in}/k(X)$ of $k(\widetilde{Z})/k(X)$. Consider the object

$$\widetilde{\mathcal{T}} := \sum_{\varphi \in \mathrm{Mor}_X(\widetilde{Z}, Z)} \varphi^*(\mathcal{T}) \in \mathcal{PHS}(\widetilde{Z}, A),$$

which is defined up to unique isomorphism. Starting from any trivialization of $\mathcal{T}$ over $(\pi^{-1}(U_i) \to Z)_{i \in I}$, we obtain a trivialization of the restriction of $\widetilde{\mathcal{T}}$ to $(\widetilde{\pi}^{-1}(U_i) \to \widetilde{Z})_{i \in I}$ of the form

$$\widetilde{U}|_{\widetilde{\pi}^{-1}(U_i)} \cong \widetilde{\pi}^{-1}(U_i) \times A,$$

where $G = \mathrm{Aut}_X(\widetilde{Z})$ acts on the right hand side in the canonical way on $\widetilde{\pi}^{-1}(U_i)$ and trivially on A. Therefore the quotient scheme $\widetilde{\mathcal{T}}/G$ is an A-torsor on $\widetilde{Z}/G = X_{in}$ in a natural way. Since $X_{in} \to X$ is a topological isomorphism, $\widetilde{\mathcal{T}}/G$ comes by base change from a unique A-torsor $\mathcal{T}'$ on X.

Definition 2.1. *The push-forward A-torsor $\pi_*(\mathcal{T})$ on X is defined by*

$$\pi_*(\mathcal{T}) = [k(Z) : k(X)]_{in} \cdot \mathcal{T}'.$$

The assignment $\mathcal{T} \mapsto \pi_*(\mathcal{T})$ defines a functor

$$\pi_* : \mathcal{PHS}(Z, A) \longrightarrow \mathcal{PHS}(X, A).$$

The functor π_* is additive in the sense that it commutes with the functors "$+$" and "(-1)" up to a natural functor isomorphism.

Let $\mathcal{T} \in \mathcal{PHS}(Z, A)$ and assume that there exists a section $s : Z \to \mathcal{T}$ to the projection $\mathcal{T} \to Z$ (so $\mathcal{T}$ is trivial and s gives a trivialization). Let again $\pi : Z \to X$ be finite and surjective, Z integral, X normal, connected. Then

$$\widetilde{\mathcal{T}} := \sum_{\varphi \in \mathrm{Mor}_X(\widetilde{Z}, Z)} \varphi^*(\mathcal{T}) \in \mathcal{PHS}(\widetilde{Z}, A)$$

has the canonical section $\sum_{\varphi \in \mathrm{Mor}_X(\widetilde{Z}, Z)} \varphi^*(s)$ over $\widetilde{Z}$. It descends to a section of $\mathcal{T}/G$ over $\widetilde{Z}/G = X_{in}$. Descending to X and multiplying by $[k(X_{in} : k(X)]$, we obtain a section

$$\pi_*(s) : X \to \pi_*(\mathcal{T}).$$

In other words, we obtain a map

$$\pi_* : \Gamma(Z, \mathcal{T}) \longrightarrow \Gamma(X, \pi_*(\mathcal{T}));$$

hence every trivialization of $\mathcal{T}$ gives a trivialization of $\pi_*(\mathcal{T})$ in a natural way.

In order to see that π_* induces the transfer map $\pi_* : H^1_{\mathrm{et}}(Z, A) \to H^1_{\mathrm{et}}(X, A)$ after passing to isomorphism classes, we formulate the construction of π_* on the level of Čech 1-cocycles. As explained above, we find an étale cover $(U_i \to X)_{i \in I}$ such that $\mathcal{T}$ trivializes over the étale cover $(\pi^{-1}(U_i) \to Z)_{i \in I}$ of Z. We fix a trivialization and obtain a Čech 1-cocycle

$$a = (a_{ij} \in \Gamma(\pi^{-1}(U_i \times_X U_j), A))$$

over $(\pi^{-1}(U_i) \to Z)_{i \in I}$ which defines $\mathcal{T}$. As before choose a pseudo-Galois covering $\widetilde{\pi} : \widetilde{Z} \to X$ dominating $Z \to X$. Now for all i, j consider the element

$$\sum_{\varphi \in \mathrm{Mor}_X(\widetilde{Z}, Z)} \varphi^*(a_{ij}) \in \Gamma(\widetilde{\pi}^{-1}(U_i \times_X U_j), A)$$

which, by Galois invariance, lies in

$$\Gamma(\pi_{in}^{-1}(U_i \times_X U_j), A) = \Gamma(U_i \times_X U_j, A).$$

The Čech 1-cocycle given by

$$[k(Z) : k(X)]_{in} \cdot \Big(\sum_{\varphi \in \mathrm{Mor}_X(\widetilde{Z}, Z)} \varphi^*(a_{ij}) \Big) \in \Gamma(U_i \times_X U_j, A).$$

now defines a trivialization of $\pi_*(\mathcal{T})$ over $(U_i \to X)_{i \in I}$. Since the transfer map on étale cohomology is defined on Čech cocycles in exactly this way (see [MVW], 6.11, 6.21), we obtain

Lemma 2.2. *Passing to isomorphism classes, the functor* $\pi_* : \mathcal{PHS}(Z, A) \to \mathcal{PHS}(X, A)$ *constructed above induces the transfer homomorphism*

$$\pi_* : H^1_{\mathrm{et}}(Z, A) \to H^1_{\mathrm{et}}(X, A).$$

If any finite subset of points of X is contained in an affine open (e.g., if X is quasi-projective), another description of the push-forward for torsors is the following: Associated with the finite morphism $\pi : Z \to X$ of degree d, there is a section $s_\pi : X \to \mathrm{Sym}^d(Z/X)$ to the natural projection $\mathrm{Sym}^d(Z/X) \to X$ (see ([SV1], p. 81). We denote the composite of s_π with $pr : \mathrm{Sym}^d(Z/X) \to \mathrm{Sym}^d(Z)$ by S_π. Defining $f : \widetilde{Z} \to \mathrm{Sym}^d(Z/X)$ by repeating each element in $\mathrm{Mor}_X(\widetilde{Z}, Z)$ exactly $[k(Z) : k(X)]_{in}$-times, the diagram

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "f"] \arrow[dr, "\widetilde{\pi}"'] & \mathrm{Sym}^d(Z/X) \arrow[r, "pr"] & \mathrm{Sym}^d(Z) \\
& X \arrow[u, "s_\pi"] \arrow[ur, "S_\pi"'] &
\end{tikzcd}$$

commutes. For an A-torsor $\mathcal{T} \to Z$, the quotient of $\mathcal{T} \times_k \cdots \times_k \mathcal{T}$ (d times) by the action of the symmetric group S_d is an A^d-torsor over $\mathrm{Sym}^d(Z)$ in a natural way. Taking the quotient by the A^{d-1}-action

$$(a_1, \ldots, a_{d-1})(t_1, \ldots, t_d) = (t_1 + a_1, t_2 - a_1 + a_2, t_3 - a_2 + a_3, \ldots, t_d - a_{d-1}),$$

we obtain an A-torsor over $\mathrm{Sym}^d(Z)$ and denote it by $\mathrm{Sym}^d(\mathcal{T})$. We obtain natural isomorphisms in $\mathcal{PHS}(\widetilde{Z}, A)$:

$$\begin{aligned}
[k(Z) : k(X)]_{in} \cdot \sum_{\varphi \in \mathrm{Mor}_X(\widetilde{Z}, Z)} \varphi^*(\mathcal{T}) &\cong (pr \circ f)^* \mathrm{Sym}^d(\mathcal{T}) \\
&\cong \widetilde{\pi}^* \circ (pr \circ s_\pi)^* \mathrm{Sym}^d(\mathcal{T}).
\end{aligned}$$

By our construction of $\pi_*(\mathcal{T})$ we obtain

Lemma 2.3. *We have a natural isomorphism in* $\mathcal{PHS}(X, A)$*:*

$$\pi_*(\mathcal{T}) \cong S_\pi^*(\mathrm{Sym}^d(\mathcal{T})),$$

where $S_\pi = pr \circ s_\pi : X \to \mathrm{Sym}^d(Z)$.

Assume now that X is regular and Y arbitrary. The group of finite correspondences $\mathrm{Cor}(X, Y)$ is defined as the free abelian group on the set of integral subschemes $Z \subset X \times Y$ which project finitely and surjectively to a connected component of X. For such a Z, we define $p_{[Z \to X]\,*} : \mathcal{PHS}(Z, A) \to \mathcal{PHS}(X, A)$ by extending (if X is not connected) the push-forward torsor defined above in a trivial way to those connected components of X which are not dominated by Z. We consider the functor

$$[Z]^* = p_{[Z \to X]\,*} \circ p^*_{[Z \to Y]} : \mathcal{PHS}(Y, A) \longrightarrow \mathcal{PHS}(X, A).$$

Using the operations "+" and "(-1)" we extend this construction to arbitrary finite correspondences.

Definition 2.4. *Let X be regular, Y arbitrary and $\alpha = \sum n_i Z_i \in \mathrm{Cor}(X,Y)$ a finite correspondence. Then*

$$\alpha^* : \mathcal{PHS}(Y,A) \longrightarrow \mathcal{PHS}(X,A)$$

is defined by setting

$$\alpha^*(\mathcal{T}) := \sum n_i [Z_i]^*(\mathcal{T}).$$

Using the isomorphism (4) above, we immediately obtain

Lemma 2.5. *For $\alpha_1, \alpha_2 \in Cor(X,Y)$ and $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{PHS}(Y,A)$, $n_1, n_2 \in \mathbb{Z}$, we have a natural isomorphism*

$$(\alpha_1 + \alpha_2)^*(n_1\mathcal{T}_1 + n_2\mathcal{T}_2) \cong n_1\alpha_1^*(\mathcal{T}_1) + n_1\alpha_2^*(\mathcal{T}_1) + n_2\alpha_1^*(\mathcal{T}_2) + n_2\alpha_2^*(\mathcal{T}_2).$$

If X and Y are regular and Z is arbitrary, we have a natural composition law

$$\mathrm{Cor}(X,Y) \times \mathrm{Cor}(Y,Z) \longrightarrow \mathrm{Cor}(X,Z),\ (\alpha,\beta) \mapsto \beta \circ \alpha,$$

(see [MVW], Lecture 1). A straightforward but lengthy computation unfolding the definitions shows

Proposition 2.6. *Let X and Y be regular and Z arbitrary. Let $\alpha \in \mathrm{Cor}(X,Y)$ and $\beta \in \mathrm{Cor}(Y,Z)$. Then, for any $\mathcal{T} \in \mathcal{PHS}(Z,A)$, we have a canonical isomorphism*

$$\alpha^*(\beta^*(\mathcal{T})) \cong (\beta \circ \alpha)^*(\mathcal{T}).$$

Finally, assume that $mA = 0$ for some natural number m. Then (using the isomorphism (3) above), we have for any $\alpha, \beta \in Cor(X,Y)$, $\mathcal{T} \in \mathcal{PHS}(Y,A)$, a natural isomorphism

$$(\alpha + m\beta)^*(\mathcal{T}) \cong \alpha^*(\mathcal{T}).$$

Therefore, we have an A-torsor

$$\bar{\alpha}^*(\mathcal{T}) \in \mathcal{PHS}(X,A)$$

given up to unique isomorphism for any $\bar{\alpha} \in \mathrm{Cor}(X,Y) \otimes \mathbb{Z}/m\mathbb{Z}$. In other words, we obtain the

Lemma 2.7. *Assume that $mA = 0$, and let $\alpha, \beta \in \mathrm{Cor}(X,Y)$ have the same image in $\mathrm{Cor}(X,Y) \otimes \mathbb{Z}/m\mathbb{Z}$. Then there is a natural isomorphism of functors*

$$\alpha^* \cong \beta^* : \mathcal{PHS}(Y,A) \to \mathcal{PHS}(X,A).$$

For a regular connected curve C we consider the subgroup $H_t^1(C,A) \subseteq H_{\mathrm{et}}^1(C,A)$ of tame cohomology classes (corresponding to those continuous homomorphisms $\pi_1^{\mathrm{et}}(C) \to A$ which factor through the tame fundamental group $\pi_1^t(\bar{C}, \bar{C} - C)$, where $\bar{C}$ is the unique regular compactification of C).

For a general scheme X over k we call a cohomology class in $a \in H^1_{\mathrm{et}}(X,A)$ curve-tame (or just tame) if for any morphism $f: C \to X$ with C a regular curve, we have $f^*(a) \in H^1_t(C,A)$. The tame cohomology classes form a subgroup

$$H^1_t(X,A) \subseteq H^1_{\mathrm{et}}(X,A).$$

The groups coincide if X is proper or if $p=0$ or if $p>0$ and A is p-torsion free, where p is the characteristic of the base field k.

Definition 2.8. *We call an étale A-torsor $\mathcal{T}$ on X* tame *if its isomorphism class lies in $H^1_t(X,A) \subseteq H^1_{\mathrm{et}}(X,A)$.*

Lemma 2.9. *Let Z be integral, X normal, connected, $\pi : Z \to X$ finite, surjective and $f: Z \to Y$ any morphism. Let $\mathcal{T}$ be a tame torsor on Y. Then $\pi_*(f^*(\mathcal{T}))$ is a tame torsor on X.*

Proof. By definition, f^* preserves curve-tameness. So we may assume $Z=Y$, $f=\mathrm{id}$. Again by the definition of curve-tameness and using Proposition 2.6, we may reduce to the case that X is a regular curve. Since étale cohomology commutes with direct limits of coefficients, we may assume that A is a finitely generated abelian group. Furthermore, we may assume that $\mathrm{char}(k)=p>0$ and $A=\mathbb{Z}/p^r\mathbb{Z}$, $r \geq 1$.

Let $\bar{Z}$ be the canonical compactification of Z, i.e., the unique proper curve over k which contains Z as a dense open subscheme and such that all points of $\bar{Z} \smallsetminus Z$ are regular points of $\bar{Z}$. By the definition of tame coverings of curves, $\mathcal{T}$ extends to a $\mathbb{Z}/p^r\mathbb{Z}$-torsor on $\bar{Z}$. Hence also $\pi_*(\mathcal{T})$ extends to the canonical compactification $\bar{X}$ of X and so is tame. □

Proposition 2.10. *Let $\bar{X}$ be a proper and regular scheme over k and let $X \subset \bar{X}$ be a dense open subscheme. Let $p=\mathrm{char}(k)>0$. Then for any $r \geq 1$ the natural inclusion*

$$H^1_{\mathrm{et}}(\bar{X},\mathbb{Z}/p^r\mathbb{Z}) \hookrightarrow H^1_{\mathrm{et}}(X,\mathbb{Z}/p^r\mathbb{Z})$$

induces an isomorphism

$$H^1_{\mathrm{et}}(\bar{X},\mathbb{Z}/p^r\mathbb{Z}) = H^1_t(\bar{X},\mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{\sim} H^1_t(X,\mathbb{Z}/p^r\mathbb{Z}) \subseteq H^1_{\mathrm{et}}(X,\mathbb{Z}/p^r\mathbb{Z}).$$

Proof. Let $\mathcal{T}_0$ be any connected component of a tame $\mathbb{Z}/p^r\mathbb{Z}$-torsor $\mathcal{T}$ on X. Then the morphism $\mathcal{T}_0 \to X$ is curve-tame in the sense of [KS], §4, and $\mathcal{T}_0$ is the normalization of X in the abelian field extension of p-power degree $k(\mathcal{T}_0)/k(X)$. By [KS], Thm. 5.4. (b), $\mathcal{T}_0 \to X$ is numerically tamely ramified along $\bar{X} \smallsetminus X$. This means that the inertia groups in $\mathrm{Gal}(k(\mathcal{T}_0)/k(X))$ of all points $\bar{x} \in \bar{X} \smallsetminus X$ are of order prime to p, hence trivial. Therefore $\mathcal{T}_0$, and thus $\mathcal{T}$ extends to $\bar{X}$. □

Corollary 2.11. *Let $\Delta^n = Spec(k[T_0,\dots,T_n]/\sum T_i = 1)$ be the n-dimensional standard simplex over k and let A be an abelian group. Then*

$$H^1_t(\Delta^n,A) \cong H^1_{\mathrm{et}}(k,A).$$

In particular, $H^1_t(\Delta^n,A)=0$ if k is separably closed.

Proof. Since tame cohomology commutes with direct limits of coefficients, and since $H^1_{\mathrm{et}}(\Delta^n, \mathbb{Z}) = 0$, we may assume that $A \cong \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 1$. If $p \nmid m$, we obtain:

$$H^1_t(\Delta^n, \mathbb{Z}/m\mathbb{Z}) \cong H^1_{\mathrm{et}}(\mathbb{A}^n, \mathbb{Z}/m\mathbb{Z}) \cong H^1_{\mathrm{et}}(k, \mathbb{Z}/m\mathbb{Z}).$$

If $p = \mathrm{char}(k) > 0$ and $m = p^r$, $r \geq 1$, Proposition 2.10 yields

$$H^1_t(\Delta^n, \mathbb{Z}/p^r\mathbb{Z}) \cong H^1_t(\mathbb{A}^n, \mathbb{Z}/p^r\mathbb{Z}) \xleftarrow{\sim} H^1_t(\mathbb{P}^n, \mathbb{Z}/p^r\mathbb{Z}) = H^1_{\mathrm{et}}(\mathbb{P}^n, \mathbb{Z}/p^r\mathbb{Z}).$$

Finally note that $H^1_{\mathrm{et}}(\mathbb{P}^n, \mathbb{Z}/p^r\mathbb{Z}) \cong H^1_{\mathrm{et}}(k, \mathbb{Z}/p^r\mathbb{Z})$. $\square$

In the following, let k be an algebraically closed field of characteristic $p \geq 0$ and let X be a separated scheme of finite type over k. Let $H^S_i(X, \mathbb{Z}/m\mathbb{Z})$ denote the mod-m Suslin homology, i.e., the i-th homology group of the complex

$$\mathrm{Cor}(\Delta^\bullet, X) \otimes \mathbb{Z}/m\mathbb{Z}.$$

Let A be an abelian group with $mA = 0$. We are going to construct a pairing

$$H^S_1(X, \mathbb{Z}/m\mathbb{Z}) \times H^1_t(X, A) \longrightarrow A$$

as follows: let $\mathcal{T} \to X$ be a tame A-torsor representing a class in $H^1_t(X, A)$ and let $\alpha \in \mathrm{Cor}(\Delta^1, X)$ be a finite correspondence representing a 1-cocycle in the mod-m Suslin complex. Then

$$\alpha^*(\mathcal{T})$$

is a torsor over Δ^1. Since α is a cocycle modulo m, $(0^* - 1^*)(\alpha)$ is of the form $m \cdot z$ for some $z \in \mathrm{Cor}(\Delta^0, X) = \mathbb{Z}^{(X(k))}$. We therefore obtain a canonical identification

$$\Phi_{taut} : 0^*(\alpha^*(\mathcal{T})) \xrightarrow{\sim} 1^*(\alpha^*(\mathcal{T}))$$

of A-torsors over $\Delta^0 = \mathit{Spec}(k)$. Furthermore, by Corollary 2.11, the tame torsor $\alpha^*(\mathcal{T})$ on Δ^1 is trivial, hence a disjoint union of copies of Δ^1. By parallel transport, we obtain another identification

$$\Phi_{par} : 0^*(\alpha^*(\mathcal{T})) \xrightarrow{\sim} 1^*(\alpha^*(\mathcal{T})).$$

Hence there is a unique $\gamma(\alpha, \mathcal{T}) \in A$ such that

$$\Phi_{par} = (\text{translation by } \gamma(\alpha, \mathcal{T})) \circ \Phi_{taut}.$$

Proposition 2.12. *The element $\gamma(\alpha, \mathcal{T}) \in \mathbb{Z}/m\mathbb{Z}$ only depends on the class of $\mathcal{T}$ in $H^1_t(X, A)$ and on the class of α in $H^S_1(X, \mathbb{Z}/m\mathbb{Z})$. We obtain a bilinear pairing*

$$\langle \cdot, \cdot \rangle : H^S_1(X, \mathbb{Z}/m\mathbb{Z}) \times H^1_t(X, A) \longrightarrow A.$$

Proof. Replacing $\mathcal{T}$ by another torsor isomorphic to $\mathcal{T}$ does not change anything. The nontrivial statement is that $\langle \alpha, \mathcal{T} \rangle$ only depends on the class of α in $H^S_1(X, \mathbb{Z}/m\mathbb{Z})$. For $\beta \in \mathrm{Cor}(\Delta^1, X)$, we have

$$\langle \alpha + m\beta, \mathcal{T} \rangle = \langle \alpha, \mathcal{T} \rangle + m\langle \beta, \mathcal{T} \rangle = \langle \alpha, \mathcal{T} \rangle.$$

It therefore remains to show that

$$\langle \partial^*(\Phi), \mathcal{T} \rangle = 0,$$

for all $\Phi \in \mathrm{Cor}(\Delta^2, X)$, where $\partial_i : \Delta^1 \to \Delta^2$, $i = 0, 1, 2$, are the face maps and $\partial^*(\Phi) = \Phi \circ \partial_0 - \Phi \circ \partial_1 + \Phi \circ \partial_2$. Considering $\partial = \partial_0 - \partial_1 + \partial_2$ as a finite correspondence from Δ^1 to Δ^2, it represents a cocycle in the singular complex $\mathrm{Cor}(\Delta^\bullet, \Delta^2)$. Proposition 2.6 implies that

$$\langle \partial^*(\Phi), \mathcal{T} \rangle = \langle \Phi \circ \partial, \mathcal{T} \rangle = \langle \partial, \Phi^*(\mathcal{T}) \rangle.$$

By Corollary 2.11, the tame torsor $\Phi^*(\mathcal{T})$ is trivial on Δ^2. Hence $\langle \partial, \Phi^*(\mathcal{T}) \rangle = 0$. $\square$

In the following, we use the notation $\pi_1^{t,ab}(X) := H_t^1(X, \mathbb{Q}/\mathbb{Z})^*$. If X is connected, then $\pi_1^{t,ab}(X)$ is the abelianized (curve-)tame fundamental group of X, see [KS], §4.

Definition 2.13. *For $m \geq 1$ we define*

$$rec_X : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \pi_1^{t,ab}(X)/m$$

as the homomorphism induced by the pairing of Proposition 2.12 for $A = \mathbb{Z}/m\mathbb{Z}$ combined with the isomorphism $H_t^1(X, \mathbb{Z}/m\mathbb{Z})^* \cong \pi_1^{t,ab}(X)/m$.

The statement of the next lemma immediately follows from the definition of rec.

Lemma 2.14. *Let $f : X' \to X$ be a morphism of separated schemes of finite type over k. Then the induced diagram*

$$\begin{array}{ccc} H_1^S(X', \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{rec_{X'}} & \pi_1^{t,ab}(X')/m \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_1^S(X, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{rec_X} & \pi_1^{t,ab}(X)/m \end{array}$$

commutes.

3 Rigid Čech complexes

We consider étale sheaves F on the category Sch/k of separated schemes of finite type over a field k. By a result of M. Artin, Čech cohomology $\check{H}^\bullet(X, F)$ and sheaf cohomology $H^\bullet_{\mathrm{et}}(X, F)$ coincide in degree ≤ 1 and in arbitrary degree if X is quasi-projective (cf. [Mi], III Thm. 2.17). Comparing the Čech complex for a covering $\mathcal{U}$ and that for a finer covering $\mathcal{V}$, the refinement homomorphism

$$\check{C}^\bullet(\mathcal{U}, F) \longrightarrow \check{C}^\bullet(\mathcal{V}, F)$$

is canonical only up to chain homotopy and hence only the induced map $\check{H}^\bullet(\mathcal{U}, F) \to \check{H}^\bullet(\mathcal{V}, F)$ is well-defined. We can remedy this problem in the spirit of Friedlander [Fr], chap.4, by using rigid coverings:

We fix an algebraic closure $\bar{k}/k$. A *rigid étale covering* $\mathcal{U}$ of X is a family of pointed separated étale morphisms

$$(U_x, u_x) \longrightarrow (X, x), \quad x \in X(\bar{k}),$$

with U_x connected and $u_x \in U_x(\bar{k})$ mapping to x. For an étale sheaf F the rigid Čech complex is defined by

$$\check{C}^\bullet(\mathcal{U}, F) : \quad \check{C}^n(\mathcal{U}, F) = \prod_{(x_0,\ldots,x_n)\in X(\bar{k})^{n+1}} \Gamma(U_{x_0} \times_X \cdots \times_X U_{x_n}, F)$$

with the usual differentials. It is clear what it means for a rigid covering $\mathcal{V}$ to be a refinement of $\mathcal{U}$. Because the marked points map to each other, there is exactly one refinement morphism, hence we obtain a canonical refinement morphism on the level of complexes

$$\check{C}^\bullet(\mathcal{U}, F) \to \check{C}^\bullet(\mathcal{V}, F).$$

The set of rigid coverings is cofiltered (form the fibre product for each $x \in X(\bar{k})$ and restrict to the connected components of the marked points). Therefore we can define the rigid Čech complex of X with values in F as the filtered direct limit

$$\check{C}^\bullet(X, F) := \varinjlim_{\mathcal{U}} \check{C}^\bullet(\mathcal{U}, F),$$

where $\mathcal{U}$ runs through all rigid coverings of X. Forgetting the marking, we can view a rigid covering as a usual covering. Every covering can be refined by a covering which arises by forgetting the marking of a rigid covering. Hence the cohomology of the rigid Čech complex coincides with the usual Čech cohomology of X with values in F.

For a morphism $f : Y \to X$ and a rigid Čech covering $\mathcal{U}/X$, we obtain a rigid Čech covering $f^*\mathcal{U}/Y$ by taking base extension to Y and restricting to the connected components of the marked points, and in the limit we obtain a homomorphism

$$f^* : \check{C}^\bullet(X, F) \longrightarrow \check{C}^\bullet(Y, F).$$

Lemma 3.1. *If $\pi : Y \to X$ is quasi-finite, then the rigid coverings of the form $\pi^*\mathcal{U}$ are cofinal among the rigid coverings of Y.*

Proof. This is an immediate consequence of the fact that a quasi-finite and separated scheme Y over the spectrum X of a henselian ring is of the form $Y = Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_r$ with $Y_0 \to X$ not surjective and $Y_i \to X$ finite surjective with Y_i the spectrum of a henselian ring, $i = 1, \ldots, r$, cf. [Mi], I, Thm. 4.2. □

Lemma 3.2. *If F is qfh-sheaf on Sch/k, then for any $n \geq 0$ the presheaf $\underline{\check{C}}^n(-, F)$ given by*

$$X \longmapsto \check{C}^n(X, F)$$

is a qfh-sheaf. The obvious sequence

$$0 \to F \to \underline{\check{C}}^0(-, F) \to \underline{\check{C}}^1(-, F) \to \underline{\check{C}}^2(-, F) \to \cdots$$

is exact as a sequence of étale (and hence also of qfh) sheaves.

Proof. We show that each $\underline{\check{C}}^n(-, F)$ is a qfh-sheaf. For this, let $\pi : Y \to X$ be a qfh-covering, i.e., a quasi-finite universal topological epimorphism. We denote the projection by $\Pi : Y \times_X Y \to X$. By Lemma 3.1, we have to show that the sequence

$$\varinjlim_{\mathcal{U}} \check{C}^n(\mathcal{U}, F) \to \varinjlim_{\mathcal{U}} \check{C}^n(\pi^*\mathcal{U}, F) \rightrightarrows \varinjlim_{\mathcal{U}} \check{C}^n(\Pi^*\mathcal{U}, F)$$

is an equalizer, where $\mathcal{U}$ runs through the rigid coverings of X. Since filtered colimits commute with finite limits, it suffices to show the exactness for a single, sufficiently small $\mathcal{U}$. This, however, follows from the assumption that F is a qfh-sheaf.

Finally, the exactness of $0 \to F \to \underline{\check{C}}^0(-, F) \to \underline{\check{C}}^1(-, F) \to \cdots$ as a sequence of étale sheaves follows by considering stalks. □

Being qfh-sheaves, the sheaves F and $\underline{\check{C}}^n(-, F)$ admit transfer maps, see [SV1], §5. For later use, we make the relation between the transfers of F and of $\underline{\check{C}}^n(-, F)$ explicit: Let Z be integral, X regular and $\pi : Z \to X$ finite and surjective. Let F be a qfh-sheaf on Sch/k. For $x \in X(\bar{k})$ we have

$$X_x^{sh} \times_X Z = \coprod_{z \in \pi^{-1}(x)} Z_z^{sh},$$

where $\pi^{-1}(x)$ denotes the set of morphisms $z : \mathit{Spec}(\bar{k}) \to Z$ with $\pi \circ z = x$. For sufficiently small étale $(U_x, u_x) \to (X, x)$, the set of connected components of $U_x \times_X Z$ is in 1-1-correspondence with the set $\pi^{-1}(x)$, and to each family of étale morphisms

$$(V_z, v_z) \longrightarrow (Z, z), \quad z \in \pi^{-1}(x),$$

there is (after possibly making U_x smaller) a unique morphism

$$U_x \times_X Z \longrightarrow \coprod_{z \in \pi^{-1}(x)} V_z,$$

over Z, which sends the connected component associated with z of $U_x \times_X Z$ to V_z, and the point (u_x, z) to v_z.

In this way we obtain, for finitely many points $(x_0, \ldots, x_n)$, $n \geq 0$, and for every family

$$(V_{z_i, v_{z_i}}) \longrightarrow (Z, z_i^j), \quad z_i^j \in \pi^{-1}(x_i),$$

and sufficiently small chosen

$$(U_{x_i}, u_{x_i}) \longrightarrow (X, x_i), \quad i = 0, \ldots, n,$$

a homomorphism

$$\prod_{\substack{i=0,\ldots,n \\ z_i^j \in \pi^{-1}(x_i)}} \Gamma\big(V_{z_0^j} \times_Z \cdots \times_Z V_{z_n^j}, F\big) \longrightarrow \Gamma\big(U_{x_0} \times_X \cdots \times_X U_{x_n} \times_X Z, F\big).$$

Since F is a qfh-sheaf, we can compose this with the transfer map associated with the finite morphism

$$U_{x_0} \times_X \cdots \times_X U_{x_n} \times_X Z \to U_{x_0} \times_X \cdots \times_X U_{x_n}.$$

Forming for fixed n the product over all $(x_0, \ldots, x_n) \in X(\bar{k})^{n+1}$ and passing to the limit over all rigid coverings, we obtain the transfer homomorphism

$$\pi_* : \check{C}^\bullet(Z, F) \longrightarrow \check{C}^\bullet(X, F).$$

Passing to cohomology, we obtain the usual transfer on étale cohomology in degree 0 and 1, and in any degree if the schemes are quasi-projective.

Next we give the pairing

$$\langle \cdot, \cdot \rangle : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \times H_t^1(X, A) \longrightarrow A.$$

constructed in Proposition 2.12 for k algebraically closed and an abelian group A with $mA = 0$ the following interpretation in terms of the rigid Čech complex:

Let $a \in H_1^S(X, \mathbb{Z}/m\mathbb{Z})$ and $b \in H_t^1(X, A)$ be given, and let $\alpha \in \mathrm{Cor}_k(\Delta^1, X)$ and $\beta \in \ker(\check{C}^1(X, A) \xrightarrow{d} \check{C}^2(X, A))$ be representing elements. Note that $(0^* - 1^*)(\alpha) \in m\,\mathrm{Cor}(\Delta^0, X)$ by assumption. Consider the diagram

$$\begin{array}{ccccccc}
 & & \check{C}^0(X, A) & \xrightarrow{d} & \check{C}^1(X, A) & \xrightarrow{d} & \check{C}^2(X, A) \\
 & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\
 & & \check{C}^0(\Delta^1, A) & \xrightarrow{d} & \check{C}^1(\Delta^1, A) & \xrightarrow{d} & \check{C}^2(\Delta^1, A) \\
 & & \downarrow{\scriptstyle 0^* - 1^*} & & \downarrow{\scriptstyle 0^* - 1^*} & & \downarrow{\scriptstyle 0^* - 1^*} \\
A & \hookrightarrow & \check{C}^0(\Delta^0, A) & \xrightarrow{d} & \check{C}^1(\Delta^0, A) & \xrightarrow{d} & \check{C}^2(\Delta^0, A)
\end{array}$$

Since β represents a tame torsor $\mathcal{T}$ on X, $\alpha^*(\beta)$ represents the torsor $\alpha^*(\mathcal{T})$, which is tame by Lemma 2.9. By Corollary 2.11, there exists $\gamma \in \check{C}^0(\Delta^1, A)$ with $d\gamma = \alpha^*(\beta)$. Since

$$d(0^* - 1^*)(\gamma) = (0^* - 1^*)\alpha^*(\beta) = 0,$$

we conclude that $(0^* - 1^*)(\gamma)$ lies in

$$A = H^0(\Delta^0, A) = \ker(\check{C}^0(\Delta^0, A) \xrightarrow{d} \check{C}^1(\Delta^0, A)).$$

It is easy to verify that the assignment

$$\langle \cdot, \cdot \rangle : (a, b) \longmapsto (0^* - 1^*)(\gamma) \in A$$

does not depend on the choices made. By the explicit geometric relation between Čech 1-cocycles and torsors, and since our construction of finite push-forwards of torsors is compatible with the construction of transfers for qfh-sheaves given in [SV1], §5, we see that the pairing constructed above coincides with the one constructed in Proposition 2.12.

Finally, let

$$A \hookrightarrow I^0 \to I^1 \to I^2 \tag{5}$$

be a (partial) injective resolution of the constant sheaf A in the category of $\mathbb{Z}/m\mathbb{Z}$-module sheaves on $(\mathrm{Sch}/k)_{\mathrm{qfh}}$. Let $\phi : (\mathrm{Sch}/k)_{\mathrm{qfh}} \to (\mathrm{Sch}/k)_{\mathrm{et}}$ denote the natural map of sites. Since ϕ^* is exact, ϕ_* sends injective sheaves to injective sheaves. By [SV1], Thm. 10.2, we have $R^0\phi_*(A) = A$ and $R^i\phi_*(A) = 0$ for $i \geq 1$. Hence (5) is also a partial resolution of A by injective, étale sheaves of $\mathbb{Z}/m\mathbb{Z}$-modules. We choose a quasi-isomorphism

$$[0 \to \underline{\check{C}}^0(-, A) \to \underline{\check{C}}^1(-, A) \to \underline{\check{C}}^2(-, A)] \longrightarrow [0 \to I^0 \to I^1 \to I^2]$$

of complexes of qfh-sheaves. Since Čech- and étale cohomology agree in dimension ≤ 1, the induced map on global sections is a quasi-isomorphism, too. Hence the pairing of Proposition 2.12 can also be obtained by the same procedure as above but using the diagram

$$\begin{tikzcd}
 & I^0(X) \arrow[r, "d"] \arrow[d, "\alpha^*"] & I^1(X) \arrow[r, "d"] \arrow[d, "\alpha^*"] & I^2(X) \arrow[d, "\alpha^*"] \\
 & I^0(\Delta^1) \arrow[r, "d"] \arrow[d, "0^* - 1^*"] & I^1(\Delta^1) \arrow[r, "d"] \arrow[d, "0^* - 1^*"] & I^2(\Delta^1) \arrow[d, "0^* - 1^*"] \\
A \arrow[r, hook] & I^0(\Delta^0) \arrow[r, "d"] & I^1(\Delta^0) \arrow[r, "d"] & I^2(\Delta^0).
\end{tikzcd}$$

By [SV1], Theorem 10.7, the same argument applies with a partial injective resolution of the constant sheaf A in the category of $\mathbb{Z}/m\mathbb{Z}$-module sheaves on $(\mathrm{Sch}/k)_h$.

4 The case of smooth curves

In this section we prove Theorem 1.1 in the case that $X = C$ is a smooth curve.

Let k be an algebraically closed field of characteristic $p \geq 0$, and let C be a smooth, but not necessarily projective, curve over k. Let the semi-abelian variety $\mathcal{A}$ be the generalized Jacobian of C with respect to the modulus given by the sum of the points on the boundary of the regular compactification $\bar{C}$ of C (cf. [Se], Ch. 5). The group $\mathcal{A}(k)$ is the subgroup of degree zero elements of the relative Picard group $\mathrm{Pic}(\bar{C}, \bar{C} \smallsetminus C)$. By [SV1], Thm. 3.1 (see [Li], for the case $C = \bar{C}$), there is an isomorphism

$$H_0^S(C,\mathbb{Z})^0 := \ker(H_0^S(C,\mathbb{Z}) \stackrel{\deg}{\longrightarrow} \mathbb{Z}) \cong \mathcal{A}(k),$$

in particular, $\mathcal{A}(k)$ is a quotient of the group of zero cycles of degree zero on C. From the coefficient sequence together with the divisibility of $H_1^S(C,\mathbb{Z})$ (which is isomorphic to $k^\times$ if C is proper and zero otherwise), we obtain an isomorphism

$$H_1^S(C,\mathbb{Z}/m\mathbb{Z}) \xrightarrow[\sim]{\delta} {}_mH_0^S(C,\mathbb{Z}) \cong {}_m\mathcal{A}(k). \tag{6}$$

After fixing a closed point P_0 of C, the morphism $C \to \mathcal{A}$, $P \mapsto P - P_0$, is universal for morphisms of C to semi-abelian varieties, i.e., $\mathcal{A}$ is the generalized Albanese variety of C ([Se], V, Th. 2).

Consider the m-multiplication map $\mathcal{A} \stackrel{m}{\to} \mathcal{A}$. Its maximal étale subcovering $\widetilde{\mathcal{A}} \to \mathcal{A}$ is the quotient of $\mathcal{A}$ by the connected component of the finite group scheme ${}_m\mathcal{A}$ (if $(p,m) = 1$, the connected component is trivial). The projection $\mathcal{A} \to \widetilde{\mathcal{A}}$ induces an isomorphism $\mathcal{A}(k) \stackrel{\sim}{\to} \widetilde{\mathcal{A}}(k)$ on rational points, and we identify $\mathcal{A}(k)$ and $\widetilde{\mathcal{A}}(k)$ via this isomorphism. With respect to this identification, the projection $\widetilde{\mathcal{A}}(k) \to \mathcal{A}(k)$ is the m-multiplication map on $\mathcal{A}(k)$.

By [Se], Ch. IV, $\widetilde{C} := C \times_{\mathcal{A}} \widetilde{\mathcal{A}}$ is the maximal abelian tame étale covering of C with Galois group annihilated by m. Because $\mathrm{Aut}_{\mathcal{A}}(\widetilde{\mathcal{A}}) \cong {}_m\mathcal{A}(k)$, we obtain an isomorphism

$$\mathrm{Hom}({}_m\mathcal{A}(k), A) \xrightarrow[\sim]{\tau} H_t^1(C, A) \tag{7}$$

for any finite abelian group A with $mA = 0$.

Theorem 4.1. *For any finite abelian group A with $mA = 0$, the diagram*

$$\begin{array}{ccccc}
H_1^S(C,\mathbb{Z}/m\mathbb{Z}) & \times & H_t^1(C,A) & \xrightarrow{\langle\ ,\ \rangle} & A \\
\delta \big\downarrow \wr & & \wr \big\uparrow \tau & & \Big\| \\
{}_m\mathcal{A}(k) & \times & \mathrm{Hom}({}_m\mathcal{A}(k), A) & \xrightarrow{eval} & A
\end{array}$$

where $\langle\ ,\ \rangle$ is the pairing from Proposition 2.12 and $eval$ is the evaluation map, commutes. In particular, the upper pairing is perfect and the induced homomorphism $H_1^S(C,\mathbb{Z}/m\mathbb{Z}) \to \pi_1^{t,ab}(C)/m$ is an isomorphism.

Proof. We have to show that $\phi(\delta(\zeta)) = \langle \zeta, \tau(\phi)\rangle$ for any $\zeta \in H_1^S(C,\mathbb{Z}/m\mathbb{Z})$ and any $\phi \in \mathrm{Hom}({}_m\mathcal{A}(k), A)$. By functoriality, it suffices to consider the universal case $A = {}_m\mathcal{A}(k)$, $\phi = \mathrm{id}$. In this case $\tau(\mathrm{id})$ is the torsor $\widetilde{\pi} : \widetilde{C} \to C$.

Let C' be the regular compactification of C. By [SV1], Thm. 3.1, $\delta(\zeta) \in {}_mH_0^S(C,\mathbb{Z}) = {}_m\mathcal{A}(k)$ is the class $[z]$ of some $z \in Z_0(C)$ (the group of zero-cycles on C) such that

$$mz = \gamma^*(0) - \gamma^*(1)$$

for some finite morphism $\gamma : C' \to \mathbb{P}^1$ with $C' \smallsetminus C \subset \gamma^{-1}(\infty)$. The diagram

$$\begin{array}{ccccc}
C' \smallsetminus \gamma^{-1}(\infty) & \hookrightarrow & C & \hookrightarrow & C' \\
\big\downarrow {\scriptstyle \gamma|_{C' \smallsetminus \gamma^{-1}(\infty)}} & & & & \big\downarrow {\scriptstyle \gamma} \\
\Delta^1 & & \hookrightarrow & & \mathbb{P}^1
\end{array}$$

shows that γ induces a finite correspondence, say g, from Δ^1 to C. The class of g in $H_1^S(C, \mathbb{Z}/m\mathbb{Z})$ is a pre-image of $\delta(\zeta)$ under $H_1^S(C, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} {}_mH_0^S(C, \mathbb{Z})$, i.e., ζ is represented by g. It therefore suffices to show that

$$[z] = \langle g, \widetilde{C} \rangle.$$

Let d be the degree of γ and $\gamma^*(0) = \sum_{i=1}^d P_i$, $\gamma^*(1) = \sum_{i=1}^d Q_i$. Each point in $\gamma^*(0)$ and $\gamma^*(1)$ occurs with multiplicity divisible by m, in particular $d = mr$ for some integer r. After reindexing, we may assume that $P_i = P_j$ and $Q_i = Q_j$ for $i \equiv j \bmod r$, hence

$$z = \sum_{i=1}^r P_i - \sum_{i=1}^r Q_i.$$

On the level of closed points, $\widetilde{C} = C \times_{\mathcal{A}} \widetilde{\mathcal{A}}$ can be identified with the set of $a \in \widetilde{\mathcal{A}}(k) = \mathcal{A}(k)$ such that $ma = P - P_0$ for some point $P \in C$ (a projects to P in C, i.e., $\widetilde{\pi}(a) = P$). The ${}_m\mathcal{A}(k)$-principal homogeneous space $0^*g^*\widetilde{C}$ can be identified with the quotient of the set

$$\prod_{i=1}^d \widetilde{\pi}^{-1}(P_i)$$

by the action of ${}_m\mathcal{A}(k)^{d-1}$ given by

$$(\beta_1, \dots, \beta_{d-1})(a_1, \dots, a_d) = (a_1 + \beta_1, a_2 - \beta_1 + \beta_2, \dots, a_d - \beta_{d-1}).$$

We fix points $a_1, \dots, a_d \in \widetilde{C}$ over $P_1, \dots, P_d$ subject to the condition $a_i = a_j$ for $P_i = P_j$. Then $0^*g^*\widetilde{C}$ is identified with the quotient of the set

$$(a_1 + {}_m\mathcal{A}(k)) \times \cdots \times (a_d + {}_m\mathcal{A}(k))$$

by the action of ${}_m\mathcal{A}(k)^{d-1}$. Since each a_i occurs with multiplicity divisible by m, the trivialization $0^*g^*(\widetilde{C}) \xrightarrow{\sim} {}_m\mathcal{A}(k)$ given by

$$\overline{(a_1 + \alpha_1, \dots, a_d + \alpha_d)} \longmapsto \alpha_1 + \cdots + \alpha_d \in {}_m\mathcal{A}(k)$$

does not depend on the choice of the a_i. We do the same with $1^*g^*(\widetilde{C})$ by choosing $b_i \in \widetilde{C}$ over Q_i. Then we see that the tautological identification $\Phi_{taut} : 0^*g^*(\widetilde{C}) \xrightarrow{\sim} 1^*g^*(\widetilde{C})$ is given by

$$\overline{(a_1 + \alpha_1, \dots, a_d + \alpha_d)} \longmapsto \overline{(b_1 + \alpha_1, \dots, b_d + \alpha_d)}.$$

Now consider the morphism

$$\Sigma : \mathrm{Sym}^d(C) \longrightarrow \mathcal{A},\ (x_1, \dots, x_d) \longmapsto [\textstyle\sum(x_i - P_0)].$$

The commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^d(\widetilde{C}) & \xrightarrow{\Sigma} & \widetilde{\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathrm{Sym}^d(C) & \xrightarrow{\Sigma} & \mathcal{A} \end{array}$$

induces a map (hence an isomorphism) of ${}_m\mathcal{A}(k)$-torsors $\mathrm{Sym}^d(\widetilde{C}) \xrightarrow{\sim} \widetilde{\mathcal{A}} \times_{\mathcal{A}} \mathrm{Sym}^d(C)$. Consider the morphism $S_g : \Delta_k^1 \to \mathrm{Sym}^d(C)$ associated with the finite correspondence g. Since the generalized Jacobian of $\Delta_k^1 \cong \mathbb{A}_k^1$ is $Spec(k)$, the composite

$$\Delta_k^1 \xrightarrow{S_g} \mathrm{Sym}^d(C) \xrightarrow{\Sigma} \mathcal{A}$$

is constant with value $a := [\sum_{i=1}^{d}(P_i - P_0)] = [\sum_{i=1}^{d}(Q_i - P_0)] \in \mathcal{A}(k)$. By Lemma 2.3, we obtain an isomorphism

$$g^*(\widetilde{C}) = S_g^*(\mathrm{Sym}^d(\widetilde{C})) = \Sigma^* S_g^* \widetilde{\mathcal{A}} = \Delta_k^1 \times \widetilde{\pi}^{-1}(a)$$

(giving a trivialization after choosing a point in $\widetilde{\pi}^{-1}(a)$). On the fibre over 0 it is given by

$$\overline{(a_1 + \alpha_1, \ldots, a_d + \alpha_d)} \longmapsto \sum_{i=1}^{d}(a_i + \alpha_i) \in \widetilde{\pi}^{-1}(a) \subset \widetilde{\mathcal{A}}$$

and similarly on the fibre over 1. We conclude that $\Phi_{par} \circ \Phi_{taut}^{-1}$ is translation by

$$\sum_{i=1}^{d}(a_i - b_i) = \sum_{i=1}^{r} m(a_i - b_i) = \sum_{i=1}^{r}[P_i - Q_i] = [z].$$

This concludes the proof. □

5 The blow-up sequences

All schemes in this section are separated schemes of finite type over the spectrum of a perfect field k. A curve on a scheme X is a closed one-dimensional subscheme. The normalization of a curve C is denoted by $\widetilde{C}$.

Now let

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\ i\ } & X \end{array}$$

be an abstract blow-up square, i.e., a cartesian diagram of schemes such that $\pi : X' \to X$ is proper, $i : Z \to X$ is a closed embedding and π induces an isomorphism $(X' \smallsetminus Z')_{\mathrm{red}} \xrightarrow{\sim} (X \smallsetminus Z)_{\mathrm{red}}$.

Proposition 5.1. *Given an abstract blow-up square and an abelian group A, assume that π is finite or A is torsion. Then there is a natural exact sequence*

$$0 \to H^0_{\mathrm{et}}(X, A) \to H^0_{\mathrm{et}}(X', A) \oplus H^0_{\mathrm{et}}(Z, A) \to H^0_{\mathrm{et}}(Z', A)$$

$$\xrightarrow{\delta} H^1_t(X, A) \to H^1_t(X', A) \oplus H^1_t(Z, A) \to H^1_t(Z', A).$$

Proof. We call an abstract blow-up square trivial, if i is surjective (i.e., s_{red} is an isomorphism) or if $\pi_{\mathrm{red}} : X'_{\mathrm{red}} \to X_{\mathrm{red}}$ has a section. Every abstract blow-up square with X a connected regular curve is trivial.

Now let an arbitrary abstract blow-up square be given. If A is torsion, the proper base change theorem implies (cf. [Ge2], 3.2 and 3.6) that we have a long exact sequence

$$\cdots \to H^i_{\mathrm{et}}(X, A) \to H^i_{\mathrm{et}}(X', A) \oplus H^i_{\mathrm{et}}(Z, A) \to H^i_{\mathrm{et}}(Z', A) \to H^{i+1}_{\mathrm{et}}(X, A) \to \cdots$$

If π is finite, the same is true for arbitrary A since π_* is exact. If the blow-up square is trivial, this long exact sequence splits into short exact sequences $0 \to H^i_{\mathrm{et}}(X, A) \to H^i_{\mathrm{et}}(X', A) \oplus H^i_{\mathrm{et}}(Z, A) \to H^i_{\mathrm{et}}(Z', A) \to 0$ for all i.

Next we show the exact sequence of the proposition. We omit the coefficients A and put $H^0_t(X) = H^0_{\mathrm{et}}(X)$. We first show, that the image of the boundary map $\delta : H^0_{\mathrm{et}}(Z') \to H^1_{\mathrm{et}}(X)$ has image in $H^1_t(X)$, thus showing the existence of $H^0_t(Z') \to H^1_t(X)$ and, at the same time, the exactness of the sequence at $H^1_t(X)$. Let $\widetilde{C} \to X$ be the normalization of a curve in X. The base change

$$\begin{tikzcd} Z'_{\widetilde{C}} \arrow[r] \arrow[d] & X'_{\widetilde{C}} \arrow[d, "\widetilde{\pi}"] \\ Z_{\widetilde{C}} \arrow[r, "\widetilde{i}"] & \widetilde{C} \end{tikzcd}$$

of our abstract blow-up square to $\widetilde{C}$ is a trivial abstract blow-up square. Therefore, for any $\alpha \in H^0_{\mathrm{et}}(Z')$, the pull-back of α to $H^0_{\mathrm{et}}(Z'_{\widetilde{C}})$ lies in the image of $H^0_{\mathrm{et}}(X'_{\widetilde{C}}) \oplus H^0_{\mathrm{et}}(Z_{\widetilde{C}}) \to H^0_{\mathrm{et}}(Z'_{\widetilde{C}})$ and has therefore trivial image under $\delta : H^0_{\mathrm{et}}(Z'_{\widetilde{C}}) \to H^1_{\mathrm{et}}(\widetilde{C})$. Therefore, $\delta(\alpha) \in H^1_{\mathrm{et}}(X)$ has trivial image in $H^1_{\mathrm{et}}(\widetilde{C})$ for every curve $C \subset X$, in particular, it lies in $H^1_t(X)$.

It remains to show exactness at $H^1_t(X') \oplus H^1_t(Z)$. Let α be in this group with trivial image in $H^1_t(Z')$. Then there exists $\beta \in H^1_{\mathrm{et}}(X)$ mapping to α and it remains to show that β lies in the subgroup $H^1_t(X)$. But this is clear, because for every curve $C \subset X$ we have $H^1_t(\widetilde{C}) = \ker(H^1_t(X'_{\widetilde{C}}) \oplus H^1_t(Z_{\widetilde{C}}) \to H^1_t(Z'_{\widetilde{C}}))$. □

Proposition 5.2. *Given an abstract blow-up square*

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

and an abelian group A, there is a natural exact sequence of Suslin homology groups

$$\begin{aligned} &H_1^S(Z', A) \to H_1^S(X', A) \oplus H_1^S(Z, A) \to H_1^S(X, A) \\ &\quad \xrightarrow{\delta} H_0^S(Z', A) \to H_0^S(X', A) \oplus H_0^S(Z, A) \to H_0^S(X, A) \to 0. \end{aligned}$$

Proof. Consider the exact sequences

$$C_\bullet(Z', A) \hookrightarrow C_\bullet(X', A) \oplus C_\bullet(Z, A) \to C_\bullet(X, A) \twoheadrightarrow K_\bullet^A$$

and

$$C_\bullet(Z') \hookrightarrow C_\bullet(X') \oplus C_\bullet(Z) \to C_\bullet(X) \twoheadrightarrow K_\bullet,$$

where $K_\bullet^A$ and $K_\bullet$ are defined to make the sequences exact. Since the complexes $C_\bullet(-)$ consist of free abelian groups, in order the show the statement of the proposition, it suffices to show that $H_i(K_\bullet) = 0$ for $i \le 2$. Let Sm/k be the full subcategory of Sch/k consisting of smooth schemes. For $Y \in \mathrm{Sch}/k$ we consider the presheaf $c(Y)$ on Sm/k given by $c(Y)(U) = \mathrm{Cor}(U, Y)$. Then, by [SV2], Thm. 5.2, 4.7 and its proof, the sequence

$$0 \to c(Z') \to c(X') \oplus c(Z) \stackrel{(\pi_*, i_*)}{\to} c(X)$$

is exact and $F := \mathit{coker}(\pi_*, i_*)$ has the property that, for any $U \in \mathrm{Sm}/k$ of dimension ≤ 2 and any $x \in F(U)$, there exists a proper birational morphism $\phi : V \to U$ with V smooth such that $\phi^*(x) = 0$. Let $F_\bullet$ be the complex of presheaves given by $F_n(U) = F(U \times \Delta^n)$

with the obvious differentials and let $(F_\bullet)_{\mathrm{Nis}}$ be the associated complex of sheaves on $(\mathrm{Sm}/k)_{\mathrm{Nis}}$. Then by [SS], Thm. 2.4, the Nisnevich sheaves

$$\mathcal{H}_i((F_\bullet)_{\mathrm{Nis}})$$

vanish for $i \le 2$. Evaluating at $U = \mathit{Spec}(k)$ yields the result. □

Now assume that k is algebraically closed. Let

$$\mathit{rec}_{1,X} : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \to H_t^1(X, \mathbb{Z}/m\mathbb{Z})^*$$

be the reciprocity map constructed in Section 2 and let

$$\mathit{rec}_{0,X} : H_0^S(X, \mathbb{Z}/m\mathbb{Z}) \to H_{\mathrm{et}}^0(X, \mathbb{Z}/m\mathbb{Z})^*$$

be the homomorphism induced by the pairing

$$\langle \cdot, \cdot \rangle : \ H_0^S(X, m\mathbb{Z}) \times H_{\mathrm{et}}^0(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

defined as follows: Given $a \in H_0^S(X, \mathbb{Z}/m\mathbb{Z})$ and $b \in H_{\mathrm{et}}^0(X, \mathbb{Z}/m\mathbb{Z})$, we represent a by a correspondence $\alpha \in \mathrm{Cor}(\Delta^0, X)$ and put $\langle a, b\rangle = \alpha^*(b) \in H_{\mathrm{et}}^0(\Delta^0, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$. This is well-defined since the homomorphisms $0^*, 1^* : H_{\mathrm{et}}^0(\Delta^1, \mathbb{Z}/m\mathbb{Z}) \to H_{\mathrm{et}}^0(\Delta^0, \mathbb{Z}/m\mathbb{Z})$ agree.

Lemma 5.3. *For any m, $\mathit{rec}_{0,X}$ is an isomorphism.*

Proof. For connected X, we have the commutative diagram

$$\begin{array}{ccc} H_0^S(X, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{\mathit{rec}_{0,X}} & H_{\mathrm{et}}^0(X, \mathbb{Z}/m\mathbb{Z})^* \\ \Big\downarrow{\scriptstyle \deg} & & \Big\downarrow{\scriptstyle \wr} \\ H_0^S(k, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow[\sim]{\mathit{rec}_{0,k}} & H_{\mathrm{et}}^0(k, \mathbb{Z}/m\mathbb{Z})^*. \end{array}$$

Hence, for connected X, it suffices by functoriality to consider the mod m degree map. In particular, $\mathit{rec}_{0,X}$ is surjective for arbitrary X and is an isomorphism if $\dim X = 0$. If X is a smooth connected curve, then $H_0^S(X, \mathbb{Z}) = \mathrm{Pic}(\bar{X}, \bar{X} \smallsetminus X)$, where $\bar{X}$ is the smooth compactification of X (cf. [SV1], Thm. 3.1). The subgroup $\mathrm{Pic}^0(\bar{X}, \bar{X} \smallsetminus X)$ of degree zero elements is the group of k-rational points of the Albanese of X, and hence divisible. Therefore, $\mathit{rec}_{0,X}$ is an isomorphism for connected, and hence for all smooth curves. Considering the normalization morphism of an arbitrary scheme of dimension 1 and the exact sequences of Propositions 5.1 and 5.2, the five-lemma shows that $\mathit{rec}_{0,X}$ is a isomorphism for $\dim X \le 1$.

It remains to show that $\mathit{rec}_{0,X}$ is injective for arbitrary X. We may assume X to be connected. Let $a \in \ker(\mathit{rec}_{0,X})$ and let $\alpha \in Z_0(X)$ be a representing 0-cycle. Since $\mathit{supp}(\alpha)$ is finite, we can find a connected 1-dimensional closed subscheme $Z \subset X$ containing $\mathit{supp}(\alpha)$ (use, e.g., [Mu], II §6 Lemma). Since $\mathit{rec}_{0,Z}$ is injective and a is in the image of $H_0^S(Z, \mathbb{Z}/m\mathbb{Z}) \to H_0^S(X, \mathbb{Z}/m\mathbb{Z})$, we conclude that $a = 0$. □

Corollary 5.4. *Let k be an algebraically closed field and let $X \in \mathrm{Sch}/k$ be connected. Then the kernel of the degree map*

$$\deg : H_0^S(X, \mathbb{Z}) \longrightarrow H_0^S(k, \mathbb{Z}) \cong \mathbb{Z}$$

is divisible.

Proposition 5.5. *Let k be algebraically closed and let*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "\pi'"] & X' \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

be an abstract blow-up square. Then for any integer $m \geq 1$ the diagram

$$\begin{tikzcd} H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "rec_{1,X}"] & H_0^S(Z', \mathbb{Z}/m\mathbb{Z}) \arrow[d, "rec_{0,X}"] \\ H_t^1(X, \mathbb{Z}/m\mathbb{Z})^* \arrow[r, "\delta^*"] & H_{\mathrm{et}}^0(Z', \mathbb{Z}/m\mathbb{Z})^*, \end{tikzcd}$$

commutes. Here δ is the boundary map of Proposition 5.2 and δ^* is the dual of the boundary map of Proposition 5.1.

Proof. We have to show that the diagram

$$\begin{tikzcd} H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \arrow[d, "\delta"'] & \times & H_t^1(X, \mathbb{Z}/m\mathbb{Z}) \arrow[r, "\langle\ ,\ \rangle"] & \mathbb{Z}/m\mathbb{Z} \arrow[d, equal] \\ H_0^S(Z', \mathbb{Z}/m\mathbb{Z}) & \times & H_{\mathrm{et}}^0(Z', \mathbb{Z}/m\mathbb{Z}) \arrow[u, "\delta"] \arrow[r, "\langle\ ,\ \rangle"] & \mathbb{Z}/m\mathbb{Z} \end{tikzcd}$$

commutes. Given $a \in H_1^S(X, \mathbb{Z}/m\mathbb{Z})$ and $b \in H_{\mathrm{et}}^0(Z', \mathbb{Z}/m\mathbb{Z})$, we choose a representing correspondence $\alpha \in C_1(X, \mathbb{Z}/m\mathbb{Z}) = \mathrm{Cor}(\Delta^1, X) \otimes \mathbb{Z}/m\mathbb{Z}$ in such a way that it has a pre-image $\widehat{\alpha} \in C_1(X', \mathbb{Z}/m\mathbb{Z}) \oplus C_1(Z, \mathbb{Z}/m\mathbb{Z})$ (see the proof of Proposition 5.2). By definition, $\delta a \in H_0^S(Z', \mathbb{Z}/m\mathbb{Z})$ is represented by a correspondence $\gamma \in C_0(Z', \mathbb{Z}/m\mathbb{Z})$ such that the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "0-1"] \arrow[d, "\gamma"'] & \Delta^1 \arrow[d, "\widehat{\alpha}"] \\ Z' \arrow[r, "i'-\pi'"] & X' \amalg Z \end{tikzcd}$$

of correspondences commutes modulo m. Next choose an injective resolution $\mathbb{Z}/m\mathbb{Z} \to I^\bullet$ of $\mathbb{Z}/m\mathbb{Z}$ in the category of sheaves of $\mathbb{Z}/m\mathbb{Z}$-modules on $(\mathrm{Sch}/k)_h$ in order to compute the pairings (cf. the end of section 3). Consider the following diagram

$$
\begin{tikzcd}
 & & I^0(X') \oplus I^0(Z) \arrow[r, "i'^* - \pi'^*"] \arrow[d, "d"] \arrow[dddll, bend right, "\widehat{\alpha}^*"'] & I^0(Z') \arrow[d, "d"] \\
 & I^1(X) \arrow[r, hook, "(\pi^*, i^*)"] \arrow[d, "\alpha^*"] & I^1(X') \oplus I^1(Z) \arrow[r, "i'^* - \pi'^*"] \arrow[d, "\widehat{\alpha}^*"] & I^1(Z') \\
I^0(\Delta^1) \arrow[r, "d"] \arrow[d, "0^* - 1^*"] & I^1(\Delta^1) \arrow[r, equal] \arrow[d, "0^* - 1^*"] & I^1(\Delta^1) \arrow[r, "d"] \arrow[d, "0^* - 1^*"] & I^2(\Delta^1) \arrow[d, "0^* - 1^*"] \\
I^0(\Delta^0) \arrow[r, "d"] & I^1(\Delta^0) \arrow[r, equal] & I^1(\Delta^0) \arrow[r, "d"] & I^2(\Delta^0)
\end{tikzcd}
$$

By the argument of [MVW] Lemma 12.7, the sequence

$$0 \to F(X) \to F(X') \oplus F(Z) \to F(Z')$$

is exact for every h-sheaf F. Therefore the second line in the diagram is exact. The proper base change theorem implies (cf. [Ge2], 3.2 and 3.6) that

$$I^\bullet(X) \longrightarrow I^\bullet(X') \oplus I^\bullet(Z) \longrightarrow I^\bullet(Z') \xrightarrow{[1]}$$

is an exact triangle in $D(Ab)$. For the exact sequence of complexes

$$0 \to I^\bullet(X) \to I^\bullet(X') \oplus I^\bullet(Z) \to I^\bullet(Z') \to \mathit{coker}^\bullet \to 0,$$

this implies that the complex $\mathit{coker}^\bullet$ is exact. Therefore, $b \in \ker(I^0(Z') \to I^1(Z'))$ has a pre-image $\widehat{\beta} \in I^0(X') \oplus I^0(Z)$. Then

$$d\widehat{\beta} \in \ker(I^1(X') \oplus I^1(Z) \to I^1(Z')),$$

and there exists a unique $\varepsilon \in I^1(X)$ with $(\pi^*, i^*)(\varepsilon) = d\widehat{\beta}$ representing $\delta b \in H^1_t(X)$. We see that $\widehat{\alpha}^*(d\widehat{\beta}) = \alpha^*(\varepsilon)$. It follows that

$$d(\widehat{\alpha}^*(\widehat{\beta})) = \widehat{\alpha}^*(d\widehat{\beta}) = \alpha^*(\varepsilon) \in \ker(I^1(\Delta^1) \xrightarrow{0^* - 1^*} I^1(\Delta^0)).$$

By definition of $\langle\ ,\ \rangle$, we obtain

$$\langle a, \delta(b)\rangle = (0^* - 1^*)\widehat{\alpha}^*\widehat{\beta} \in \ker(I^0(\Delta^0) \to I^1(\Delta^0)) = \mathbb{Z}/m\mathbb{Z}.$$

On the other hand, $\langle \delta a, b\rangle = \gamma^*(b) \in H^0_{\text{et}}(\Delta^0)$ is represented by $\gamma^*\beta \in I^0(\Delta^0)$ and the commutative diagram of correspondences above implies

$$\gamma^*\beta = \gamma^*(i'^* - \pi'^*)(\widehat{\beta}) = (0^* - 1^*)\widehat{\alpha}^*\widehat{\beta}.$$

This finishes the proof. $\square$

Proposition 5.6. *Let X be a normal, generically smooth, connected scheme of finite type over a field k and let $A \subseteq H^1_{\text{et}}(X, \mathbb{Z}/m\mathbb{Z})$ be a finite subgroup. Then there exists a regular curve C over k and a finite morphism $\phi : C \to X$ such that A has trivial intersection with the kernel of $\phi^* : H^1_{\text{et}}(X, \mathbb{Z}/m\mathbb{Z}) \to H^1_{\text{et}}(C, \mathbb{Z}/m\mathbb{Z})$.*

Proof. For any normal scheme Z and dense open subscheme $Z' \subset Z$, the induced map $H^1_{\text{et}}(Z, \mathbb{Z}/m\mathbb{Z}) \to H^1_{\text{et}}(Z', \mathbb{Z}/m\mathbb{Z})$ is injective. Hence we may replace X by an open subscheme and assume that X is smooth. Let $Y \to X$ be the finite abelian étale covering corresponding to the kernel of $\pi_1^{ab}(X) \twoheadrightarrow A^*$. We have to find a regular curve C and a finite morphism $C \to X$ such that $C \times_X Y$ is connected.

Choose a separating transcendence basis $t_1, \dots, t_d$ of $k(X)$ over k. This yields a rational map $X \to \mathbb{P}^d_k$. Let t be another indeterminate and let X_t (resp. Y_t) be the base change of X (resp. Y) to the rational function field $k(t)$. Consider the composition $\phi : Y_t \to X_t \to \mathbb{P}^d_{k(t)}$. Since $k(t)$ is Hilbertian [FJ], Thm. 12.10, we can find a rational point $P \in \mathbb{P}^d_{k(t)}$ over which ϕ is defined and such that P has exactly one pre-image y_t in Y_t. The image $x_t \in X_t$ of y_t has exactly one pre-image in Y_t. Let x be the image of x_t in X. If $\operatorname{trdeg}_k k(x) = 1$ put $x' = x$, if $\operatorname{trdeg}_k k(x) = 0$ (i.e., x is a closed point in X) choose any $x' \in X$ with $\operatorname{trdeg}_k k(x') = 1$ such that x is a regular point of the closure of x'. In both cases the normalization C of the closure of x' in X is a regular curve with the desired property. □

6 Proof of the main theorem

In this section we prove our main result. We say that "resolution of singularities holds for schemes of dimension $\leq d$ over k" if the following two conditions are satisfied.

(1) For any integral separated scheme of finite type X of dimension $\leq d$ over k, there exists a projective birational morphism $Y \to X$ with Y smooth over k which is an isomorphism over the regular locus of X.

(2) For any integral smooth scheme X of dimension $\leq d$ over k and any birational proper morphism $Y \to X$ there exists a tower of morphisms $X_n \to X_{n-1} \to \cdots \to X_0 = X$, such that $X_n \to X_{n-1}$ is a blow-up with a smooth center for $i = 1, \dots, n$, and such that the composite morphism $X_n \to X$ factors through $Y \to X$.

Theorem 6.1 (=Theorem 1.1). *Let k be an algebraically closed field of characteristic $p \geq 0$, X a separated scheme of finite type over k and m a natural number. Then*

$$rec_X : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \pi_1^{t,ab}(X)/m$$

is surjective. It is an isomorphism of finite abelian groups if $(m,p) = 1$, and for general m if resolution of singularities holds for schemes of dimension $\leq \dim X + 1$ over k.

The proof will occupy the rest of this section. Following the notation of Section 5, we write $H^0_t = H^0_{\text{et}}$ and consider the maps

$$rec_{i,X} : H_i^S(X, \mathbb{Z}/m\mathbb{Z}) \to H^i_t(X, \mathbb{Z}/m\mathbb{Z})^*$$

for $i = 0, 1$ (i.e., $rec_X = rec_{1,X}$). Given a morphism $X' \to X$, we have a commutative diagram of pairings defining rec_i for $i = 0, 1$.

$$\begin{array}{ccccc}
H_i^S(X', \mathbb{Z}/m\mathbb{Z}) & \times & H^i_t(X', \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{\langle\ ,\ \rangle} & \mathbb{Z}/m\mathbb{Z} \\
\phi_* \downarrow & & \uparrow \phi^* & & \| \\
H_i^S(X, \mathbb{Z}/m\mathbb{Z}) & \times & H^i_t(X, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{\langle\ ,\ \rangle} & \mathbb{Z}/m\mathbb{Z}.
\end{array}$$

Step 1: $rec_{1,X}$ is surjective for arbitrary X.

We may assume that X is reduced and proceed by induction on $d = \dim X$. The case $\dim X = 0$ is trivial. Consider the normalization morphism $X' \to X$, which is an isomorphism outside a closed subscheme $Z \subset X$ of dimension $\leq d-1$. Using the exact sequences of Propositions 5.1 and 5.2, which are compatible by Proposition 5.5 and the fact that $rec_{0,X}$ is an isomorphism by Lemma 5.3, a diagram chase shows that it suffices to show surjectivity of $rec_{1,X}$ for normal schemes.

Let X be normal. Since $H^1_t(X, \mathbb{Z}/m\mathbb{Z})$ is finite, it suffices to show that the pairing defining $rec_{1,X}$ has a trivial right kernel. We may assume that X is connected. Let $b \in H^1_t(X, \mathbb{Z}/m\mathbb{Z})$ be arbitrary but non-zero. By Proposition 5.6, we find a morphism $\phi : C \to X$ with C a smooth curve such that $\phi^*(b) \in H^1_{\mathrm{et}}(C, \mathbb{Z}/m\mathbb{Z})$ is non-zero. Since the pairing for C is perfect by Theorem 4.1, the pairing for X has a trivial right kernel.

Step 2: Theorem 6.1 holds if $(m,p) = 1$.

If $(m,p) = 1$, $H^S_1(X, \mathbb{Z}/m\mathbb{Z})$ and $H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z})^*$ are isomorphic finite abelian groups by [SV1]. In particular, they have the same order. Hence the surjective homomorphism $rec_{1,X}$ is an isomorphism.

Step 3: Theorem 6.1 holds for arbitrary X if $m = p^r$ and resolution of singularities holds for schemes of dimension $\leq \dim X + 1$ over k.

We may assume that X is reduced. Using resolution of singularities and Chow's Lemma, we obtain a morphism $X' \to X$ with X' smooth and quasi-projective, which is an isomorphism over a dense open subscheme of X. Using the exact sequences of Propositions 5.1 and 5.2, Lemma 5.3, *Step 1*, induction on the dimension and the five-lemma, it suffices to show the result for smooth, quasi-projective schemes.

Let X be smooth, quasi-projective and let $\bar{X}$ be a smooth, projective variety containing X as a dense open subscheme. Then we have isomorphisms

$$\begin{array}{lcll} H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}/p^r\mathbb{Z})^* & \cong & \mathrm{CH}_0(\bar{X}, 1, \mathbb{Z}/p^r\mathbb{Z}) & [\text{Ge3}, \S 5], \\ \mathrm{CH}_0(\bar{X}, 1, \mathbb{Z}/p^r\mathbb{Z}) & \cong & H^S_1(\bar{X}, \mathbb{Z}/p^r\mathbb{Z}) & [\text{SS, Thm. } 2.7]. \end{array}$$

By Proposition 6.2 below the natural homomorphism

$$H^S_1(X, \mathbb{Z}/p^r\mathbb{Z}) \to H^S_1(\bar{X}, \mathbb{Z}/p^r\mathbb{Z})$$

is an isomorphism of finite abelian groups and by Proposition 2.10, we have an isomorphism

$$H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{\sim} H^1_t(X, \mathbb{Z}/p^r\mathbb{Z}).$$

Hence the finite abelian groups $H^1_t(X, \mathbb{Z}/p^r\mathbb{Z})^*$ and $H^S_1(X, \mathbb{Z}/p^r\mathbb{Z})$ are isomorphic, in particular, they have the same order. Since $rec_{1,X}$ is surjective, it is an isomorphism.

In order to conclude the proof of Theorem 6.1 it remains to show

Proposition 6.2. *Let k be a perfect field, $X \in \mathrm{Sch}/k$ smooth, $U \subset X$ a dense open subscheme and $n \geq 0$ an integer. Assume that resolution of singularities holds for schemes of dimension $\leq \dim X + n$ over k. Then for any $r \geq 1$ the natural map*

$$H^S_i(U, \mathbb{Z}/p^r\mathbb{Z}) \to H^S_i(X, \mathbb{Z}/p^r\mathbb{Z})$$

is an isomorphism of finite abelian groups for $i = 0, \ldots, n$.

Remark 6.3. A proof of Proposition 6.2 for $n = 1$ and k algebraically closed independent of the assumption on resolution of singularities would relax the condition in Theorem 6.1 to:

There exists a smooth, projective scheme $\bar{X}' \in \mathrm{Sch}/k$, dense open subschemes $U' \subset X' \subset \bar{X}'$, $U \subset X$, and a surjective, proper morphism $X' \to X$ which induces an isomorphism $U'_{\mathrm{red}} \to U_{\mathrm{red}}$.
In particular, Theorem 6.1 would hold for $\dim X \leq 3$ without any assumption on resolution of singularities [CV].

Proof of Proposition 6.2. We set $R = \mathbb{Z}/p^r\mathbb{Z}$. By [MVW], Lecture 14, we have

$$H_i^S(X, R) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)}(R[i], M(X, R)).$$

Let $d = \dim X$. Choose a series of open subschemes $U = X_d \subset \cdots X_1 \subset X_0 = X$ such that $Z_j := X_j \smallsetminus X_{j+1}$ is smooth of dimension j for $j = 0, \ldots, d-1$. Using the exact Gysin triangles [MVW, 15.15]

$$M(X_{j+1}, R) \to M(X_j, R) \to M(Z_j)(d-j)[2d-2j] \overset{[1]}{\to} M(X_{j+1}, R)[1]$$

and induction, it suffices to show that

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)}(R[i], M(Z_j, R)(s)[2s]) = 0$$

for $j = 0, \ldots, d-1$, $i = 0, \ldots, n+1$ and $s \geq 1$. Using smooth compactifications of the Z_j and induction again, it suffices to show

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)}(R[i], M(Z, R)(s)[2s]) = 0$$

for Z connected, smooth, projective, $i = 0, \ldots, d - d_Z + n$ and $s \geq 1$.

By the comparison of higher Chow groups and motivic cohomology [V] and by [GL], Thm. 8.5, the restriction of $R(s)$ to the small Nisnevich site of a smooth scheme Y is isomorphic to $\nu_r^s[-s]$, where ν_r^s is the logarithmic de Rham Witt sheaf of Milne and Illusie. In particular, $R(s)|_Y$ is trivial for $s > \dim Y$.

For an étale k-scheme Z we obtain

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)}(R[i], M(Z, R)(s)[2s]) = H^{2s-i}_{\mathrm{Nis}}(Z, R(s)) = 0$$

for $s \geq 1$ and all $i \geq 0$. Now assume $\dim Z \geq 1$. Using resolution of singularities for schemes of dimension $\leq d+n$, the same method as in the proof of [SS], Thm. 2.7 yields isomorphisms

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)}(R[i], M(Z, R)) \cong \mathrm{CH}^{d_Z}(Z, i, R)$$

for $i = 0, \ldots, d-1+n$. Applying this to $Z \times \mathbb{P}^s$ and using the decompositions given by the projective bundle theorem on both sides implies isomorphisms

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)}(R[i], M(Z, R)(s)[2s]) \cong \mathrm{CH}^{d_Z+s}(Z, i, R)$$

for $i = 0, \ldots, d-1+n$. By [V], the latter group is isomorphic to

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(k,R)}(M(Z, R)[2d_Z + 2s - i], R(d_Z + s)) \cong H^{2d_Z+2s-i}_{\mathrm{Nis}}(Z, R(d_Z + s)),$$

which vanishes for $s \geq 1$. This finishes the proof. □

Remark 6.4. The assertion of Proposition 6.2 remains true for non-smooth X if U contains the singular locus of X (see [Ge4], Prop. 3.3).

7 Comparison with the isomorphism of Suslin-Voevodsky

Theorem 7.1. *Let k be an algebraically closed field, $X \in \mathrm{Sch}/k$ and m an integer prime to $\mathrm{char}(k)$. Then the reciprocity isomorphism*

$$rec_X : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \pi_1^{ab}(X)/m$$

is the dual of the isomorphism

$$\alpha_X : H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow H^1_S(X, \mathbb{Z}/m\mathbb{Z})$$

of [SV1], Cor. 7.8.

The proof will occupy the rest of this section. Let $i : \mathbb{Z}/m\mathbb{Z} \hookrightarrow I^0$ be an injection into an injective sheaf in the category of $\mathbb{Z}/m\mathbb{Z}$-module sheaves on $(\mathrm{Sch}/k)_{\mathrm{qfh}}$ and put $J^1 = \mathrm{coker}(i)$. Then (see the end of section 3) the pairing between $H_1^S(X, \mathbb{Z}/m\mathbb{Z})$ and $H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z})$ constructed in Proposition 2.12 can be given as follows: For $a \in H_1^S(X, \mathbb{Z}/m\mathbb{Z})$ choose a representing correspondence $\alpha \in \mathrm{Cor}(\Delta^1, X)$ and for $b \in H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z})$ a pre-image $\beta \in J^1(X)$. Consider the diagram

$$\begin{tikzcd}
 & I^0(X) \arrow[r] \arrow[d, "\alpha^*"] & J^1(X) \arrow[d, "\alpha^*"] \\
 & I^0(\Delta^1) \arrow[r, two heads] \arrow[d, "0^* - 1^*"] & J^1(\Delta^1) \arrow[d, "0^* - 1^*"] \\
\mathbb{Z}/m\mathbb{Z} \arrow[r, hook] & I^0(\Delta^0) \arrow[r] & J^0(\Delta^0).
\end{tikzcd} \tag{8}$$

Then $\alpha^*(\beta)$ is the image of some element $\gamma \in I^0(\Delta^1)$ and $(0^* - 1^*)(\gamma) \in \mathbb{Z}/m\mathbb{Z} = \ker(I^0(\Delta^0) \to J^1(\Delta^0))$ equals $\langle a, b \rangle$.

For $Y \in \mathrm{Sch}/k$ let $\mathbb{Z}_Y^{\mathrm{qfh}}$ be the free qfh-sheaf generated by Y. We set $A = \mathbb{Z}[1/\mathrm{char}(k)]$ and $L_Y = \mathbb{Z}_Y^{\mathrm{qfh}} \otimes A$. For smooth U the homomorphism

$$\mathrm{Cor}(U, X) \otimes A \to \mathrm{Hom}_{\mathrm{qfh}}(L_U, L_X)$$

is an isomorphism by [SV1], Thm. 6.7. We have

$$\begin{aligned} H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z}) &= H^1_{\mathrm{qfh}}(X, \mathbb{Z}/m\mathbb{Z}) = \mathrm{Ext}^1_{\mathrm{qfh}}(L_X, \mathbb{Z}/m\mathbb{Z}) \\ &= \mathrm{coker}(\mathrm{Hom}_{\mathrm{qfh}}(L_X, I^0) \to \mathrm{Hom}_{\mathrm{qfh}}(L_X, J^1)). \end{aligned}$$

The diagram (8) can be rewritten in terms of Hom-groups as follows:

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathrm{qfh}}(L_X, I^0) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{Hom}_{\mathrm{qfh}}(L_X, J^1) \arrow[d, "\alpha^*"] \\
 & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^1}, I^0) \arrow[r, two heads] \arrow[d, "0^* - 1^*"] & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^1}, J^1) \arrow[d, "0^* - 1*"] \\
\mathbb{Z}/m\mathbb{Z} \arrow[r, hook] & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^0}, I^0) \arrow[r] & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^0}, J^1).
\end{tikzcd} \tag{9}$$

We denote the morphism $L_X \to J^1$ corresponding to $\beta \in J^1(X) \cong \mathrm{Hom}_{\mathrm{qfh}}(L_X, J^1)$ by the same letter β. Putting $E := I^0 \times_{J_1,\beta} L_X$, the extension

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow E \longrightarrow L_X \longrightarrow 0$$

represents $b \in \mathrm{Ext}^1_{\mathrm{qfh}}(L_X, \mathbb{Z}/m\mathbb{Z})$. Consider the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathrm{qfh}}(L_X, E) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{Hom}_{\mathrm{qfh}}(L_X, L_X) \arrow[d, "\alpha^*"] \\
 & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^1}, E) \arrow[r, two heads] \arrow[d, "0^* - 1^*"] & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^1}, L_X) \arrow[d, "0^* - 1^*"] \\
\mathbb{Z}/m\mathbb{Z} \arrow[r, hook] & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^0}, E) \arrow[r] & \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^0}, L_X).
\end{tikzcd} \tag{10}$$

Because diagram (10) maps to diagram (9) via β_* and $\mathrm{id} \in \mathrm{Hom}_{\mathrm{qfh}}(L_X, L_X)$ maps under β_* to $\beta \in \mathrm{Hom}_{\mathrm{qfh}}(L_X, J^1)$, we can calculate the pairing using diagram (10) after replacing β by id. Since id maps to $\alpha \in \mathrm{Hom}_{\mathrm{qfh}}(L_{\Delta^1}, L_X)$ under α^*, we see, writing the lower part of diagram (10) in the form

$$\begin{tikzcd}
\mathbb{Z}/m\mathbb{Z} \arrow[r, hook] \arrow[d, "0"] & E(\Delta^1) \arrow[r, two heads] \arrow[d, "0^* - 1^*"] & L_X(\Delta^1) \arrow[d, "0^* - 1^*"] \arrow[dl, dashed, "h"] \\
\mathbb{Z}/m\mathbb{Z} \arrow[r, hook] & E(\Delta^0) \arrow[r] & L_X(\Delta^0),
\end{tikzcd} \tag{11}$$

that

$$\langle a, b\rangle = h(\alpha) \bmod m \in \ker(E(\Delta^0)/m \to L_X(\Delta^0)/m) = \mathbb{Z}/m\mathbb{Z},$$

where h is the unique homomorphism making diagram (11) commutative. We consider the complex $C_\bullet(X) = \mathrm{Cor}(\Delta^\bullet, X) \otimes A = L_X(\Delta^\bullet)$ with the obvious differentials. By the above considerations, the homomorphism induced by the pairing of Proposition 2.12

$$H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z}) = H^1_{\mathrm{qfh}}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow$$
$$H_1^S(X, \mathbb{Z}/m\mathbb{Z})^* = \mathrm{Ext}^1_A(C_\bullet(X), \mathbb{Z}/m\mathbb{Z}) = \mathrm{Hom}_{D(A)}(C_\bullet(X), \mathbb{Z}/m\mathbb{Z}[1]),$$

is given by sending an extension class $[\mathbb{Z}/m\mathbb{Z} \hookrightarrow E \twoheadrightarrow L_X]$ to the morphism $C_\bullet(X) \to \mathbb{Z}/m\mathbb{Z}[1]$ in the derived category of A-modules represented by the morphism

$$C_\bullet(X) \to [0 \to E(\Delta^0) \to L_X(\Delta^0) \to 0]$$

which is given by $\mathrm{id} : L_X(\Delta^0) \to L_X(\Delta^0)$ in degree zero and by $h : L_X(\Delta^1) \to E(\Delta^0)$ in degree one.

The same construction works for any qfh-sheaf of A-modules F instead of L_X, i.e., setting $C_\bullet(F) = F(\Delta^\bullet)$ and starting from an element

$$[\mathbb{Z}/m\mathbb{Z} \hookrightarrow E \twoheadrightarrow F] \in \mathrm{Ext}^1_{\mathrm{qfh}}(F, \mathbb{Z}/m\mathbb{Z}),$$

we get a map $C_\bullet(F) \to \mathbb{Z}/m\mathbb{Z}[1]$ in the derived category of A-modules. We thus constructed a homomorphism

$$\mathrm{Ext}^1_{\mathrm{qfh}}(F, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathrm{Ext}^1(C_\bullet(F), \mathbb{Z}/m\mathbb{Z}), \tag{12}$$

which for $F = L_X$ and under the canonical identifications coincides with the map

$$H^1_{\text{et}}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow H^S_1(X, \mathbb{Z}/m\mathbb{Z})^*$$

induced by the pairing constructed in Proposition 2.12.

Now we compare the map (12) with the map

$$\alpha_X : \mathrm{Ext}^1_{\mathrm{qfh}}(F, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathrm{Ext}^1_A(C_\bullet(F), \mathbb{Z}/m\mathbb{Z}) \tag{13}$$

constructed by Suslin-Voevodsky [SV1] (cf. [Ge1] for the case of positive characteristic). Let $F^\sim_\bullet$ be the complex of qfh-sheaves associated with the complex of presheaves $F_\bullet(U) = F(U \times \Delta^\bullet)$. By [SV1], the inclusion $F \to F^\sim_\bullet$ induces an isomorphism

$$\mathrm{Ext}^1_{\mathrm{qfh}}(F^\sim_\bullet, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{qfh}}(F, \mathbb{Z}/m\mathbb{Z}), \tag{14}$$

and evaluation at $Spec(k)$ induces an isomorphism

$$\mathrm{Ext}^1_{\mathrm{qfh}}(F^\sim_\bullet, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} \mathrm{Ext}^1_A(C_\bullet(F), \mathbb{Z}/m\mathbb{Z}). \tag{15}$$

The map (13) of Suslin-Voevodsky is the composite of the inverse of (14) with (15).

We construct the inverse of (14). Let a class $[\mathbb{Z}/m\mathbb{Z} \hookrightarrow E \twoheadrightarrow F] \in \mathrm{Ext}^1_{\mathrm{qfh}}(F, \mathbb{Z}/m\mathbb{Z})$ be given. As a morphism in the derived category this class is given by the homomorphism

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0
\end{array} \quad .$$

We therefore have to construct a homomorphism $F_1 \longrightarrow E$ making the diagram

$$\begin{array}{ccccccc}
F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

commutative. The construction is a sheafified version of what we did before. Let $U \in \mathrm{Sch}/k$ be arbitrary. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}(U) & \longrightarrow & E(U \times \Delta^2) & \longrightarrow & F(U \times \Delta^2) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \delta^0 - \delta^1 + \delta^2} & & \downarrow{\scriptstyle \delta^0 - \delta^1 + \delta^2} & & \\
0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}(U) & \longrightarrow & E(U \times \Delta^1) & \longrightarrow & F(U \times \Delta^1) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0^* - 1^*} & & \downarrow{\scriptstyle 0^* - 1^*} & & \\
0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}(U) & \longrightarrow & E(U) & \longrightarrow & F(U) & \longrightarrow & 0
\end{array} \quad .$$

Let $\alpha_1 \in F(U \times \Delta^1)$ be given. By the smooth base change theorem and since $H^1_{\mathrm{et}}(\Delta^1, \mathbb{Z}/m\mathbb{Z}) = 0$, we can lift α_1 to $E(U \times \Delta^1)$ after replacing U by a sufficiently fine étale cover. Applying $0^* - 1^*$ to this lift, we get an element in $E(U)$. This gives the homomorphism $F_1 \to E$. Now let $\alpha_2 \in F(U \times \Delta^2)$ be arbitrary. After replacing U by a sufficiently fine étale cover, we can lift α_2 to $E(U\times\Delta^2)$. Since $(0^*-1^*)(\delta^0-\delta^1+\delta^2) = 0$ this shows that $(\delta^0 - \delta^1 + \delta^2)(\alpha_2)$ maps to zero in $E(U)$.

This describes the inverse isomorphism to (14). Evaluating at $U = \mathit{Spec}(k)$ gives back our original construction, hence (12) and (13) are the same maps. This finishes the proof.

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