Cremer Non-linearization Theorem and Continued Fractions

Erik Lewerenz

Universität Heidelberg

kg225@stud.uni-heidelberg.de

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Overview

Quick Recap

Let f be a rational function of degree greater or equal to 2. If z_0 is an indifferent fixed point of ($|f'(z)| = 1$) then the following are equivalent

(1) f is locally linerarizable around z_0
(2) $z_0 \in \hat{\mathbb{C}} \setminus J(f)$
(3) the connected component U of $\hat{\mathbb{C}} \setminus J(f)$ containing z_0 is conformally isomorphic to $\mathbb{E}$ under an isomorphism which conjugates f on to multiplication by λ on the disk.

Main Goal

Investigate maps of the form

$$f(z) = \lambda z + \sum_{n=2}^{N} a_n z^n, \quad a_n \in \mathbb{C}$$

Where $\lambda = e^{2\pi i \xi}, \quad \xi \in \mathbb{R} \setminus \mathbb{Q}$. We will determine all such ξ where we can find $z = h(\omega)$ with $f(h(\omega)) = h(\lambda\omega)$.

Definition

A property is true for generic (in the topological sense) $\lambda \in \mathbb{S}^1 \Leftrightarrow$ The set of all such λ for which said property is true contains a countable intersection of dense open subsets of $\mathbb{S}^1$.

Note that the set of generic λ is not necessarily a non null set.

Cremer Non-linearization Theorem (1927)

For generic $\lambda \in \mathbb{S}^1$ it is true that if z_0 is a fixed point of any rational function with multiplier λ, then z_0 is the limit of an infinite sequence of periodic points. This means there is no linearizing coordinate in any neighborhood of z_0.

Siegel's Linearization Theorem (1942)

For almost every $\lambda \in \mathbb{S}^1$ any holomorphic function $f : U \to \mathbb{C}$ with fixed point mulitplier λ can be linearized by a local holomophic change of variables.

Siegel's and Cremer's Theorems

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Observation

Any irrational λ is either a "Cremer Point" or a "Siegel Point".

Reminder (Siegel Disks)

Recall that a connected $U \subset \hat{\mathbb{C}} \setminus J$ is called a Siegel disk.

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Diophantine Condition

$\xi \in \mathbb{R} \setminus \mathbb{Q}$ satisfies a Diophantine Condition of order $\kappa \Leftrightarrow \exists \varepsilon = \varepsilon(\xi)$ such that $\left|\xi - \frac{p}{q}\right| > \frac{\varepsilon}{q^\kappa} \quad \forall \frac{p}{q} \in \mathbb{Q}$. Or by setting $\lambda = e^{2\pi i \xi} \Rightarrow |\lambda^q - 1| > \frac{\varepsilon'}{q^{\kappa-1}}$.

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Diophantine Numbers

Let $D_\kappa \subset \mathbb{R} \setminus \mathbb{Q}$ denote all such numbers. Note that $D_\kappa \subset D_\eta \Leftrightarrow \kappa < \eta$. Now we define $Si := \bigcup_{\kappa \in \mathbb{N}} D_\kappa$.

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Similarly we can define

Roth Numbers

$Ro := \bigcap_{\kappa \in \mathbb{N}_{\geq 2}} D_\kappa$

Continued fractions

Pick $\xi \in \mathbb{R} \setminus \mathbb{Q} \cap (0, 1)$. We can investigate the fractional expansion

$$\xi = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}, \quad a_i \in \mathbb{N}$$

Continued fractions

If we chop off the expansion at any a_{n-1} we get $\frac{p_n}{q_n}$ the "n-th convergent" of ξ. Interestingly it holds true that

$$q_{n+1} > q_n > \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} > 1, \quad n > 2$$

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Additionally, by construction, $\frac{p_n}{q_n}$ will be the best approximation of ξ with denominator at most q_n.

Continued fractions

As before set $\lambda = e^{2\pi i \xi}$. We will need 2 facts (without Proof)

Fact 1

$|\lambda^k - 1| < |\lambda^{q_n} - 1|, \quad k = 1, 2, ..., q_n - 1$

Fact 2

There exist constants $0 < c_1 < c_2 < \infty$ which satisfy

$$\frac{c_1}{q_{n+1}} \leq |\lambda^{q_n} - 1| \leq \frac{c_2}{q_{n+1}}$$

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\lambda \in Si &\Leftrightarrow \lambda \in \bigcup_{\kappa \in \mathbb{N}} D_\kappa \\
&\Leftrightarrow \exists \kappa \in \mathbb{N} : \lambda \in D_\kappa \\
&\Leftrightarrow \exists \kappa \in \mathbb{N}, \varepsilon'(\xi) \in \mathbb{R} : |\lambda^q - 1| < \frac{\varepsilon'}{q^{\kappa-1}} \\
&\Leftrightarrow |\lambda^{q_{n+1}} - 1| \leq \frac{c_1}{q_{n+1}^{\kappa-1}} < |\lambda^{q_n} - 1| < \frac{\varepsilon'}{q_n^{\kappa-1}} \\
&\Leftrightarrow \sup_{n \in \mathbb{N}} \frac{\log(q_{n+1})}{\log(q_n)} < \infty
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\end{aligned}$$

Equivalently one gets for $\lambda \in Ro$

$$\lim_{n \to \infty} \frac{\log(q_{n+1})}{\log(q_n)} = 1$$

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Note that we have $Ro \Rightarrow Si \Rightarrow Br \Rightarrow PM$

Sharper Theorems

With those new sets we can formulate several theorems

Theorem of Bryuno (1972)

If ξ satisfies Br then any holomorphic function $f : U \to \mathbb{C}, z \mapsto e^{2\pi i \xi} z + \sum_{\nu=2}^{N} a_\nu z^\nu$ can be linearized by a holomorphic variable change.

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Theorem of Yoccoz (1987)

If for ξ we have $\sum_{n\in\mathbb{N}} \frac{\log(q_{n+1})}{q_n} = \infty$ then every neighborhood of the origin will contain infinitely many periodic orbits. Hence the origin is a Cremer Point.

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Theorem of Perez-Marco (1990)

If ξ satisfies the Perez-Marco (PM) condition any holomorphic function $f : U \to \mathbb{C}, z \mapsto e^{2\pi i \xi} z + \sum_{\nu=2}^{N} a_\nu z^\nu$ which cannot be linearized will contain infinitely many orbits in any neighborhood of the origin.

Cremers Non-linearization theorem

We will define one last set of irrational numbers. We say $\xi \in \mathbb{R} \setminus \mathbb{Q}$ satisfies a Cremer Condition of degree d if

$$Cr_d := \limsup_{q \to \infty} \frac{\log(\log(|\lambda^q - 1|^{-1}))}{q} > \log(d)$$

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We can now state Cremers Theorem more formally

Cremer Non-linearization Theorem

If ξ satisfies Cr_d for $d \geq 2$, then for an arbitrary rational function of degree d any neighborhood of a fixed point of multiplier $\lambda := e^{2\pi i \xi}$ must contain infinitely many periodic orbits. No linearization is possible.

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Now set $f(z) = \frac{P(z)}{Q(z)}$. It follows that $\deg(P) < d$. Applying some algebra we may also assume $P(z) = \lambda z + \mathcal{O}(z^s), \quad s < d$.

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$$f^{\circ q}(z) = \frac{\mathcal{O}(z^{s^q}) + \lambda^q z}{\sum_{\nu=0}^{d^q} a_\nu z^\nu} \Leftrightarrow 0 = z\left(\sum_{\nu=0}^{d^q} a_\nu z^\nu - \mathcal{O}(z^{s^q}) + \lambda^q z\right)$$

But this is just an ordinary monic Polynomial! Now we can proceed to show the polynomial case.

Polynomial Case Part 1

Let $f(z) = \lambda z + \sum_{\nu=2}^{d} a_\nu z^\nu$, $d > 1, a_d = 1$, with $\lambda := e^{2\pi i \xi}$, where ξ satisfies Cr_d.

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$$\sum_{\nu=2}^{d^q-1} a_\nu z^\nu = \prod_{\sum n_i = d^q - 1} (z - z_i)^{n_i} = \pm(\lambda^q - 1)$$

Polynomial Case Part 2

Now there must exist a root z_j with

$$0 < |z_j| < |\lambda^q - 1|^{1/(d^q-1)} < |\lambda^{d^q} - 1|^{1/(d^q-1)} < |\lambda^{d^q} - 1|^{1/d^q}$$

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By Hypothesis we can choose $\varepsilon > 0$ such that for arbitrarily large q

$$\frac{\log(\log(1/|\lambda^q - 1|))}{q} > \log(d) + \varepsilon \Leftrightarrow |\lambda^q - 1|^{1/(d^q-1)} < e^{-e^{\varepsilon q}} < e^{-\varepsilon q} \tag{1}$$

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Obviously $e^{-\varepsilon q} \to 0$ for large q. By Taylor we can choose $\delta > 0$ such that $|f(z)| < e^{\varepsilon}|z|$ for $|z| < \delta$. Thus,

$$|f^{\circ k}(z)| < \delta, \quad 1 \leq k \leq q \text{ and } |z| < e^{-\varepsilon q}\delta$$

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We have already seen that there exist periodic points (1) for any q. Therefore the entire orbit must also be contained in this neighborhood. This completes the proof.

Literature

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