SYLLABUS FOR THE STUDENT SEMINAR: GROUPS ACTING ON TREES

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Trees (graphs without loops) are very simple mathematical object, yet the groups that can act on them have remarkable algebraic properties. In the seminar we will discuss algebraic properties of groups that act on trees, and we will use geometry to construct, starting from easy building blocks, more complicated groups with interesting algebraic property.

This will lead us to the Bass-Serre theory of graphs of groups, one of the milestones of geometric group theory, and will allow us to give easy geometric proofs of purely algebraic facts, as for example that subgroups of free groups are necessarily free, as well as the fact that a torsion-free subgroup of $\mathrm{SL}(2, \mathbb{Q}_p)$ is free.

Talk 1: Amalgams

The aim of the first talk will be to describe universal constructions that allow to define more complicated groups starting from easier building blocks. The speaker should focus on the construction of direct limits of groups, and present amalgams as a particular case of that construction. Time permitting the theory can be applied to present an example, due to Higman, of an infinite group with no finite index subgroups. [Serre p. 1-10]

Talk 2: Trees

Switching to geometry, the speaker should give a formal definition of a tree, and explain the concept with many pictures to foster our intuition. Given a generic graph, the speaker should explain how to construct maximal subtrees, and breafly discuss the Cayley graph of a group with respect to a generating set. [Serre p. 13-24]

Talk 3: Trees and free groups

After recalling the definition of a free group, the speaker should prove that a group that acts freely on a tree is free, and deduce that every subgroup of a free group is free. The talk can also include a discussion of the Schreier index formula relating the rank of a free subgroup and its index. [Serre p. 25-31]

Date: May 8, 2018.

Talk 4: Trees and amalgams

We should understand the notion of fundamental domain for more general groups acting on trees with non-trivial edge and vertex stabilizers. The special cases in which the fundamental domain is an edge should be described in detail, and related with amalgams described in Talk 1. Time permitting the speaker can discuss the relation between loops and HNN extension. [Serre, p. 32-37]

Talk 5: Trees of groups

The talk deals with trees of groups and their limits. The speaker should also recall the statement of Van Kampen theorem, and describe the topological counterpart of the more algebraic structure description of the fundamental group of the tree of groups. [Serre, p. 37-40, Hatcher p. 92-96]

Talk 6: Fundamental groups of graphs of groups

The speaker should give two constructions of the fundamental group of a graph of groups, and discuss the notion of reduced words in this context. [Serre p. 41-50]

Talk 7: Universal coverings and subgroups of amalgams

In this talk the speaker should construct the universal tree on which the fundamental group of a graph of groups acts, and discuss Kurosh's theorem. [Serre p. 50-57]

Talk 8:Amalgams and property (FA)

After introducing the notion of property (FA), the speaker should discuss criteria for establishing that a group acting on a tree has a global fixed point. Time permitting the speaker can show that $\mathrm{SL}(3, \mathbb{Z})$ has property (FA). [Serre, Ch. I .6]

Talk 9: The tree associated to $\mathrm{SL}_2(\mathbb{Q}_p)$

The goal of the talk is to give a complete discussion of the geometric description of the tree associated to $\mathrm{SL}_2(\mathbb{Q}_p)$, describing the point stabilizers for this action. Time permitting the speaker can discuss what happens for more general local fields. [Serre, Ch. II.1.1-II.1.3]

Talk 10*: The automorphism group of a tree

The goal of this talk is to study the automorphism group of a tree and define a topology on it that turns it in a locally compact group. The speaker should also discuss the fact that, in the case of the $p+1$-regular tree $\mathcal{T}_{p+1}$ associated to $SL_2(\mathbb{Q}_p)$, the group $\mathrm{Aut}(\mathcal{T}_{p+1})$ contains $SL_2(\mathbb{Q}_p)$ as a closed subgroup, but is much bigger.

Talk 11*: Tree lattices

The speaker should define the notion of lattice in a locally compact group, and construct examples of lattices in $\mathrm{Aut}(\mathcal{T}_{p+1})$ whose associated graph of groups is not finite.

Talk 12*: Stalling's theorem

After recalling the notion of ends of a group, the speaker should show that a group G has more than one end if and only if it admits a nontrivial (that is, without a global fixed vertex) action on a simplicial tree with finite edge-stabilizers and without edge-inversions. [Wall p. 24-27]

References

[1] A. Hatcher *Algebraic topology*

[2] J.P. Serre *Trees*, Springer-Verlag, 1980. ISBN 3-540-10103-9

[3] C. T. C. Wall. *The geometry of abstract groups and their splittings.* Revista Matemática Complutense vol. 16(2003), no. 1, pp. 5-101