SYLLABUS FOR THE STUDENT SEMINAR: DYNAMICS IN ONE COMPLEX VARIABLE

In this seminar, we study both theoretically and experimentally the dynamics of holomorphic mappings $f : S \to S$ on a Riemann surface S. Key examples will be rational functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere $\widehat{\mathbb{C}}$ as simple as $f(z) = z^2 + 2$. The crucial question in this framework is to understand the behaviour of the sequence $z, f(z), f(f(z)), ...$ with starting point $z \in S$. Is the sequence periodic? Is it converging? How does its behaviour change if we move the starting point? The starting points giving rise to a chaotic behaviour under the above procedure form the so-called Julia set of the mapping f. More generally, we will consider a family f_λ of functions parametrized by a complex number λ. The values of λ for which the Julia set is connected will form the so-called Mandelbrot set in parameter space. The Julia and Mandelbrot sets have a beautiful fractal structure and complicated topological properties. We aim to understand them rigorously and visualize them experimentally with a computer during the seminar.

1. Talks

Unless otherwise stated, the numbering of chapters, theorems, etc. is referred to [Milnor]. If you have difficulties in finding any of the books, let us know.

Thursday, June 10

Talk 1 (16.00 – Visualizing Fatu and Julia sets; Devaney, Section 16)**. Marcel** Define Julia and Fatu set for a holomorphic map of the Riemann sphere. Explain the basic tools to visualize Julia sets for holomorphic mappings of the Riemann sphere (Appendix H). Write a computer program that allows for visualization or learn to use a program from the internet. Use it to discuss various examples and observe how subtly the shape of the Julia set depends on the initial parameter. Illustrate the self similarity of Julia sets (Problem 4-d). See also Devaney Section 16.

Saturday, June 12

Talk 2 (9.30 – The Poincaré disk and the theorem of Pick; Section 2)**. Monika**

Define Riemann surfaces and state the uniformization theorem (Theorem 1.1). Recall Schwarz lemma (Lemma 1.2) and Liouville Theorem (Theorem 1.3) from complex analysis without a proof. Introduce hyperbolic Riemann

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surfaces, and show that the triply punctured sphere is hyperbolic (Lemma 2.5). Introduce the Poincaré metric on the Poincaré disk (Lemma 2.7). State and prove the theorem of Pick (Theorem 2.11).

Talk 3 (11.00 – Montel's theorem; Section 3)**. Burku**
Recall from complex analysis Weierstrass uniform convergence theorem (Theorem 1.4) without a proof. Introduce the topology of locally uniform convergence (Lemma 3.1). Discuss the notion of normal family of maps, providing examples and counterexamples, and prove Montel's theorem (Theorem 3.7).

Talk 4 (14.00 – The Mandelbrot set; Devaney, Section 17 – English)**. Anna**
Discuss the definition of the Mandelbrot set for the family of holomorphic maps $f_c(z) = z^2 + c$. Build a visualization tool that allows to picture it, and zoom in at various points. Relate with what we discussed so far. [Devaney, Section 17, with some experiments]

Saturday, June 19

Talk 5 (9.30 – Fatu and Julia set; Section 4 – Deutsch)**. Franziska**
Recall the definition of the Fatu and Julia set for a holomorphic map of the Riemann sphere. Introduce the grand orbit of such a map. Use it to show that iterates of an arbitrary neighborhood of an arbitrary point in the Julia set contain the complement of at most two points in the Riemann sphere. Deduce topological transitivity of the restriction of the map to the Julia set.

Talk 6 (11.00 – Siegel disk; Section 5)**. Cornelius**
Discuss fixed points of holomorphic maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the Riemann sphere, and give examples of maps f admitting indifferent fixed points (visualizing the associated Julia sets). Define the Siegel disk associated to such a fixed point, and deduce its structure from Theorem 5.2, the classification of possible dynamics on hyperbolic surfaces.

Talk 7 (13.30 – Attracting and repelling points and cycles; Section 8 – Deutsch)**. Britta**
Introduce the multiplier at a fixed point and show that it locally determines the dynamic around the fixed point if its absolute value is not zero or one (Theorem 8.2). Define the basin of attraction and the immediate basin of attraction, and show that there is an upper bound on the number of attracting periodic cycles by finding critical points in the imediate basin of attraction

Talk 8 (14.45 – Parabolic fixed points; Section 10)**. Julia**
Local dynamics of a holomorphic function near a parabolic fixed point whose multiplier is a q-th root of unity for some q. The Leau-Fatou Flower theorem. Statement of the parabolic linearization theorem (without proof), finiteness of the number of parabolic periodic points (Section 10)

Talk 9 (16.00 – Cremer non-linearization theorem; Section 11)**. Erik**

Local dynamics of a holomorphic function near a parabolic fixed point whose multiplier is not a q-th root of unity for any q: Cremer non-linearization theorem, diophantine approximation, continued fractions (Section 11)

2. Prerequisites:

Linear Algebra 1 and 2, Function Theory

3. Useful literature:

(1) John Milnor, *Dynamics in one complex variable. Third edition*. Annals of Mathematics Studies, 160. Princeton University Press, Princeton, NJ, viii+304 pp., 2006.
(2) Curtis McMullen, *Complex Dynamics and Renormalization*. Annals of Mathematics Studies, 135. Princeton University Press, Princeton, NJ, x+214 pp., 1994.
(3) Robert L. Devaney, *A first course in chaotic dynamical systems: theory and experiment. Second edition*. Boca Raton: CRC Press, Taylor & Francis Group, x+318 pp., 2020.
(4) Alan F. Beardon, *Iteration of rational functions, Complex analytic dynamical systems*. Graduate Texts in Mathematics, 132. Springer-Verlag, New York, 1991. xvi+280 pp.
(5) Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, x+175 pp., 1993.
(6) Will Merry, *1. The Definition of a Dynamical System.* Available at `https://www.merry.io/dynamical-systems/1-the-definition-of-a-dynamical-system/`.