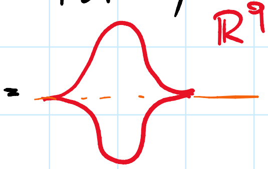
Today: if M^m, Q^q mfds $m \geq q$, then

Thm 1 $\boxed{WS(M,Q) \xrightarrow{R} FS(M,Q) \text{ is a w.h.e.}}$

Recall: $\bar f : \overbrace{\mathbb{R}^{m-q} \times \mathbb{R}^{q-1} \times \mathbb{R}}^{\mathbb{R}^m} \to \overbrace{\mathbb{R}^{q-1} \times \mathbb{R}}^{\mathbb{R}^q}$ local model around $\{0\} \times D^q_{\max\{0,t\}}$, $t \in \mathbb{R}$

$$(x, y, z) \mapsto \left(y,\ z^3 + 3(|y|^2 - t)z + \left(-\sum_{i=1}^{s} x_i^2 + \sum_{i=s+1}^{m-q} x_i^2\right)\right)$$

For $t>0$ $\bar f$ singular at $\{0\} \times S^{q-1}_t =: \Sigma_{\bar f}$ and $\bar f(\Sigma_{\bar f})$ = 

$\bar f$ is called wrinkle

For $t=0$ $\bar f$ singular at $(0,0)$, $\bar f$ is called an embryo

For $t<0$ $\bar f$ is a submersion.

Remark

If $t>0$ (WLOG $t=1$) and write $y = (y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^{q-k-1}$ we can treat y_1 as parameter

$$\bar f_{y_1} : \mathbb{R}^{m-q} \times \mathbb{R}^{q-k-1} \times \mathbb{R} \to \mathbb{R}^{q-k-1} \times \mathbb{R}$$

$$(x, y_2, z) \mapsto \left(y_2,\ z^3 + 3\Big(|y_2|^2 - \overbrace{(1-|y_1|^2)}^{=t}\Big)z + \left(-\sum_{i=1}^{s} x_i^2 + \sum_{i=s+1}^{m-q} x_i^2\right)\right)$$

$\bar f_{y_1}$ is a wrinkle for $|y_1|^2 < 1$, embryo for $|y_1|^2 = 1$ and subm. for $|y_1|^2 > 1$.

Def $WS(M,Q) = \{ f : M \to Q \mid \text{locally } f = \bar f \}$

$FS(M,Q) = \{ F : M \to J^1(M,Q) \mid F(p) = (p, f(p), \overbrace{F_p : T_pM \to T_{f(p)}Q}^{\text{surj.}}) \}$

Proving Thm 1 means: $\forall$ $k \in I^k \mapsto F_k \in FS(M,Q)$ with $F_k = df_k$ for $k \in \partial I^k$

$\exists$ $k \in I^k \mapsto \tilde f_k \in WS(M,Q)$ with $\tilde f_k = f_k$ for $k \in \partial I^k$

and $F_k \overset{\text{hom.}}{\sim} R(d\tilde f_k)$ $\forall k \in I^k$.

Note: out of $k \mapsto \tilde f_k$ one can define $\tilde{\tilde f} : M \times I^k \to Q \times I^k$, $(p,k) \mapsto (\tilde f_k(p), k)$. The proof of Thm 1 will show that $\tilde{\tilde f} \in WS(M \times I^k, Q \times I^k)$, then $\tilde f_k \in WS(M,Q)$ by the Remark and some transversality ($\tilde{\tilde f}$ has no embryos and the wrinkles of $\tilde{\tilde f}$ lie with respect to the maps $M \times \{k\} \hookrightarrow M \times I^k \to M$ as in the local model of the Remark.

of f the with respect to the maps $M\times\{k\} \hookrightarrow M\times I \to M$ as in the local model of the Remark.

Below we are going to assume $k=0$ ($I^k\times\{0\}$) for simplicity. We reduce Thm 1 to

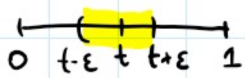
Thm 2 Let $F\in FS(I^m, I^q)$ with $F=df$ in $Op(\partial I^m)$, then $\exists\ \tilde{f}\in WS(I^m, I^q)$ with $\tilde{f}=f$ in $Op(\partial I^m)$ and $F\sim R(d\tilde{f})$.

Proof of Thm 1 from Thm 2: Find small open sets $U\subset M$ and $V\subset Q$ with $U\cong I^m$ and $V\cong I^q$, $f(U)\subset V$. Since being a submersion is a Diff-invariant, open diff. rel. and $M\setminus U$ is an open mfd, by hol. approx. $\exists\ \tilde{f}^{(1)}: M\setminus U\to Q$ submersion s.that $d\tilde{f}^{(1)}\sim F$ on $M\setminus U$ and $\|\tilde{f}^{(1)}_1 - f\|_{C^0}$ small on ∂U. Now apply Thm 2 on U to $F_{|U}$ to extend $\tilde{f}^{(1)}$ inside U to $\tilde{f}\in WS(M,Q)$ with $\tilde{f}\sim F$ on U. ☒

Proof of Thm 2

Idea: use one coordinate in I^m as parameter $I^m = I\times I^{m-1}$. Write I^m as 1-par. fam.
$p=(t,y)$

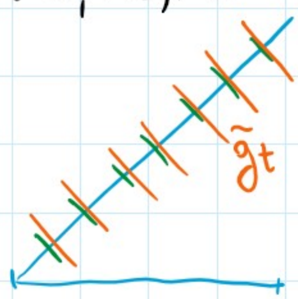
of open mfds: $I^m = \bigcup_{t\in I} (t-\varepsilon, t+\varepsilon)\times I^{m-1}$ for $\varepsilon>0$



Set $\tilde{F}_t := F_{|(t-\varepsilon,t+\varepsilon)\times I^{m-1}} \in FS((t-\varepsilon,t+\varepsilon)\times I^{m-1}, I^q)$. By hol. approx:

$\exists\ \tilde{g}_t : (t-\varepsilon, t+\varepsilon)\times I^{m-1}\to I^q$ subm., $d\tilde{g}_t\sim \tilde{F}_t$ and $\tilde{g}_t = f$ on $op(\partial I^m)$.

Example
$m=q=1$



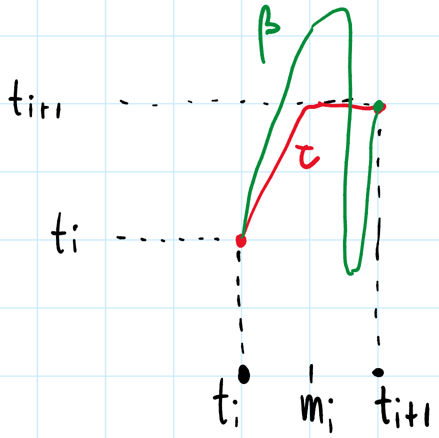
Now we interpolate the $\tilde{g}_t$ fixing a grid $t_i = i/N$ $i=0,\dots,N$ where $\varepsilon N\gg 1$. Call m_i the midpoint of $[t_i, t_{i+1}]$. We construct functions $\tau: I\to I$ and $\beta: I\to I$ as follows.

On $[t_i, m_i]$: τ increasing with $\tau(t_i)=t_i$ and $\tau(m_i)=t_{i+1}$ ($\tau'<3$)
β increasing with $\beta(t_i)=t_i$ and $\beta(m_i)=t_i+\varepsilon$ ($\beta'>\varepsilon N$)

On $[t_i, m_i]$: τ increasing with $\tau(t_i)=t_i$ and $\tau(m_i)=t_{i+1}$ $(\tau'<3)$

β increasing with $\beta(t_i)=t_i$ and $\beta(m_i)=t_i+\varepsilon$ $\left(\beta' > \frac{\varepsilon N}{3}\right)$

On $[m_i, t_i]$: $\tau \equiv t_{i+1}$, β is a 1-dim wrinkle $\beta(m_i)=t_i+\varepsilon$, $\beta(t_{i+1})=t_{i+1}$,
$\beta(t)\in[t_{i+1}-\varepsilon, t_i+2\varepsilon]$, β' is convex



We define $\alpha : I^m \to I^m$, $\alpha(x,y) = \big(\rho(y)\beta(x) + (1-\rho(y))x, y\big)$.
where ρ is a cut-off function

Remark α has a wrinkle in every $[m_i, t_{i+1}]\times I^{m-1}$. $d\alpha$ sing. $\iff \rho\cdot\beta' + (1-\rho) = 0$

$\iff \left\{\beta' = \overbrace{1-\frac{1}{\rho}}^{\leq 0}\right\}$ is a sphere with equator at $y\in S^{m-1}$ where $1-\frac{1}{\rho} = \min \beta'$.

We define finally : $\tilde f(x,y) = \tilde g_{\tau(x)}(\alpha(x,y))$ $\forall (x,y)\in I^m$.

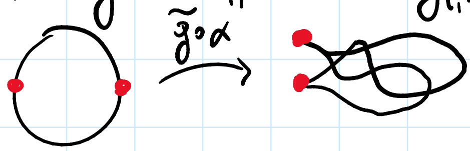
Claim $\tilde f$ is a subm. on $[t_i, m_i]$ for N big enough. We check it where $\rho = 1$.

There: $\tilde f(x,y) = \tilde g_{\tau(x)}(\beta(x), y)$ $\quad d\tilde f = \left(\frac{\partial \tilde g}{\partial t}\cdot\tau' + \frac{\partial \tilde g}{\partial x}\cdot\beta' \,\middle|\, \frac{\partial \tilde g}{\partial y}\right)$ surj. iff

$\left(\frac{\partial \tilde g}{\partial t}\cdot\frac{\tau'}{\beta'} + \frac{\partial \tilde g}{\partial x} \,\middle|\, \frac{\partial \tilde g}{\partial y}\right) = \overbrace{\frac{\tau'}{\beta'}\left(\frac{\partial \tilde g}{\partial t} \,\middle|\, 0\right)}^{\text{small}} + \overbrace{d\tilde g}^{\text{surj.}}$ surj. ✓

Remark On $[m_i, t_{i+1}]$ $\tilde f = \underbrace{\tilde g_{t_{i+1}}}_{\text{subm.}} \circ \overbrace{\alpha}^{\text{wrinkle}}$. Composition might not be a wrinkle!

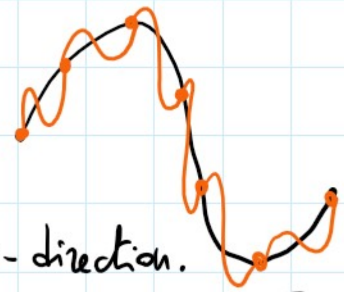
$d\tilde f$ singular iff $\operatorname{Ker} dg_{t_{i+1}} \subset T(\alpha(\Sigma_\alpha))$ and $g_{t_{i+1}}$ might not be inj. on $\alpha(\Sigma_\alpha)$



. Idea: chop the wrinkles of α into smaller ones.

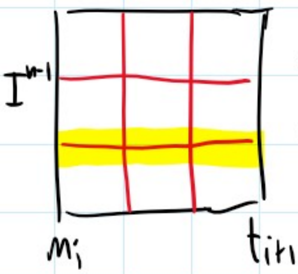
If $m=q$ this concludes because $g_{t_{i+1}}$ is locally a diffeo. If $m>q$ we have to further perturb (wrinkle) the small wrinkles.

Chopping wrinkles

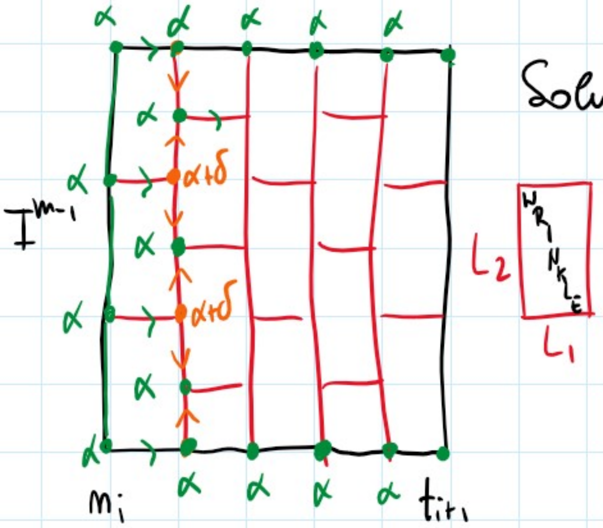


Reducing the size in the x-direction is easy.
However we have to reduce the size also in the y-direction.
Idea: subdivide $[m_i, t_{i+1}]$ into union of smaller rectangles R_J and find $\tilde{\alpha}$ on $\mathcal{O}p\left(\bigcup_J \partial R_J\right)$ with $\frac{\partial \tilde{\alpha}}{\partial x}>0$ and $|\tilde{\alpha}-\alpha|_{C^0}$ small. Then extend $\tilde{\alpha}$ inside each R_J with a wrinkle.

R_J must be chosen carefully:



: not good because $\tilde{\alpha}$ increases along the long horizontal lines, so $|\tilde{\alpha}-\alpha|_{C^0}$ cannot be guaranteed.



Solution: go zig-zag to recover on the vertical lines.

Wrinkling the wrinkles for $m>q$

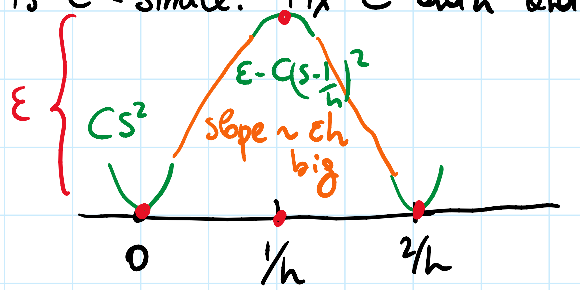
Now that the wrinkles of $\tilde{\alpha}$ are small we can suppose at these wrinkles the map $\tilde{\alpha}$ is equivalent to $(w,z)\mapsto\left(w, z^3+3(|w|^2-1)z\right)$ $(w,z)\in\mathbb{R}^{m-1}\times\mathbb{R}$, and $\tilde{g}_{x_{i+1}}$ is equivalent to $(w,\tilde{z})\mapsto(y,\tilde{z})$ where $w=(x,y)\in\mathbb{R}^{m-q}\times\mathbb{R}^{q-1}$.

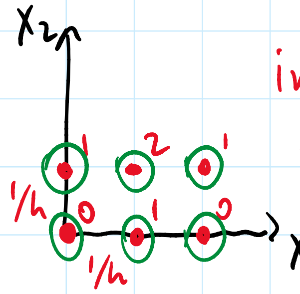
Then: $\tilde{g}_{x_{i+1}}\circ\tilde{\alpha}$ equivalent to $(x,y,z)\mapsto\left(y, z^3+3(|y|^2-1)z+3|x|^2z\right)$

Then: $\tilde g_{x_{i+1}} \circ \tilde\alpha$ equivalent to $(x, y, z) \mapsto \big(y,\ z^3 + 3(|y|^2-1)z + \underbrace{3|x|^2 z}_{\text{not a family of Morse functions in } x}\big)$.

We perturb $\tilde\alpha$ to $\Big(w,\ z^3 + 3(|w|^2-1)z + \Big[\sum_{i=1}^{m-q} \gamma(x_i)\Big]\cdot \underbrace{\rho(w,z)}_{\text{cutoff outside } \Sigma_{\tilde\alpha}}\Big)$

for some $\gamma : \mathbb{R} \to \mathbb{R}$ to be determined so that $\sum \gamma(x_i) + 3|x|^2 z$ is a family of Morse functions in x and $\sum \gamma(x_i)$ is C^0-small. Fix C and h and define γ to be $\frac{2}{h}$ periodic and

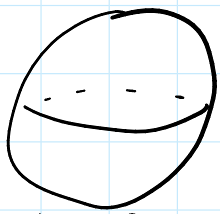




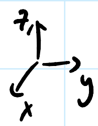
index of critical points of $\gamma(x_1) + \gamma(x_2)$.

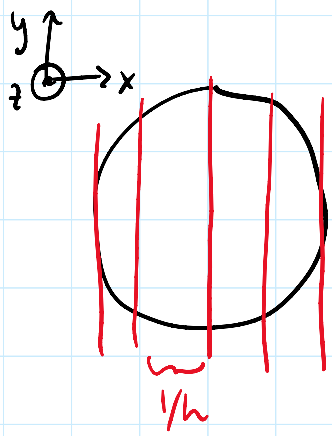
Then $\sum \gamma(x_i) + 3|x|^2 z$ has critical points only in the green regions if slope εh is big. In each of the regions the critical point is unique and Morse if C is big.

Picture for $m=3$ and $q=2$

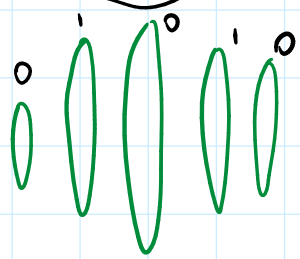


$\dim \Sigma_{\tilde\alpha} = 2$. Seen from top:





perturb $\longrightarrow$



new wrinkles with alternating indices.