Wrinkled Submersions

Lets now consider the case $m \geq q$:

- $FSubm(M,Q) := \left\{ (f,F) \,\middle|\, \begin{tikzcd} M \arrow[r, "f"] & Q \\ TM \arrow[u] \arrow[r, "F"] & TQ \arrow[u] \end{tikzcd} \text{ of rank } q \right\}$
- $Subm(M,Q) := \{ f : M \to Q \mid Tf : TM \to TQ \text{ has rank } q \}$

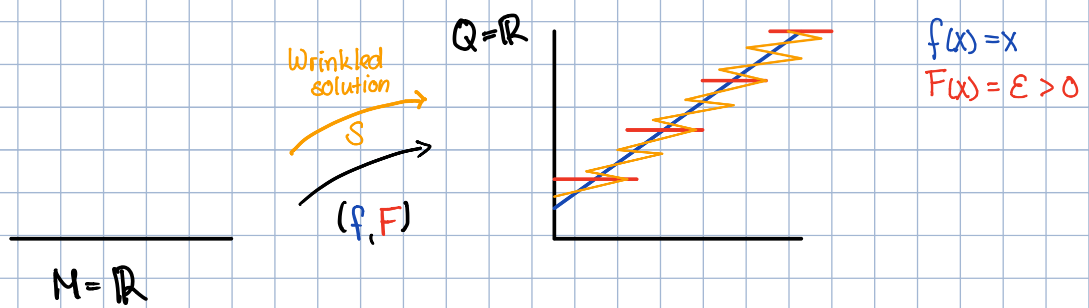
And ask the analogous question:

$$\text{Is } \begin{array}{ccc} Subm(M,Q) & \longrightarrow & FSubm(M,Q) \\ f & \longmapsto & (f, Tf) \end{array} \text{ a w.h.e. ?}$$

Ex 1: Take M closed, $Q = \mathbb{R}^q$ $\Rightarrow$ no actual submersion $f: M \to Q$ possible, as $\|f\|$ has a local maximum.

Thus the answer to the above question is NO!

Ex 2:



Goal: Define a set $WSubm(M,Q)$ (wrinkled) of subm. with "simple" singularities s.t.

$$WSubm(M,Q) \hookrightarrow FSubm(M,Q)$$
"some inclusion"

is a weak homotopy equivalence.

Stratification of $(M \times Q)^{(1)}$

If $p \in M$ is a singularity of $f: M \to Q$, then $\operatorname{rank} Tf < q$.

Observe:

$$\Sigma^{a}_{q \times m} := \{\, q \times m \text{ matrices of } \overset{= q - \text{rank}}{\text{corank}}\ a \,\}$$

is a $(q-a)\cdot q + (m-q+a)\cdot(q-a) = mq - a(m-q+a)$ dimensional closed submfd of $M_{q \times m}$.

Definition:

A closed subset S of a mfd. M is called stratified if:

- $S = \bigcup_{j=0}^{N} S_j$, S_j locally closed submfds
- $\overline{S_k} = \bigcup_{j=k}^{N} S_j$

The set $M_{q \times m}$ is stratified by

$$M^{a}_{q \times m} := \bigcup_{i=a}^{q} \Sigma^{i}_{q \times m}$$

and this stratification induces a stratification of 1-jets

$$(M \times Q)^{(1)} \longrightarrow M.$$

Thom-Boardman theorem:

Let $X \to M$ be a smooth fibration, and Σ a stratified subset of the jet-space $X^{(r)}$. Then for a generic section $f: M \to X$ its jet-extension $J_f^r: M \to X^{(r)}$ is transversal to Σ.

Rmk: This means that the stratification of $(M\times Q)^{(1)}$ pulls back to a stratification of M, as $J_f^{(1)}$ is transverse to all strata.

We define
$$M^a := \{p\in M \mid J_f^{(1)}(p) \in \Sigma^a_{q\times m} \longleftrightarrow \begin{matrix} Tf: T_pM \to T_{f(p)}Q \\ \text{has corank } a \end{matrix} \}.$$

these are the strata of the stratification of M. Observe that
$$\operatorname{codim}(M^a \subseteq M) = \operatorname{codim}(\Sigma^a_{q\times m} \subseteq M_{q\times m})$$

$f: M^m \to N^n$ transv to $S \subset N$
$\Rightarrow f^{-1}(S) \subset M$ submfd
& $\operatorname{codim} f^{-1}(S) = \operatorname{codim} S$

Similarly one defines
$$M^{a,b} := \{p\in M^a \mid \begin{matrix} T_pf: T_pM^a \to T_{f(p)}Q \\ \text{s.t. } \dim\ker T_pf = b \end{matrix} \}$$

corank + dim $f(M)\cap S$
$m - rk + rk + s - n = m - n + s$
$\Rightarrow$ codim = $n - s$

Cor: A generic function $f: M \to \mathbb{R}$ is Morse.

proof:
- $\operatorname{codim}(\Sigma^1_{m\times 1} \subset M_{m\times 1}) = m$

  $\Rightarrow \operatorname{codim}(M^1) = m$ for generic f

  $\Rightarrow$ a generic fct. has isolated singularities

- J_f^1 is transverse to the strat. of $(M\times\mathbb{R})^{(1)}$

  $\Rightarrow T_p J_f^1(T_pM) \oplus T_{J_f^1(p)}\Sigma^1 = T_{J_f^1(p)}(M\times\mathbb{R})^{(1)}$ $\quad p\in M$ critical point of f

  subbundle of $(M\times\mathbb{R})^{(1)}$ of corank 1 matrice

  $\Rightarrow rk\, T_p J_f^1(T_pM) = rk\, \mathrm{Hess}_p(f) = m$

  $\Rightarrow p$ is non-deg.

Obs: • $\operatorname{codim}(\Sigma^a_{q\times m}) = a(m-q+a) \leq m$

$\overset{a\geq 2}{\iff} \quad m \leq \dfrac{aq-a^2}{a-1}$

$\Rightarrow$ There are no higher sing. $(a \geq 2)$ if $m \gg q$.

Our goal is to construct maps without higher singularities, i.e.

$M^a = \emptyset$ for all $a > 1$

Simple singularities: folds, cusps, wrinkles

We will now introduce wrinkles a certain type of singularities in M^1.

Sing. are local, we will thus describe them in local coordinates,

i.e. consider $f|_B : B \overset{\text{open}}{\subset} M \to Q$, where B is small enough to be covered by a single chart ($B \cong \mathbb{R}^m$).

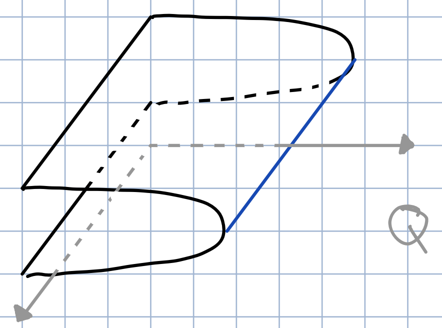
A. Folds

$m \geq q \geq 1$

$$\begin{aligned} \mathbb{R}^{q-1} \times \mathbb{R}^{m-q+1} &\longrightarrow \mathbb{R}^{q-1} \times \mathbb{R} \\ (y, x) &\longmapsto \left(y, -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{m-q+1} x_i^2\right) \end{aligned}$$

k is called the index.

Ex $q = m = 2$, $k = 1$



$\rightsquigarrow$ the singular locus is $B^1 = \{x = 0\} \subseteq M^1$

All points in a fold belong to $B^{1,0}$.

B. Cusps

$m \geq q \geq 2$

$$\begin{aligned} \mathbb{R}^{q-1} \times \mathbb{R}^{m-q} \times \mathbb{R} &\longrightarrow \mathbb{R}^{q-1} \times \mathbb{R} \\ (y, x, z) &\longmapsto \left(y, z^3 - 3yz - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{m-q} x_i^2\right) \end{aligned}$$

the derivative is:

$$\left(\begin{array}{cc|cccc} \multicolumn{2}{c|}{Id} & \multicolumn{4}{c}{0} \\ \hline -3z & 0 & -2x_1 & \cdots & 2x_{m-q} & 3(z^2 - y) \end{array}\right)$$

thus the singular locus is $B^1 = \{z^2 = y, x = 0\} \subset M^1$ a smooth submfd of dim. $q-1$, however its image is cuspidal!

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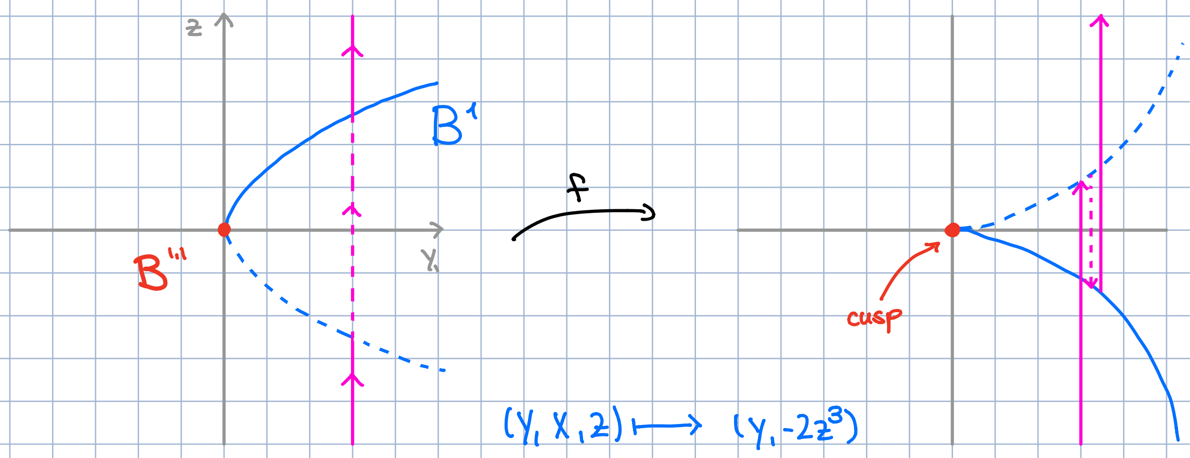
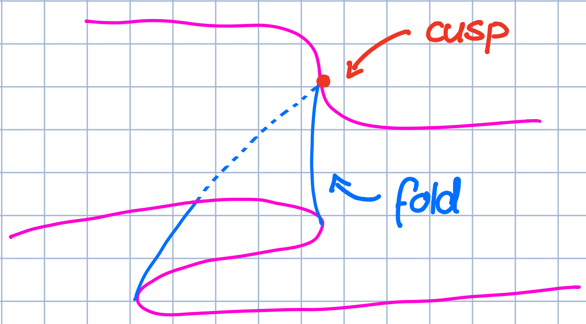
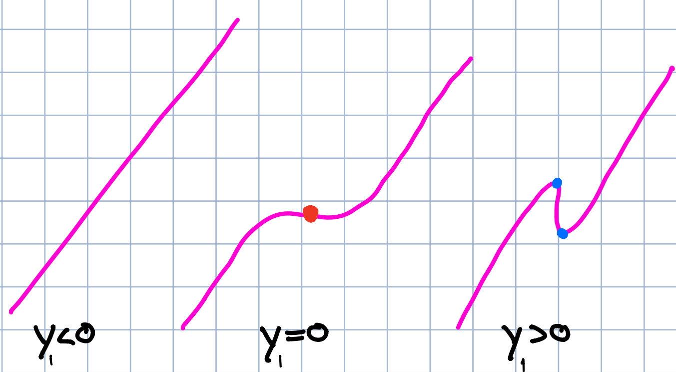


The cusps take place in $B^{1,1} = \{z=0, y_1=0, x=0\} \subseteq M^{1,1}$.
It is easier to visualize if we add a 3rd dimension.



This can be thought of as a family of functions

$$(y_1, x, z) \longmapsto (y_1, f_{x,y}(z))$$



C. Wrinkle

$m \geq q \geq 1$

$$\mathbb{R}^{q-1} \times \mathbb{R}^{m-q} \times \mathbb{R} \longrightarrow \mathbb{R}^{q-1} \times \mathbb{R}$$

$$(y, x, z) \longmapsto \left(y,\ z^3 + 3(|y|^2-1)z - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{m-q} x_i^2\right)$$

The derivative is:

$$\left(\begin{array}{ccc|ccc} & \mathrm{id} & & & 0 & \\ \hline 6y_1 z & \dots & 6y_{q-1}z & -2x_1 \dots & +2x_{m-q} & 3(z^2+|y|^2-1) \end{array}\right)$$

$$\Rightarrow \quad B^1 = \{x=0,\ z^2+|y|^2=1\}$$

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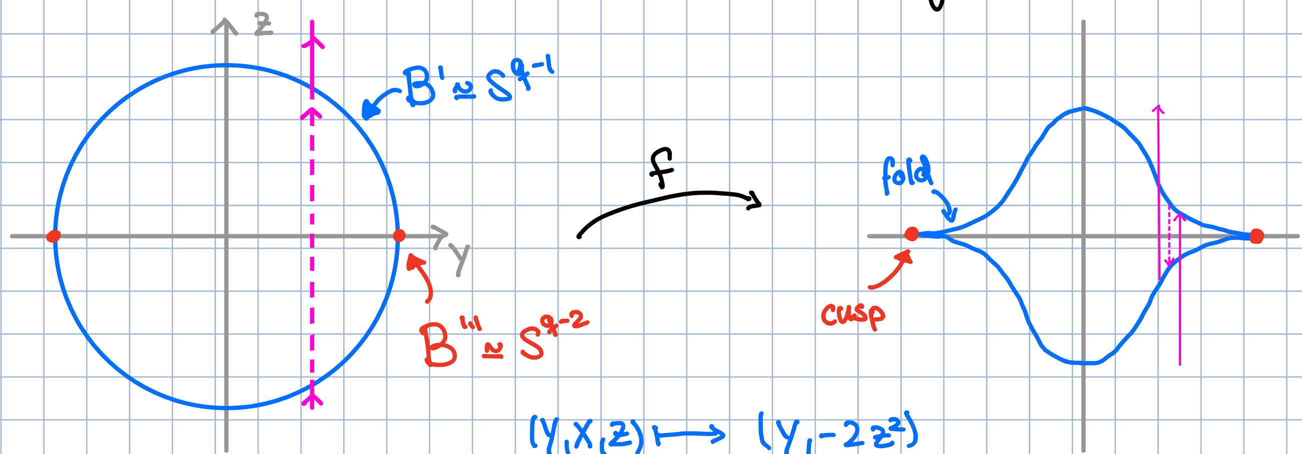
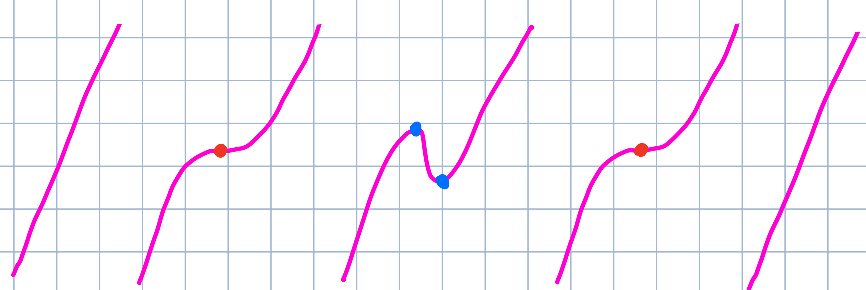


Thus the cusps take place in $B'' = \{z=0,\ |y|^2=0,\ x=0\}$.

A wrinkle thought of as a family of functions is



A more general example of a wrinkle could be:

pairs are born and then they cancel in a different way

This corresponds to:

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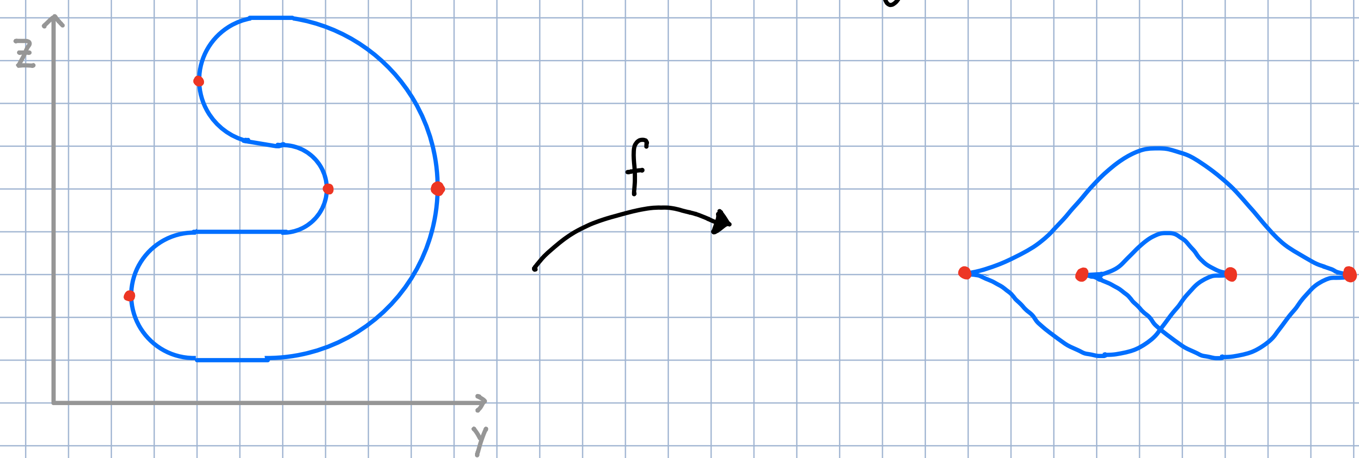
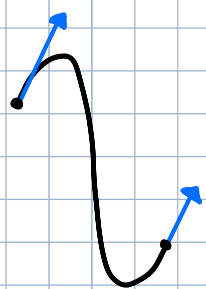
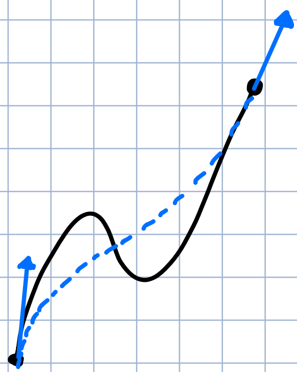


<u>Def</u> $f|_B : B \to Q$ is a wrinkle if it agrees with the given model up to diffeo. in domain and target.
In particular it agrees with the model on an arb. small nbhd. of $D = \{z^2 + |y|^2 \leq 1\}$. D is called membrane.

Rmk: Wrinkels look like the singularities could be canceled. But in general this is not the case:



There is no increasing fct. agreeing with at the end-points.



But there are wrinkles that can be canceld, but this requires finding a "large" nbhd.

D Embryos

Conversly we can introduce a wrinkle if there was none:

$$\mathbb{R}^{q-1} \times \mathbb{R}^{m-q+1} \times \mathbb{R} \longrightarrow \mathbb{R}^{q-1} \times \mathbb{R}$$

$$(y, x, z) \longmapsto \left(y,\ z^3 - 3|y|^2 z - \sum^{k} x_i^2 + \sum_{k+1} x_i^2\right)$$

By perturbing this map to yield

$$(y, x, z) \longmapsto \left(y,\ z^3 - 3(|y|^2 - \varepsilon) z - \sum^{k} x_i^2 + \sum_{k+1} x_i^2\right)$$

then:
- if $\varepsilon > 0$ a wrinkle appears
- if $\varepsilon < 0$ the map becomes regular

Wrinkled submersions

$$WSubm := \left\{ f : M \to Q \;\middle|\; \begin{array}{l} f|_{M \setminus (\sqcup B_i)} \text{ is a submersion} \\ f|_{B_i} \text{ is a wrinkle} \end{array} \right\}$$

Regularisation:

We claim

$$WSubm(M,Q) \xhookrightarrow{\text{inclusion by regularisation}} FSubm(M,Q).$$

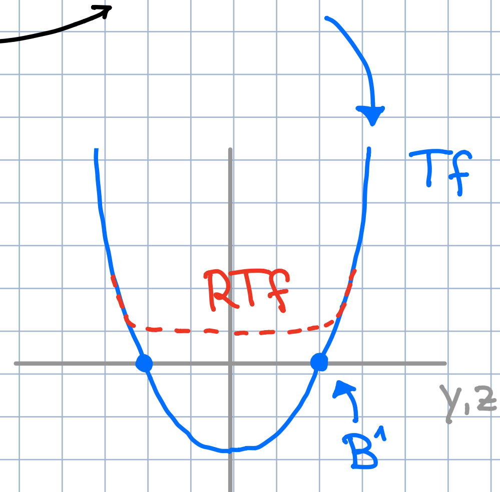
Consider the differential of a wrinkle:

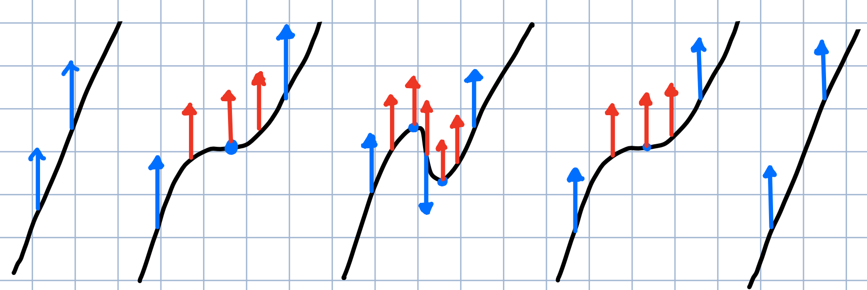
$$\left(\begin{array}{ccc:ccc}
 & id & & & 0 & \\ \hdashline
6y_1 z & \dots & 6y_{q-1} z & -2x_1 & \dots & +2x_{m-q} \quad \boxed{3(z^2+|y|^2-1)}
\end{array}\right)$$

We can modify this term around the membrane to obtain a map

$$RTf : TM \longrightarrow TQ$$

corresponding to a formal submersion.





We can finally state the main theorem, which Gabriele will prove next week!

Theorem:

The regularization map

$$\mathcal{W}\mathrm{Subm}(M,Q) \hookrightarrow \mathcal{F}\mathrm{Subm}(M,Q)$$

is a weak homotopy equivalence.

Thom-Boardman theorem:

Let $X \to M$ be a smooth fibration, and Σ a stratified subset of the jet-space $X^{(r)}$. Then for a generic section $f: M \to X$ its jet-extension $J^r_f : M \to X^{(r)}$ is transversal to Σ.

at least in C^r

Idea of proof:

1.) proof the case $\operatorname{codim} \Sigma > m$, because then transv. means $\Sigma \cap J^r_f(M) = \emptyset$.

- consider $M = D^m$ a closed disc

C^0-topology

$\Rightarrow$ $Sec(X^{(r)} \setminus \Sigma) \subset Sec(X^{(r)})$ open as $\Sigma \subset X^{(r)}$ closed and D^n compact.

$\Rightarrow$ $Hol(X^{(r)} \setminus \Sigma) \subset Hol(X^{(r)})$ open

$\Rightarrow$ need to show $Hol(X^{(r)} \setminus \Sigma) \subset Hol(X^{(r)})$ everywhere dense

- Take $F: D^m \to X^{(r)}$ hol. sec.

$\Rightarrow$ F has hol trivialized tub. nbhd.

$$Y = D^m \times P_r(m,q) = D^n \times \mathbb{R}^K.$$

polynomials $\mathbb{R}^n \to \mathbb{R}^q$ of degree $\leq r$

Let $\pi: Y \to \mathbb{R}^K$ proj.

$\Rightarrow$ $\pi(\Sigma) \subseteq \mathbb{R}^K$ consists of all hol. sect. which are non-transversal to Σ.

- $\dim \Sigma \leq \dim X^{(r)} - m = K$

$\Rightarrow$ $\pi(\Sigma) \subset \mathbb{R}^K$ has measure zero

$\Rightarrow$ $\mathbb{R}^K \setminus \pi(\Sigma)$ is everywhere dense in $\mathbb{R}^K$

$\Rightarrow$ Any open nbhd of F contains a section $\tilde{F} \in Hol(X^{(r)} \setminus \Sigma)$

2.) Denote $\Sigma^{(1)} \subseteq X^{(r+1)}$ subset of $(r+1)$-jets of sections $f: O_p V \to X$ for which J_f^r is not transversal to Σ.

<u>Lemma:</u> If $\operatorname{codim} \Sigma \leq m$, then $\Sigma^{(1)}$ is a stratified subset of $X^{(r+1)}$ of codim $m+1$.

3.) *Observe that:*

$$J_f^{(r)} \in \mathrm{Hol}\, X^{(r)} \text{ transv. to } \Sigma \iff J_f^{(r+1)} \in \mathrm{Hol}(X^{(r+1)} \setminus \Sigma^{(1)})$$

Thus the case $\operatorname{codim} \Sigma \leq m$ can be reduced to the previous one.