The search for a C^1-close h-princ. for emb

$\varphi : W\mathrm{emb} \hookrightarrow F\mathrm{emb}$ w.h.e. and C^1-close.

(more than for open mnfd's).

<u>Recall</u>: the grassmanian bundle $Gr_d(V)$ of d on a smooth mnfd V is defined through:

$\pi : Gr_d V \longrightarrow V$, with fibers

$\pi^{-1}\{p\} = Gr_d(T_pV)$ ⇐ grassmanian mnfd. of T_pV (tang. to T_pV)

Note that a map $F : TV \longrightarrow TW$ (tangl.; bundle map. mono.) induces a map

$GF : V \longrightarrow Gr_n W$, where $n = \dim V$

$p \longmapsto F(T_pV)$

<u>Main theorem</u> (Disc; V, W orient)

- $f_k : V \to W$, $k \in K$ a fam. of emb.
- $F_{k,s} : V \longrightarrow Gr_n W$ (or. grassm. bundle) be fam. of n-plane fields lifting

$$\begin{array}{ccc} V & \longrightarrow & Gr_n W \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

f_k, $F_{k,0} = f_k$, $s \in [0,1)$.

$\Longrightarrow \exists$ fam. of wrinkled emb $g_{k,s} : V \longrightarrow W$ s.t.

$g_{k,0} = f_k$ and f_k and $g_{k,s}$ are C^0-close and $F_{k,s}$ and $G(Dg_{k,s})$ are C^0-close

<u>Addit</u>: if $F_{k,s} = G(Dg_{k,s})$ on $U \times K' \subseteq V \times K$

$\leadsto$ $f_k = g_{k,s}$ on $U \times K'$

<u>Rmk.</u>: We take the tangent plan $\underbrace{Df_k(T_pV)}_{= G(Df_k)(p)}$ and move them via $F_{k,s}$ in $Gr_n W$. We would like to move f so that it remains close to tangent to $F_{k,s}$

In gen. this is not true see for ex.

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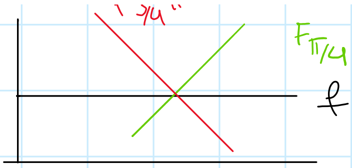
$f(x) = (x, 1)$



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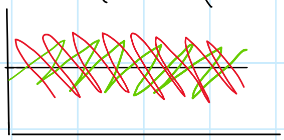

wrinkled emb.

Def.: a $f: V \to W$, $\dim V = n < \dim W = m$ is called wrinkled emb if

1) f is top. emb.

2) For each conn. comp S_i of $\Sigma(f)$ we have $S_i \cong S^{n-1}$ and bound a disk.

3) f is near S_i diffeo to

$\omega: \mathcal{O}_p(S^{n-1}) \longrightarrow \mathbb{R}^n$ — wrf. fct.

$$(x_1, \ldots, x_{n-1}, z) \longmapsto \left(x_1, \ldots, x_{n-1}, z^3 + 3(|x|^2 - 1)z, \int_0^z (s^2 + |x|^2 - 1)\, ds, 0 \ldots 0,\right)$$

Note that S_i or $f(S_i)$ are called wrinkles.

Now restr. for a wrinkle S_i, ω to S^{n-1} yields:

$$(x_1, \ldots, x_{n-1}, \pm\sqrt{1-|x|^2}) \longmapsto (x_1, \ldots, x_{n-1}, \mp z(1-|x|^2)^{3/2}, 0, \ldots, 0)$$

$\forall$ wrinkles S_i there $\exists$ a equator S_i' such that:

$\quad\quad (S_i' \cong S^{n-1} \cap \{z = 0\})$

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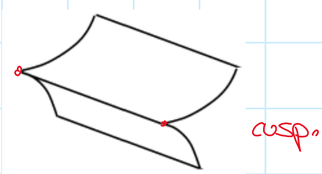

(in Johanna's term. a fold)

2) Near the equator f is given by

$$(x_1, \ldots, x_{n-1}, z) \longmapsto (x_1, \ldots, x_{n-1}, z^3 - 3x_1 z, \int^z (s^2 - x_1)^2 ds, 0, \ldots, 0)$$

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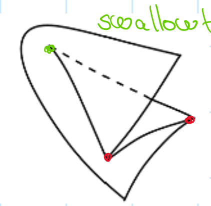

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Regularization

Changing the unfold. fct. $u(x,z) = \int_0^z (s^2 + |x|^2 - 1)^2 ds$ to a C^1-close fct. $\hat{u}(x,z)$ s.t. $\partial_z \hat{u}(x,z) > 0$ and $\hat{u} \equiv u$ on $\Sigma(f) \setminus \mathcal{O}_p S^1$.

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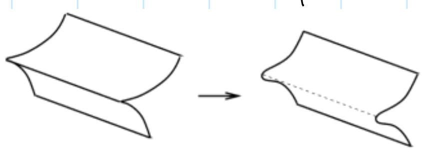	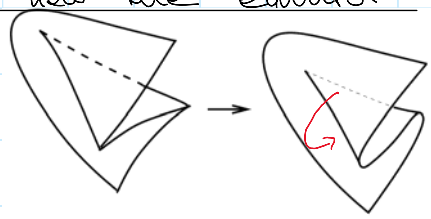

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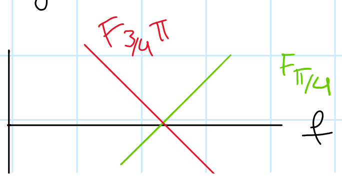

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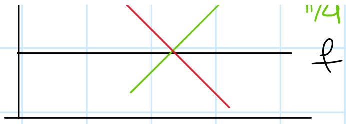

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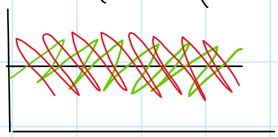

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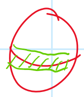

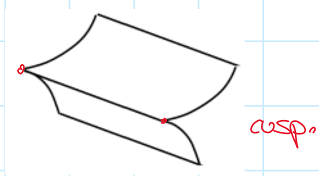

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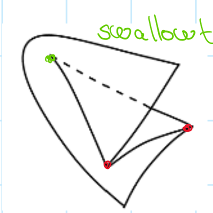

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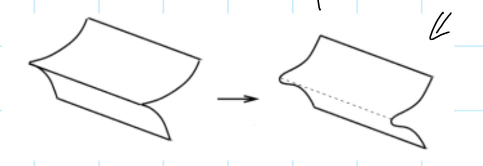

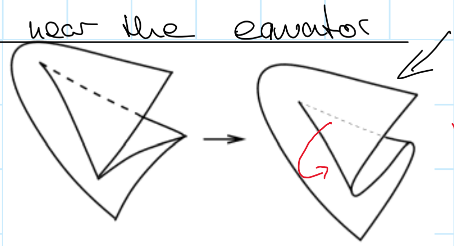

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3. Proof of main thm in the non par. case

<u>thm:</u> $f_k: V \to W$ wrinkled emb. ← rotation less than $\pi/4$

$F_{s,k}: V \to Gr_n W$ be fam. of n-plane fields lifting

$f: \tilde V \to \tilde W$ f with $F_0 = G(Df)$, $s \in [0,1)$

$\Longrightarrow$ $\exists$ fam. of wrinkled emb. $g_s : V \longrightarrow W$ s.t.
$g_0 = f$ and f and g_s are C^0-close and F_s & GDg_s C^0-close

Add. If $F_s = GDg_s$ on $U \Rightarrow f = g_s$ on U.

(Tangent rota. is a comp. of finite rot. with angle less than $\pi/4$).

Proof: $(W, g_W) = (\mathbb{R}^m, dx_1^2 + \dots + dx_m^2)$

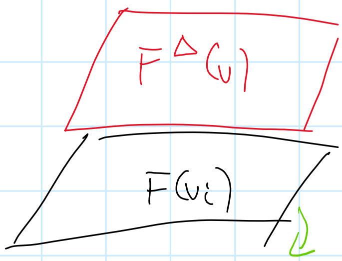
1) Choose a triang Δ of V s.t. $S^1 \subseteq (n-2)$ skeleton
$\Sigma(f) = S \subseteq (n-1)$ skeleton

2) Given a triang Δ and map $F : V \longrightarrow Gr_n W$ we define

$$F^{\Delta}|_{\Delta_i^n}(v) \parallel F|_{\Delta_i^n}(v_i)$$

bary center

$F^{\Delta}|_{\Delta_i^n} : V \longrightarrow Gr_n(W)$



(F^{Δ} is multivalued over $(n-1)$ skeleton).

Case 1 $\dim V + 1 = \dim W$

Choose a triang Δ of V s.t. the simplices are so small s.t. for all i $f(\Delta_i^n)$ is arb C^1-close to $F_0(v_i)$ and $\forall s$ F_s^{Δ} is arb. C^0-close to F_s

$\longrightarrow$ Instead of approx. F_s we approx F_s^{Δ}

Now we make use of the following thm.

thm 2.6.1 $F_s : V \longrightarrow Gr_n W$ n-plane field lifting a emb.
$f : V \longrightarrow W$ with $F_0 = G(Df)$, $s \in [0,1]$,
f is wrinkled emb.

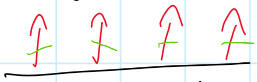
$\Longrightarrow$ $\exists$ arb. C^0-small graph. isotopy of wrinkled emb
$f_s : V \longrightarrow W$ s.t.

$GDf_s|$ is arb C^0-close to $F_s|$

$f_s : V \longrightarrow W$ s.t.

$GDf_s|_{\mathcal{O}_p \Sigma_i}$ is arb. C^0-close to $F_s|_{\mathcal{O}_p \Sigma_i}$.

graph. means in that sense $f_t \pitchfork$ normal fol $\mathcal{V}$ on V



So we apply this thm to $(n-1)$ skeleton $\overset{\Delta^{n-1}}{\vee}$ of Δ

$\Longrightarrow \exists$ graph. isotopy $\tilde{f}_s : V \longrightarrow \mathbb{R}^{n+1} = W$, $\tilde{f}_0 = f$ s.t.

$GD\tilde{f}_s|_{\mathcal{O}_p \Delta^{n-1}}$ is arb. C^0-close to $G_s^\Delta|_{\mathcal{O}_p \Delta^{n-1}}$

Note that : $S \subseteq \mathbb{R}^{n+1}$ or. hypersurface $\tilde{G} : S \longrightarrow S^n$

1) alm. hor. $\tilde{G}(S) \subseteq B_\varepsilon(n) \subseteq S^n$ (north pole) (angle metric)
2) graph $\tilde{G}(S) \subseteq B_{\pi/2}(n) \subseteq S^n$
3) alm. graph. $\tilde{G}(S) \subseteq B_{\pi/2+\varepsilon}(n) \subseteq S^n$
4) quasi graph $\tilde{G}(S) \subseteq B_\pi(n) \subseteq S^n$

Since $f(\Delta_i^n)$ is almost graph. w.r.t. $F_0^\Delta(v_i)$ and $\tilde{f}_s$ is graph.

$\Longrightarrow$ $\tilde{f}_s(\Delta_i^n)$ is alm. grap. w.r.t $F_0^\Delta(v_i)$

Since the angle between $F_s^\Delta(v_i)$ and $F_0^\Delta(v_i) < \pi/4$

$\rightsquigarrow$ $\tilde{f}_s(\Delta_i^n)$ is alm. graph. w.r.t $F_s^\Delta(v_i)$

- Moreover $\tilde{f}_s(\Delta_i^n)$ is alm. hor. w.r.t. $F_s^\Delta(v_i)$ near $\partial \tilde{f}_s(\Delta_i^n)$ (Since $GD\tilde{f}_s|_{\Delta_i^n}$ is arb. close to $G^\Delta|_{\Delta_i^n}$)

$\Longrightarrow$ So we can apply the following Lemma :

Lemma 2.7.3 Let $S_t \subseteq \mathbb{R}^{n+1}$ $t \in I$ a fam. of or. quasi graph. hypersurfaces s.t S_t is almost hor. for $t=0$ and S_t is almost hor. near the bdy ∂S_t $\forall t \in I$

$\Longrightarrow \exists$ C^0-approx of family of emb. $i_{S_t} : S_t \longrightarrow \mathbb{R}^{n+1}$ by a family of alm. hor. wrinkled emb.

$f_t : S_t \longrightarrow \mathbb{R}^{n+1}$ [with depth ≤ 1]

s.t. 1) $f_t = i_{S_t}$ $t=0$ 2) $f_t = i_{S_t}$ near ∂S_t $\forall t \in I$

We apply Lemma 2.73 over each simplex to $S_t = \tilde{f}_t(\Delta_i^n)$ and $F_t^\Delta(v_i)$ as the hor. plane.

$\Rightarrow$ We get a fam. of wrinkled emb.
$f_t : \hat{f}_t(\Delta_i^n) \to W$ which is almost hor. w.r.t. $F_t^\Delta(v_i)$

$\Rightarrow$ f_t is C^0 close to $\hat{f}_t$ and $GDf_t|_{Op\,\Delta_i^n}$ is C^0-close to $F_t^\Delta|_{\Delta_i^n}$

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<u>Case $n+1 < m$</u> Δ triang. of V $\forall i$ $f(\Delta_i^n)$ is arb. C^1-close to $F^\Delta(v_i)$

$\rightsquigarrow$ over each simplex we can work with the proj. of the isotopy $\tilde{f}_t$ to $L_{v_i} \cong \mathbb{R}^{n+1}$

$\hat{}$ this is the $(n+1)$-dim subspace where the rot. goes on.

$\to$ apply to this case 1 $\square$