S-immersions

Review: $M^m \to Q^m$ (oriented), $\mathrm{FSubm}(M,Q) \neq \emptyset \Rightarrow \mathrm{Subm}(M,Q) \overset{w.h.e}{\simeq} \mathrm{FSubm}(M,Q)$

(Johanna, Gabrielle)

It can happen that $\mathrm{FSubm}(M,Q) = \emptyset$, but there exist maps $M \to Q$ with "reasonable" singularities (not all wrinkles)

Exam:

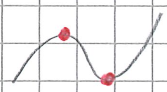
$$\begin{tikzcd} S^2 \arrow[r] \arrow[dr, dashed] & \mathbb{R}^3 \arrow[d, "pr_{x,y}"] \\ & \mathbb{R}^2 \end{tikzcd}$$

otherwise TS^2 would be trivial

map with "simple" singularity, but there are no formal submersions $S^2 \to \mathbb{R}^2$

Def: $S^{m-1} \subset M$ closed submanifold. Then $\underbrace{\mathrm{SImm}(M,Q;S)}$ is the space of S-immersions. $:= \{$maps that are immersions in $M \setminus S$ and S is the set of folds$\}$

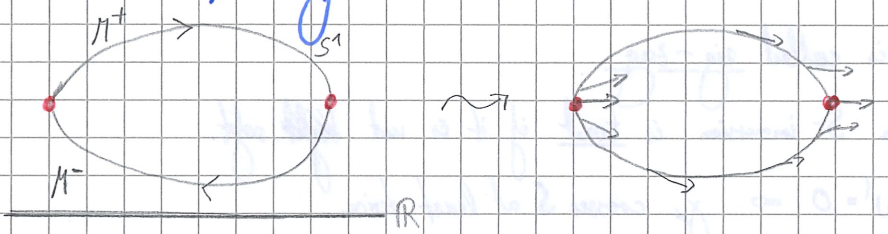
Obs: Two folds in a row can be canceled ("regularised"), but one fold can not



Search: Formal analogue of SImm

The compl. of S is divided in two regions M^+ & M^-, dep. on whether TM is sent to TQ preserving orient. or not.

Exam:



Want to glue TM^+ & TM^- in such a way that the resulting bundle maps surj. onto TQ.

Formally: $\nu \subset TM|_S$ the direction normal to S in M. Glue TM^+ and TM^- over S by setting

$$TM^+| = TS \oplus \nu \xrightarrow[\mathrm{Id} \oplus (-\mathrm{Id})]{} TS \oplus \nu = TM^-|_S$$

Result Folded tangent to M, $T^S M$

Exam: $T^{S^{n-1}} S^n \cong \pi^* T\mathbb{R}^n$

Obs. Still an oriented bundle. Keeping the orientation of TM^+ leads to reversing that of TM^-

Def: Formal S-immersions

$$\mathrm{FSImm}(M,Q;S) = \left\{ (f,F) \;\middle|\; \begin{tikzcd} T^S M \arrow[r, "F"] \arrow[d] & TQ \arrow[d] \\ M \arrow[r] & Q \end{tikzcd} \text{ surj} \right\}$$

Q: There is a natural inclusion

$$S\,\mathrm{Imm}(M,Q;S) \hookrightarrow FS\,\mathrm{Imm}(M,Q;S).$$

Is it a w.h.e.?

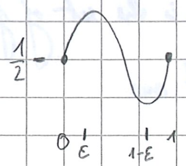
Statement of the result

We distinguish two types of S-immersions: soft and taut.

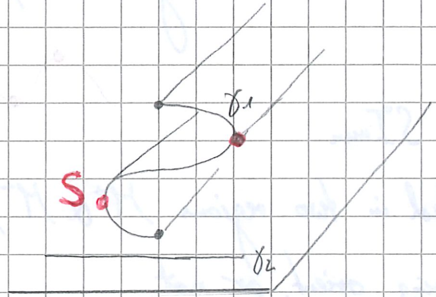
Def: An S-immersion $M \xrightarrow{f} Q$ is *soft* if there are paths

- $\gamma_1: I \to M$ embedded ($\gamma_1(0) \neq \gamma_1(1)$) and transverse to S,
- $\gamma_2: I \to Q$ immersed

s.t. $f \circ \gamma_1 = \gamma_2 \circ w$ where $w: I \to I$ is the 1-dim wrinkle



with $w(0) = w(1)$ (and hence $f \circ \gamma_1(0) = f \circ \gamma_1(1)$).



γ_1 is called *zig-zag*.

An S-immersion is *taut* if it is not ~~tight~~ soft.

Rmk: $w' = 0 \Rightarrow \gamma_1$ crosses S at least twice.

Lemma: $f: M \to Q$ is taut $\Leftrightarrow \exists\, \alpha \neq \mathrm{id}: M \to M$ involution fixing S pointwise with $f \circ \alpha = f$

Sketch of easy implication: „$\Leftarrow$" zig-zag and involution are incompatible

where should p go, if $\alpha|_{\gamma_1} \neq \mathrm{id}$ not pos. for invol $\neq \mathrm{id}$ (Why?)

Corollary: Being soft is an open & closed condition

($\Rightarrow$ soft/taut only depends on the conn. comp.)

Proof: Having invol. is open $\Rightarrow$ taut is open

Having a zig-zag is open $\Rightarrow$ soft is open. ∎

Thm: M closed, connected, $S \neq \emptyset$, $\dim(M) \geq 3$

Then $SImm(M, Q; S)$ can be subdivided by connected comp. in two sets

- $SImm^{soft}(M, Q; S)$
- $SImm^{taut}(M, Q; S)$

h-principle for soft: $SImm^{soft}(M, Q; S) \overset{w.h.e}{\simeq} \mathcal{F}SImm(M, Q; S)$

$$SImm^{taut}(M, Q; S) \cong Imm(M^+, Q) \times Involutions(M, S)$$

Obs: a) Morally: soft $\triangleq$ overtwisted, taut $\triangleq$ tight

b) h-principle for overtwisted/soft is based on flexible model (ot. disc / zig-zag)

c) h-principle for contact: isom in π_0 only

- not always possible to select a cont. family of ot. discs for a family of ot. structures
- always poss. to choose zig-zags coherently for S-immersions
- Key idea: o.t. discs can not always be displ. from one another zig-zags can.

d) tight contact structures need techniques ~~can be understood by h-principle~~ (taut S-immersions can be understood by h-principle

e) This h-principle is closely related to the one for loose Legendrians.

Proof of h-principle for soft S-immersions

Def: Each conn. comp. of $M \setminus S$ is called a chamber

C chamber:
$$WSImm(M, Q; S, C) = \left\{ f \;\middle|\; \begin{array}{l} f|_{M \setminus C \cup Op(S)} \text{ is an } SImm \text{ into } Q \\ f|_C \text{ is a wrinkled submersion} \end{array} \right\}$$

Sketch of proof (in the case $f_0, f_1 \in SImm^{soft}$, $f_t \in \mathcal{F}SImm$ / π_0-case)

~ The zig-zags of f_0, f_1 are different $\to$ make them the same

~ Modify f_t so that they lie in $WSImm$ and have the same zig-zag.

~ use zig-zigs to eliminate the wrinkles introduced

Remark: zig-zags $\triangleq$ loose charts for loose Legendrians

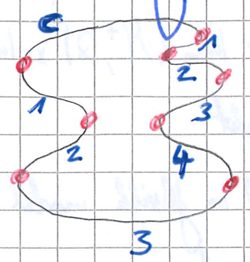
use lang. of twist-markings

Lemma 1: The inclusion

$$WS\mathrm{Imm}(M, Q; S, C) \xrightarrow{\mathcal{I}} FS\mathrm{Imm}(M, Q, S)$$

is a weak homotopy equivalence.

Proof: Assign to each chamber a number (depth) counting the (minimal) number of times a path from C has to cross S to get there



Sort them by non-increasing depth.

Given a family $(D^2, \partial D^2) \xrightarrow{\varphi} (FS\mathrm{Imm}, WS\mathrm{Imm})$, repr. an element in $\pi_2(FS\mathrm{Imm}, WS\mathrm{Imm})$.

Deform φ inductively over the chambers by h-principle for immersions in open manifolds (rel. to prev. chambers)

Because at least one of the ends (of the chamber) has less depth $\rightsquigarrow$ not defined there (yet)

Last chamber (C): use the wrinkled submersion thm. ∎

Lemma 1 reduces the thm. to

$$\pi_2(WS\mathrm{Imm}, S\mathrm{Imm}^{soft}) = 0$$

i.e. every family $(D^2, \partial D^2) \xrightarrow{\varphi} (WS\mathrm{Imm}, S\mathrm{Imm}^{soft})$ can be homotoped to lie completely in $S\mathrm{Imm}^{soft}$.

Overview in 3 Steps

1) Construct a "good" collection of zig-zags for φ (by modifying φ suitably)

Lemma 2: ~~There is a collection of disjoint intervalls $\{\gamma_i : I \to M\}$ with γ_i a zig-zag for each $\varphi(x)$, $x \in U_i$ with $\{U_i\}_0^N$ a cover of D^2.~~

~~There~~ exists a cover $\{U_i\}_0^N$ of D^2 and a collection $\{\gamma_i : I \to M\}$ s.t. γ_i is a zig-zag for each $\varphi(x)$, $x \in U_i$, and γ_i, γ_j are disjoint for $i \neq j$.

Lemma 3: Given a covering $\{U_i\}$ with zig-zags $\{\gamma_i\}$, any subcover (as in Lemma 1) $\{U_i\}$ has a choice of disjoint zig-zags $\{\tilde{\gamma}_i\}$. (meaning: refinement)

2) Move this collection so that it touches C.

Lemma 4: Let $\varphi \in WShum$ and $\gamma : [0,2] \to M$ disjoint from wrinkles with $\gamma(2) \in C$ and $\gamma|_{(0,1)}$ a zig-zag. Then there exists $\tilde{\varphi}$ homotopic to φ with $\tilde{\varphi}|_{M \setminus Op(\gamma)} = \varphi$ having a zig-zag touching C and the new curve $\tilde{\gamma}$ has the same image (in M) as γ.

Summary: After deformation $\varphi : (D^2, \partial D^2) \to (WShum, Shum)$ admits a covering $\{U_i\}$ of D^2 with disjoint intervals $\{\gamma_i : I \to M\}$ touching C s.t. γ_i is a zig-zag for each $\varphi(x)$, $x \in U_i$.

Lemma 3 was used, so that the deform. in Lemma 4 can be applied in little balls U_i, where φ was almost constant.

3) Having the zig-zags at C, we make them "grow tentacles" to swallow the wrinkles in C.

Lemma 5: Swallowing of wrinkles

Let $\varphi \in WShum(M, Q; S, C)$, $D \subset C$ the membrane of a wrinkle of φ. Let $\gamma : [0,2] \to M$ be a path s.t.

- $\gamma|_{[0,1]}$ is a zig-zag, $\gamma|_{[1,2]} \subset C$
- $\gamma(2) \in D$ and $\gamma|_{[0,2]}$ is disjoint from the wrinkles

Then φ can be deformed to $\tilde{\varphi} \in WShum(M, Q; S, C)$ in $Op(D \cup \gamma)$ so that $\tilde{\varphi}$ has one wrinkle less.