Junior Prof. Gabriele Benedetti – Research statement

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Research directions

Overview

The study of invariant sets plays a crucial role in the understanding of the properties of a dynamical system: it can be used to obtain information on the dynamics both at a local scale, for example to determine the existence of nearby **stable motions**, and at a global one, for example to detect the presence of **chaos** (see [24]). In the realm of continuous flows, **periodic orbits** are the simplest example of invariant sets and, therefore, they represent the first object of study.

My research is focused on finding periodic orbits for a **Hamiltonian flow** $t \mapsto \varphi_t^H$ on a **symplectic manifold** (W, ω) by means of **variational methods**. Here $H : W \to \mathbb{R}$ denotes the Hamiltonian function of the system, which completely determines the flow and it is constant along its trajectories. When $W = T^*M$ is the phase space of a manifold M, these flows have received special consideration, as they can be used to gain an insight on topological properties of M and of its loop space [4], and to describe the time evolution of a physical system on M [21].

From geodesic to magnetic flows

Concrete examples of such physical systems are **magnetic flows**, which model the motion of a charged particle on a Riemannian manifold (M, g) under the effect of a stationary magnetic field, described by a closed 2-form $\sigma \in \Omega^2(M)$. In this case, T^*M is endowed with a **twisted** symplectic structure

$$\omega_\sigma := dp \wedge dq + \pi^*\sigma,$$

where $dp \wedge dq$ is the canonical symplectic form and $\pi : T^*M \to M$ the foot-point projection. The Hamiltonian function determining the dynamics is $H_g(q, p) = \frac{1}{2}|p|_q^2$, the kinetic energy with respect to g. Magnetic flows represent a fair compromise between concreteness and abstractness. On the one hand, when $\sigma = 0$, they reduce to the geodesic flow of (M, g), and we can ask which properties of geodesic flows still hold for magnetic flows and which new phenomena appear. I investigated this circle of ideas in [12, 9, 13], and I will pursue it further in Projects D, E. On the other hand, magnetic flows serve as a motivation and testing ground for general results in symplectic geometry, as demonstrated in [14] and as it will be explored in Projects A, C, and F.

From the action functional to the action form

The techniques used in my work come from the interplay between differential geometry/topology and the calculus of variation. Classically, given a Hamiltonian system on a symplectic manifold (W, ω), one studies the associated **action functional** $\mathbb{A} : \Lambda W \to \mathbb{R}$ on the free loop space of W. The critical points of $\mathbb{A}$ correspond to the periodic orbits of the system and, under some additional hypotheses, the existence of critical points of particular type is proven by geometrical or topological considerations. In many interesting situations [21], however, such a functional is **not globally** defined. In these cases, one can still find a **closed action 1-form** $\alpha \in \Omega^1(\Lambda W)$ such that **locally** $\alpha = d\mathbb{A}$, and the zeros of α yield the desired periodic trajectories. The lack of a global functional brings new challenges, when it comes to proving **compactness results** and the **existence of multiple periodic orbits**. These problems were tackled in [8, 9, 10, 1, **?**] and they will play a prominent role in Projects A, and F.

Hypersurfaces of contact-type

Since the Hamiltonian function H is constant along the trajectories of the flow, we can separately study the dynamics on each hypersurface $\Sigma := \{H = c\} \subset (W, \omega)$, as c varies in $\mathbb{R}$. The trajectories of $\varphi^H|_\Sigma$ are tangent to the characteristic distribution $\ker \omega|_\Sigma \subset T\Sigma$. In particular, if we discard their parametrisation, they depend only on the hypersurface Σ and not on the particular function H. In 1979, Alan Weinstein formulated a celebrated conjecture asserting that for a wide class of hypersurfaces Σ in symplectic manifolds, the characteristic distribution **always carry a periodic orbit** [31]. They are the so-called **contact-type hypersurfaces**, namely those which admit a contact form $\lambda \in \Omega^1(\Sigma)$ such that $d\lambda = \omega|_\Sigma$. When this happens, the flow φ^H is equivalent to the **Reeb flow** φ^λ associated to λ. As the conjecture has been proved in many interesting cases (for instance, when Σ has dimension three [27]), it is imperative to decide if a hypersurface in a dynamical system of interest is of contact-type. Hypersurfaces of geodesic flows are easily seen to be of contact-type, while the situation for magnetic flows is more subtle. We have studied this problem for magnetic flows on the two-sphere in [12] and we will try to answer some remaining open questions in Project D.

Ten years after Weinstein, Helmut Hofer and Eduard Zehnder began studying the relationship between **symplectic** invariants (called **capacities**) of an open **domain** $D \subset W$ and **contact** invariants of its **boundary** $\Sigma = \partial D$, provided Σ is of contact-type [20]. They have obtained extraordinary achievements for the case of **convex domains** in euclidean space. In Project B we aim at generalizing their results in two directions: first, by considering Lagrangian-Legendrian boundary conditions; second by working on a more abstract class of symplectic manifolds, namely on **symplectizations of Zoll contact manifolds**. We recall that

- the symplectization of a contact manifold (Σ, λ) is the symplectic manifold $\big(\hat{\Sigma} := \Sigma \times (0, \infty), d(r\lambda)\big)$, where $r \in (0, \infty)$ is the coordinate on the second factor;
- a contact form is called **Zoll**, if its Reeb flow induces a free circle-action.

Concrete research projects

We pursue the research directions outlined above by means of the following six research projects.

A) Systolic inequalities in contact and symplectic geometry

The main aim of **systolic** geometry, initiated by Louwner in 1946 [15], is to answer the following

Question How long can the shortest closed geodesic of a Riemannian metric g on a fixed manifold M be?

In more precise terms, if M is a closed manifold of dimension n and g a Riemannian metric on it, one is interested in finding an upper bound (uniform in g) to the systolic ratio

$$\rho(g) := \frac{\ell_{\min}(g)}{\mathrm{vol}(M, g)^{1/n}}.$$

The bound has been established for large classes of non simply-connected spaces and for all closed surfaces [19, 17]. Consequently, one would like to understand if maximising metrics exist and, if they do, to describe them. Already for $M = S^2$, the situation is far from being understood and the naive guess that the round metric g_0 is a maximizer is false as observed by Croke. Nevertheless, Abbondandolo, Bramham, Hryniewicz and Salomão proved in [2] that g_0 is a **local maximizer** of ρ:

If g is a suitably pinched metric on S^2, then $\rho(g) \leq \rho(g_0)$ and the equality holds if and only if g is a Zoll metric.

To prove this result, the authors translated the problem in the language of contact geometry. Indeed, if (Σ^{2n-1}, λ) is a contact manifold, one can consider the shortest period $T_{\min}(\lambda)$ of a Reeb orbit and the contact volume $\mathrm{vol}(\Sigma, \lambda)$. The systolic ratio of the contact manifold is then defined as

$$\rho(\lambda) := \frac{T_{\min}(\lambda)}{\mathrm{vol}(\Sigma, \lambda)^{1/n}}.$$

The connection between **Riemannian** and **contact** geometry is prompted by the fact that the unit cotangent bundle of (M, g) inside (T^*M, ω_0) admits a contact from λ_g, and the systolic ratio of g and of λ_g are the same up to a universal constant. The systolic ratio was introduced by Álvarez-Paiva and Balacheff for the first time in [7], where a weaker form of the following conjecture is proved.

Local systolic inequality Conjecture *Every Zoll contact form λ_0 on a closed manifold Σ has a C^3-neighborhood $\mathcal{U}$ in the space of contact forms with the following property: if $\lambda \in \mathcal{U}$, then $\rho(\lambda) \le \rho(\lambda_0)$ and the equality holds if and only if λ is Zoll.*

In [3], the conjecture has been established for $\Sigma = S^3$. In a joint work with Jungsoo Kang (WWU Münster), we will settle the three-dimensional case:

Theorem 1 *Let Σ be a closed 3-manifold. There exists a Zoll contact form λ_0 on Σ if and only if Σ is a non-trivial orientable circle bundle over an orientable surface. In this case, the local systolic inequality at λ_0 holds.*

Our proof relies on two main ingredients, which also have potential applications beyond the scope of the present theorem (see Project C):

- the construction of a **surface of section** for the Reeb flow of λ;
- the notion of **generating function** for Hamiltonian diffeomorphisms on surfaces with boundary.

As a sample application of Theorem 1, we mention that it allows for a generalisation of the Riemannian systolic inequality to the magnetic case, when the magnetic field is sufficiently strong.

B) The minimal-period- and the minimal-chord-capacity

Suitable for a PhD Project

Let (Σ, λ_0) be a Zoll contact manifold and $C^\infty_+(\Sigma)$ the space of positive smooth functions on Σ. For every $F \in C^\infty_+(\Sigma)$, we can construct the contact form $\lambda_F := F\lambda_0 \in \Omega^1(\Sigma)$ and the domain $D_F \subset \Sigma \times (0, \infty)$ given by the sub-graph of F. Let use write $\mathcal{C}$ and $\mathcal{D}$ for the set of contact forms and of obtained in this way, respectively. On $\mathcal{C}$, we can define the minimal period of a periodic Reeb orbit $T_{\min}$ and the minimal length of a Reeb chord $T^\Lambda_{\min}$. On $\mathcal{D}$ we can define the Hofer-Zehnder capacity c_{HZ} [20] and the Lagrangian capacity c_L [16].

Question On which subsets $\mathcal{C}' \subset \mathcal{C}$ the functions $T_{\min}, T^\Lambda_{\min} : \mathcal{C}' \to (0, \infty]$ become contact capacities?

We recall that a function $\tau : \mathcal{C}' \to [0, \infty]$ is a **contact capacity** if it is non-trivial, 1-homogeneous, invariant under strict contactomorphisms, and there holds

$$\forall \lambda_{F_1}, \lambda_{F_2} \in \mathcal{C}', \qquad F_1 \le F_2 \quad \Longrightarrow \quad \tau(\lambda_{F_1}) \le \tau(\lambda_{F_2}).$$

Hofer and Zehnder proved in [20] that if $\Sigma = S^{2n-1}$, then $T_{\min}$ is a contact capacity on the space $\mathcal{C}_{\mathrm{conv}}$ of convex contact forms and that $c_{HZ}(D_F) = T_{\min}(\lambda_F)$, if $\lambda_F \in \mathcal{C}_{\mathrm{conv}}$. Together with Jungsoo Kang, we plan to employ a variational setting introduced by Weinstein in [30] to prove the following version of the Hofer-Zehnder theorem for arbitrary Zoll manifolds:

Theorem 2 *Let λ_0 be a Zoll contact form on a closed manifold Σ. There exists a C^2-neighbourhood $\mathcal{U}$ of λ_0 in the space of contact forms on Σ such that*

- $T_{\min}|_{\mathcal{U}}$ *is a contact capacity,*
- $\forall \lambda_F \in \mathcal{U} \quad c_{HZ}(D_F) = T_{\min}(\lambda_F)$.

We also expect that c_L and $T^\Lambda_{\min}$ are the protagonists of a similar story. In this case, the first step will be to look at convex domains in $\mathbb{R}^{2n}$ (since there is still no analogue of Hofer-Zehnder Theorem in this situation). This last result could be an excellent topic for a PhD project since it is connected with important results in the field, it requires a good knowledge of classical techniques and could lead, in a second stage, to a generalization of Theorem 2 for Lagrangian-Legendrian boundary conditions.

C) Surfaces of section for strong magnetic fields: the Conley conjeture and the Twist condition

Going beyond the Weinstein conjecture, one can ask what is the minimum number of periodic orbits that a Reeb flow on a contact manifold can have. On one hand, there is a class of contact manifolds admitting a Reeb flow with only **finitely many** periodic orbits [6]. On the other hand, one can try to identify some topological conditions assuring that every Reeb flow on a given contact manifold has **infinitely many** periodic orbits. A manifestation of this second research direction (known as the Contact Conley Conjecture) is a recent result of Ginzburg, Gürel, and Macarini showing that if (Σ, λ_0) is a Zoll contact manifold such that the orbit space $M := \Sigma/S^1$ of the Reeb flow of λ_0 is aspherical, then for all index-admissible $f : \Sigma \to (0, \infty)$, the Reeb flow of $\lambda := f\lambda_0$ possesses infinitely many periodic orbits [18]. Combining the existence of a surface of section (see Project A) with the theory of Hamiltonian homeomorphisms on closed surfaces as developed in [22], we can give an alternative proof of this result, when M is a surface different from S^2 and λ is C^3-close to λ_0. Our methods yield a surface of section S also when $M = S^2$. However, in this case there are examples of Reeb flows with only two periodic orbits (see [13] for an example in the magnetic category) as S could be an annulus. On the other hand, having some non-resonant condition on the function f allows us to prove that the return map on S is a twist map and, hence, by Poincaré Last Geometric Theorem, that the Reeb flow of λ has infinitely many periodic orbits. This abstract argument would have the following new application to magnetic systems.

Theorem 3 *Let g be a Riemannian metric on S^2 and $f : S^2 \to (0, \infty)$ a positive function. Let μ be the area form of g and take $\sigma := f\mu$ as magnetic form. For a critical point $q \in S^2$ of f, we write $\rho_f(q) := \det \mathrm{Hess}(1/f)$, where the Hessian is taken with respect to σ-symplectic coordinates. Suppose that either*

- *the function f has a critical point, which is non-degenerate and is not a global maximum or minimum of f, or*
- *the function f has a non-degenerate global minimum q_- and a non-degenerate global maximum q_+ such that*

$$\rho_f(q_-) + \rho_f(q_+) \neq 0.$$

Then, there exists $c_* > 0$ such that, for all $c \in (0, c_*)$, there are infinitely many periodic orbits of the magnetic systems with kinetic energy equal to c.

D) Magnetic curvature and hypersurfaces of contact-type

This project aims at performing a **numerical investigation** of a conjectural relation between the magnetic curvature of (M, g, σ) at energy $c > 0$, and the contact property of the hypersurface $\Sigma_c := \{|p|^2/2 = c\}$ inside (T^*M, ω_σ), when M is an oriented surface. If $f : M \to \mathbb{R}$ is the function given by $\sigma = f\mu$ (here μ is the area form of g), the **magnetic curvature** at level c is defined as

$$K_{g,\sigma,c} : \Sigma_c \to \mathbb{R}, \qquad K_{g,f,c}(q, v) = 2cf^2(q) - d_q f(v^\perp) + K(q),$$

where $v^\perp$ is the vector obtained after rotating v by $\pi/2$ and K is the Gaussian curvature of g. Such a quantity arises naturally, when considering the linearization of the magnetic flow restricted to the energy level. Furthermore, as observed in [11], when $M = S^2$ and the magnetic system has a rotational symmetry, if $K_{g,\sigma,c} > 0$, then there are only two periodic orbits at level c, which are invariant by the symmetry, and they have positive action. A useful criterion due to McDuff [23] ensures that if the action of *all* invariant measures is positive, then the level is of contact type. Therefore, we have the following

Question Does $K_{g,\sigma,c} > 0$ imply that $\Sigma_c \subset (T^*M, \omega_\sigma)$ is of contact type?

In the presence of a symmetry, the invariant measures can be explicitly described, as the dynamical system is integrable. However, their action is expressed by an integral that can be only numerically computed. Thus, as a first step towards giving an answer to the question above, we have the following

Goal Determine numerically the action of the invariant measures of a significant family of rotationally symmetric magnetic systems on S^2 with positive magnetic curvature.

E) A relation between Magnetic systems on S^2 and Symplectic structures on $S^2 \times S^2$

From an idea of Peter Albers Suitable for a Master Thesis

Let σ be the area form of the round metric on S^2 and, for every $r > 0$, let $D^*_2 S^2$ be the co-disc bundle of S^2 of radius r with respect to the round metric. It is an unpublished result of Zapolski that there exists an explicit symplectomorphism

$$(D^*_2 S^2, \omega_0) \to (S^2 \times S^2 \setminus \Delta, \sigma \oplus \sigma),$$

where Δ is the diagonal of $S^2 \times S^2$. Zapolski's construction has a natural generalization to co-disc bundles in the twisted cotangent bundle of S^2. More precisely, the following theorem holds.

Theorem 4 *For every non-negative real number $a \geq 0$, there exists a symplectomorphism*

$$(D^*_{2(1+a)} S^2, \omega_{a\sigma}) \to (S^2 \times S^2 \setminus \Delta, \sigma \oplus (1+a)\sigma),$$

which is equivariant under the natural actions of $SO(3)$.

The importance of this result relies on the fact that the symplectic manifolds $(S^2 \times S^2, \sigma \oplus (1+a)\sigma)$ have received great attention (see [19, 5]) and one could use Theorem 4 to transfer this knowledge to magnetic systems on S^2. To make a concrete example, this would yield a more elegant proof of the main theorem in [14]. This project would be ideal as a Master Thesis since it has a very geometrical flavour, it requires to perform some explicit computations, and it opens up to further study.

F) Symplectic cohomology for non-exact forms

Symplectic Cohomology is an algebraic invariant that can be associated to every compact symplectic manifold (W, ω) with convex boundary $(\partial W, \lambda)$ (see [29]). It encodes dynamical and geometric information of the symplectic manifold, since it is defined in terms of periodic orbits of Hamiltonian functions with a prescribed behaviour at the boundary. This invariant, which we denote by $SH(W, \omega)$, has received much attention when the the symplectic form ω is **exact** [26]. However, the **non-exact** case is also extremely interesting, especially for its applications to algebraic geometry [25]. A relevant theme in this context is the concept of **fragility**, first introduced by Seidel in his PhD thesis. In our situation it translates to the fact that some symplectic manifolds (W, ω) admit convex deformations $s \mapsto \omega_s$, $s \geq 0$, $\omega_0 = \omega$ with the property that $SH(W, \omega_0) \neq SH(W, \omega_s)$, for all $s > 0$. In view of this, it becomes crucial to understand under which conditions Symplectic Cohomology is a **robust** invariant. This is the topic of an ongoing collaboration with Alexander Ritter (Oxford). We are in position of proving the following result.

Theorem 5 *Let W be a compact manifold with boundary and let $s \mapsto \omega_s$, $s \geq 0$ be a convex deformation such that the associated transgression class $[\tau(\omega_s)] \in H^1(\Lambda W)$ is constant in the parameter s. Then,*

$$SH(W, \omega_0) = SH(W, \omega_s), \qquad \forall s > 0.$$

The theorem hinges on a **compactness** result for Floer cylinders proved in [11] (but tracing back to [28, Section 5]). In a second part of the project, we aim at using such a compactness result to extend the definition of Symplectic Cohomology to non-convex perturbations (W, ω) of a convex symplectic manifold (W, ω_0) for Hamiltonians growing slowly near to the boundary. This would have concrete applications to the twisted cotangent bundle (T^*M, ω_σ), when M has dimension bigger than two.

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