Integrable Systeme und das KAM Theorem

Seminar 11BMASE244

Gabriele Benedetti
gbenedetti@mathi.uni-heidelberg.de
http://www.mathi.uni-heidelberg.de/~gbenedetti
Raum 3/402, Telefon: 062215414228
Sprechstunde: Montags 15 - 17 Uhr und nach Vereinbarung

22. September 2018

Organisatorisches

- Vorbesprechung Montag den 17.12. um 16 Uhr (Büro 3/402)
- Seminar Montags und Donnerstags (11-13 Uhr): 10 Termine.
- Raum: SR3, Mathematikon 3.OG.
- Von 7.1.2019 bis 7.2.2019

Hilfreiche Vorkenntnisse

Bitte schauen Sie nach den folgenden Begriffen aus Analysis III/Differential Geometrie 1:

- Vektorfelder: Flüße, Lie Klammer, Pushforward durch Abbildungen
- Differentialformen: äußere Ableitung, Stokes Satz, Pullback durch Abbildungen
- Untermannigfaltigkeiten des euklidischen Raums

Plan des Seminars

Das Seminar besteht aus den folgenden Vorträgen.

1 ODEs and Variational principles

Vector fields and their flows, invariant sets. Lie brackets. Differential of a function. Action of vector fields on functions. First integrals. Second order equations, conservation of total energy. Example of the pendulum: configuration space is the 1-torus. *References: Moser-Zehnder 1.1 a,b and Exercise 3.*

Variational principles. Euler-Lagrange equations. Lagrangian for second order equations. Geodesics for a Riemannian metric. Minimizing geodesics for metrics of the type $dx^2 + a(x, y)dy^2$. *References: Moser-Zehnder 1.2 a and Example 3.*

Legendre condition and Legendre Transform. Hamilton equations. The Hamilton function H and L have the same regularity (as a consequence the flow exists if L is C^2). Hamiltonian for second order conservative equations and for geodesics. Translations are Hamiltonian flows. *References: Moser-Zehnder 1.2 b.*

Poisson bracket and the derivative of a function along a Hamiltonian flow. Poisson bracket of coordinate functions. *References: Moser-Zehnder 1.3 c, in particular equation (1.35).*

2 Canonical transformations and Linear theory

Definition of J and of the standard symplectic form. Characterization of canonical transformations. Hamiltonian flows are canonical. *References: Moser-Zehnder 1.2 c,d and Theorem 1.7.*

Linear theory: symplectic vector spaces, orthogonal, subspaces (isotropic, coisotropic, Lagrangian), symplectic isomorphisms. Exercise 9 in Homework 1. All symplectic vector spaces are isomorphic. The symplectic group. Alternating forms. Volume form as wedge power of the symplectic form. Pull-back of forms by endomorphism. Pull back of volume forms and the determinant. *References: Canas da Silva 1.1, 1.2 and Homework 1.*

Corollaries: Liouville theorem, Poincaré recurrence (example of the pendulum, counterexample of translations).

3 Lagrangian submanifolds and Generating functions

Lagrangian submanifolds. Examples: Clifford torus, graph of a 1-form. Associated canonical transformations. Closed 1-forms on euclidean space and on the torus.

(If there is time: Hamilton principle, Maupertuis principle. *References: Fasano-Marmi 9.4 9.5.* The Jacobi metric. *References: Arnold 45D.*)

Generating functions. Lift of a diffeomorphism of configuration space to a canonical transformation. Liouville Theorem on n Poisson-commuting functions. *References: Hofer-Zehnder Appendix A.1.*

4 Discrete dynamics and Billiards

Dynamics given by iteration of a (symplectic) map. Perturbation of shear maps and the standard map. Phase portrait using this program by James Meiss.

The billiard map. Distance as generating function (orbits connecting two points with n bounces). *References: Tabachnikov1 1.1 1.2.*

Billiards inside circles and ellipses with phase portrait and formula for the first integral. Caustics and the string construction. Lazutkin's theorem and existence of caustics *References: Tabachnikov1 2.1; Tabachnikov2 Theorem 4.4; Tabachnikov1 2.8.*

Existence of periodic trajectories. *References: Tabachnikov1 2.5.*

5 Symmetry and Integrals of motion

Properties of Poisson brackets. *References: Fasano-Marmi Theorem 10.19, Corollary 10.6, Theorem 10.18, Example 10.26.*

Hamiltonian generated by the lift of a flow of diffeomorphisms in configuration space and reduction. Example of translations, rotations, time shift. Noether's theorem in Hamiltonian formulation. *References: Fasano-Marmi Theorem 10.27; Moser 3(a).*

Closed surfaces of revolution in the euclidean space: first fundamental form and geodesic equation, the Clairaut integral. Parallels and meridians. Critical points of the integral. Regular values of the integral and invariant Lagrange tori: they yield geodesics oscillating between two parallels. Hessian at the critical points and shape of the surface of revolution. Existence of homoclinic and heteroclinic.

6 Complete integrability and Hamilton-Jacobi

Example of a Hamiltonian on the cotangent bundle of a torus depending only on momenta. Linear flows on the torus. *References: Fasano-Marmi 11.7, in particular Theorem 11.8.*

Hamilton-Jacobi equation. Method of separation of variables: application to two-centre problem. *References: Arnold 47; see also Fasano-Marmi Example 11.6.*

7 Geodesic flows on ellipsoids

Elliptic coordinates, Liouville line element and Hamilton-Jacobi method. *References: Klingenberg 3.5 up to Theorem 3.5.8; Fasano-Marmi 11.10 Problem 6.*

Periodic geodesics and geodesics through umbilics. *References: Klingenberg Theorems 3.5.15, 3.5.16 without proof.*

Sketch of the geometric approach. Limit as the ellipsoid becomes flat yields an elliptical billiard. *References: Moser 5 (in particular f and g), 6; Arnold 47C; Tabachnikov1 2.3.*

8 The Liouville-Arnold-Jost Theorem

Statement of the theorem and detailed proof. *Hofer-Zehnder: Appendix A.2.*

9 Non-existence of integrals

Introduction to Poincaré's basic problem of dynamics about perturbation of completely integrable systems. A generical Hamiltonian systems has isolated periodic orbits and therefore is not integrable (periodic orbits come in torus family for integrable systems).

Poincaré Method for the non-existence of integrals in the perturbative case. *References: Kozlov1 IV 1.1-1.4 and 2.1; see also Kozlov2 IV.*

Topological obstructions to integrability. *References: Kozlov1 III 1.1-1.5 and 2.1; see also Kozlov2 III.*

10 Dynamics of circle maps

Circle homeomorphism, totation number, Poincaré classification, Denjoy Theorem, Denjoy counterexample. *References: Katok-Hasselblatt 11, 12.1-2; see also Milnor.*

11 A toy model for the KAM Theorem

Arnold's theorem on the existence of analytical conjugacy to a rotation for Diophantine rotation numbers. *References: Wayne Section 2; see also Katok-Hasselblatt 12.3.*

12 KAM Theorem I: the statement

The classical KAM Theorem. Statement of the KAM Theorem with parameters. Derivation of the classical theorem from the parametric one. *References: Pöschel Section 1 and 2.*

13 KAM Theorem II: the idea of the proof

Scheme of proof for the KAM Theorem with parameters. Some background material about Lipschitz functions and analytic functions. *References: Pöschel Section 3, Appendix A and B.*

14 KAM Theorem III: the iterative step

The core of the proof: the iterative step and the convergence of Newton's method. *References: Pöschel Section 4 and 5.*

Literatur

- Arnold, *Mathematical methods of classical mechanics*
- Canas da Silva, *Lectures on Symplectic Geometry*
- Fasano-Marmi, *Analytical mechanics*
- Hofer-Zehnder, *Symplectic invariants and Hamiltonian Dynamics*
- Katok-Hasselblatt, *Introduction to the modern theory of dynamical systems*
- Klingenberg, *Riemannian geometry (2nd edition)*
- Kozlov1, *Symmetries, topology and resonances in Hamiltonian mechanics*
- Kozlov2, *Integrability and non-integrability in Hamiltonian mechanics*
- Moser, *Various aspects of integrable Hamiltonian systems*, in the book *Dynamical systems (C.I.M.E. Summer School, Bressanone, 1978)*, Marchioro editor
- Moser-Zehnder, *Notes on Dynamical Systems*
- Pöschel, *A lecture on the classical KAM-theorem*
- Tabachnikov1, *Billiards*
- Tabachnikov2, *Geometry and Billiards* (also available in German)
- Wayne, *An Introduction to KAM Theory*