Aufgabe 1

a) $\nu_a : (\mathbb{R}^n, g_{(\sigma_+,\sigma_-)}) \longrightarrow (\mathbb{R}^n, g_{(\sigma_+,\sigma_-)})$

$$p \longmapsto a \cdot p$$

$$\underline{\underline{\nu_a^* g_{(\sigma_+,\sigma_-)}}} = g_{(\sigma_+,\sigma_-)}(d\nu_a \cdot, d\nu_a \cdot) = \underline{\underline{a^2 \cdot g_{(\sigma_+,\sigma_-)}}} \rightsquigarrow \underline{\underline{\lambda = a}}$$

b) $\sigma^{-1} : (\mathbb{R}^n, g_{eukl}) \longrightarrow (S^n_r \setminus \{r e_{n+1}\}, g_{S^n_r})$

$$\sigma^{-1}(q) = r \cdot e_{n+1} + \frac{2r^2}{r^2+|q|^2} \cdot (q - r e_{n+1})$$

$\sigma^{-1}(q) = r e_{n+1} + s \cdot (-r e_{n+1} + q)$

Löse für s: $r^2 = s^2 \cdot |q|^2 + (r + s(-r))^2$

$$\Rightarrow s = 0 \text{ oder } s = \frac{2r}{r^2+|q|^2}$$

$$d\sigma^{-1} = \frac{2r^2}{r^2+|q|^2} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \end{pmatrix} - \frac{4r^2}{(r^2+|q|^2)^2} \begin{pmatrix} q_1^2 & q_1 q_2 & \cdots & q_1 q_n \\ q_2 q_1 & q_2^2 & & \vdots \\ \vdots & & & \vdots \\ q_n q_1 & \cdots & \cdots & q_n^2 \\ -r q_1 & \cdots & \cdots & -r q_n \end{pmatrix}$$

$$(\sigma^{-1})^* g_{S^n_r}(e_i, e_j) = \begin{cases} 0 & i \neq j \\ \dfrac{4r^4}{(r^2+|q|^2)^2} & i = j \end{cases}$$

$$\Rightarrow \underline{\underline{(\sigma^{-1})^* g_{S^n_r} = \frac{4r^4}{(r^2+|q|^2)^2} g_{eukl.}}} \quad , \quad \underline{\underline{\lambda = \frac{2r^2}{r^2+|q|^2}}}$$

c) $\rho^{-1} : (B^n_r, g_{eukl}) \longrightarrow (H^n_r, g_{H^n_r})$

$$\rho^{-1}(q) = -r e_{n+1} + \frac{2r^2}{r^2-|q|^2} \cdot q + r e_{n+1}$$

$\rho^{-1}(q) = -r e_{n+1} + s \cdot (q + r e_{n+1})$

Löse für s: $-r^2 = s^2 |q|^2 - (s \cdot r - r)^2 \Rightarrow s = 0$ oder $s = \frac{2r^2}{r^2-|q|^2}$

$$d\rho^{-1} = \frac{2r^2}{r^2-|q|^2}\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \dots & 0 \end{pmatrix} + \frac{4r^2}{(r^2-|q|^2)^2}\begin{pmatrix} q_1^2 & q_1q_2 & \cdots & q_1q_n \\ \vdots & & & \vdots \\ q_nq_1 & \cdots & \cdots & q_n^2 \\ rq_1 & \cdots & \cdots & rq_n \end{pmatrix}$$

$$(\rho^{-1})^* g_{H^n_r}(e_i,e_j) = \begin{cases} 0 & i \neq j \\ \dfrac{4r^4}{(r^2-|q|^2)^2} & i = j \end{cases}$$

$$\Rightarrow \underline{\underline{(\rho^{-1})^* g_{H^n_r} = \frac{4r^4}{(r^2-|q|^2)^2}\, g_{eucl.}}} \quad \Rightarrow \quad \underline{\underline{\lambda = \frac{2r^2}{r^2-|q|^2}}}$$

Aufgabe 2

$$\mathbb{R}^{n+1} = \underset{x}{\mathbb{R}} \times \underset{y}{\mathbb{R}^{n-1}} \times \underset{t}{\mathbb{R}}$$

$$E_r^n = \{r\} \times \mathbb{R}^{n-1} \times (0,\infty)$$

Definiere φ : $p \in H_r^n = \{|x|^2 + |y|^2 - |t|^2 = -r^2\}$

p' Schnittpunkt zwischen $\overline{Op}$ und $\{t = r\}$

$p = (x_p, y_p, t_p)$, $p' = \left(\frac{r x_p}{t_p}, \frac{r y_p}{t_p}, r\right)$

Projektion von p' auf $S_r^n = \{|x|^2 + |y|^2 + |t|^2 = r^2\}$:

$$p'' = \left(\frac{r x_p}{t_p}, \frac{r y_p}{t_p}, \sqrt{r^2 - \left(\frac{r x_p}{t_p}\right)^2 - \left(\frac{r}{t_p}\right)^2 |y|^2}\right)$$

Gerade von $(-r,0,0)$ und p'': $(-r,0,0) + s \cdot (p' - (-r,0,0))$

Schnitt mit $E_r^n = \{r\} \times \mathbb{R}^{n-1} \times (0,\infty)$

$$r = -r + s \cdot \left(\frac{r x_p}{t_p} + r\right) \Rightarrow s = \frac{2 t_p}{x_p + t_p}$$

$$\Rightarrow \varphi(p) = \frac{2t_p}{x_p + t_p} \cdot \left(\frac{r y_p}{t_p}, \sqrt{r^2 - \left(\frac{r x_p}{t_p}\right)^2 - \left(\frac{r}{t_p}\right)^2 |y_p|^2}\right)$$

$$= \frac{2t_p}{x_p + t_p}\left(\frac{r}{t_p} y_p, \frac{r}{t_p}\sqrt{t_p^2 - x_p^2 - |y_p|^2}\right)$$

$$= \frac{2}{x_p + t_p}\left(r y_p, r^2\right)$$

Beh: $\varphi^{-1}(z,s) = \left(\frac{r^2}{s} - \frac{a(z,s)}{4}, \frac{rz}{s}, \frac{r^2}{s} + \frac{a(z,s)}{4}\right)$

$$\varphi^{-1} \circ \varphi(x,y,t) = \varphi^{-1}\left(\frac{2}{t+x} r y, \frac{2r^2}{t+x}\right)$$

$$= \left(\frac{t+x}{2} - \frac{a\left(\frac{2r}{t+x} y, \frac{2r^2}{t+x}\right)}{4}, y, \frac{t+x}{2} + \frac{a\left(\frac{2r}{t+x} y, \frac{2r^2}{t+x}\right)}{4}\right)$$

$$= \left(\frac{t+x}{2} - \frac{1}{4} \cdot \left(\frac{t+x}{2r^2}\right) \cdot \left(\frac{2r}{t+x}\right)^2 \cdot |y|^2 - \frac{1}{4} \cdot \frac{2r^2}{t+x}, y,\right.$$

$$\left.\frac{t+x}{2} + \frac{1}{4}\left(\frac{t+x}{2r^2}\right)\left(\frac{2r}{t+x}\right)^2 |y|^2 + \frac{1}{4}\frac{2r^2}{t+x}\right)$$

$$= \left(\frac{t^2+2tx+x^2-r^2}{2(t+x)} - \frac{1}{2(t+x)}|y|^2 ,\ y ,\ \frac{t^2+2tx+x^2+r^2}{2(t+x)} + \frac{1}{2(t+x)}|y|^2 \right)$$

Def. H_r^n

$$= \left(\frac{2tx+2x^2+r^2-r^2}{2(t+x)} ,\ y ,\ \frac{2tx+2t^2-r^2+r^2}{2(t+x)} \right)$$

$$= \underline{\underline{(x, y, t)}}$$

$$\psi \circ \psi^{-1}(z,s) = \psi\left(\frac{r^2}{s} - \frac{a(z,s)}{4} ,\ \frac{rz}{s} ,\ \frac{r^2}{s} + \frac{a(z,s)}{4} \right)$$

$$= \frac{2}{\left(\frac{r^2}{s} - \frac{a(z,s)}{4}\right) + \left(\frac{r^2}{s} + \frac{a(z,s)}{4}\right)} \left(\frac{r^2 z}{s} , r^2 \right)$$

$$= \frac{s}{r^2} \left(\frac{r^2 z}{s} , r^2 \right)$$

$$= \underline{\underline{(z, s)}}$$

$$(\Psi^{-1})^* g_{H_r^n} (e_i, e_j) = \begin{cases} \frac{r^2}{s^2} & i = j \\ 0 & \text{sonst} \end{cases}$$

$$\rightarrow \underline{\underline{(\Psi^{-1})^* g_{H_r^n} = \frac{r^2}{s^2} g_{eukl}}} ,\quad \underline{\underline{\lambda = \frac{r}{s}}}$$

$$\frac{\partial a}{\partial z_i} = \frac{2 z_i}{s} ,\quad \frac{\partial a}{\partial s} = -\frac{1}{s^2} |z|^2_{eukl} + 1$$

$$d\Psi^{-1} = \begin{pmatrix} -\frac{1}{2s} z_1 & -\frac{1}{2s} z_2 & \cdots & -\frac{1}{2s} z_{n-1} & -\frac{r^2}{s^2} - \frac{1}{4}\left(-\frac{1}{s^2}|z|^2 + 1\right) \\ \frac{r}{s} & 0 & & 0 & -r z_1 / s^2 \\ 0 & \frac{r}{s} & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & \frac{r}{s} & -r z_{n-1} / s^2 \\ \frac{1}{2s} z_1 & \cdots & & \frac{1}{2s} z_{n-1} & -\frac{r^2}{s^2} + \frac{1}{4}\left(-\frac{1}{s^2}|z|^2 + 1\right) \end{pmatrix}$$

$$g_{H_r^n} = g_{(n,1)} = dx^2 + dy^2 - dt^2.$$

Aufgabe 2-3

a) 1: $(\mathbb{R}^n, g_{(\sigma_+,\sigma_-)})$ $e_1,\dots,e_n \in T_p\mathbb{R}^n$ $\Big\}$ orthogonal
$e_1',\dots,e_n' \in T_q\mathbb{R}^n$

$$g(e_i,e_i) = \begin{cases} >0 & i\le\sigma_+ \\ <0 & i\ge \sigma_+ +1 \end{cases}$$

~~denn Umsort. ist geg. durch $A_{ij}=(a_{ij})$: $\forall i\ \exists! j: a_{ij}\neq 0$ & $a_{ij}=1$.~~

~~AS~~

Betrachte $\Phi = \tau_{q-A(p)} \circ A$, wobei $A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}$ &

$A^+e_i = e_i'$ $1\le i\le\sigma_+$, $A^-e_i = e_i'$ $\sigma_+ + 1\le i\le n$.

$(A^+)^T = (A^+)^{-1}$ $\qquad (A^-)^T = (A^-)^{-1}$ Ex. nach L.A.

$$\begin{pmatrix} A^+ & \\ & A^- \end{pmatrix}\cdot\begin{pmatrix} +\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix} A^+ & \\ & A^-\end{pmatrix}^T = \begin{pmatrix} A^+ & 0 \\ 0 & -A^-\end{pmatrix}\begin{pmatrix} A^{+T} & 0 \\ 0 & (A^-)^T\end{pmatrix} = \begin{pmatrix}\mathbb{1} & \\ & -\mathbb{1}\end{pmatrix}.$$

$$\ulcorner - \delta_{ij} = g(e_i',e_j') = g(Ae_i, Ae_j) = g(A^TAe_i, e_j) \quad \forall j$$
$$= g((w_{\sigma_++1},\dots,w_{n}), e_j) = -w_j \Rightarrow A^TAe_i = e_i \to A^T = A^{-1}. \lrcorner$$

2: $(e_1,\dots,e_n)$ ONB von $T_pQ^{-1}(c)$

$\rightsquigarrow$ $(p,e_1,\dots,e_n)$ bzw. $(e_1,\dots,e_n,p)$ ONB von $T_p\mathbb{R}^{n+1}$, $g_{(\sigma_+,\sigma_-)}$
$\qquad c>0 \qquad\qquad c<0$

Sei $(e_1',\dots,e_n')$ ONB von $T_qQ^{-1}(c)$

$\rightsquigarrow$ $(q,e_1',\dots,e_n')$ bzw. $(e_1',\dots,e_n',q)$ ONB von $(T_q\mathbb{R}^{n+1}, g_{(\sigma_+,\sigma_-)})$

Wähle $A\in O(\sigma_+,\sigma_-)$: $A(p)=q$, $A(e_i)=e_i'$

$\rightsquigarrow$ A erhält $Q^{-1}(c)$

(b) 1. $(0,\infty)\times S_1^{n-1} \longrightarrow (\mathbb{R}^n\setminus\{0\}, g_{eukl})$

$(r, p) \longmapsto r\cdot p$

$$n\left\{\left(\begin{array}{c|cccc} p_1 & r & 0 & \dots & 0\\ \vdots & 0 & r & \dots & 0\\ \vdots & \vdots & & \ddots & 0\\ p_n & 0 & 0 & \dots & r\end{array}\right)\right.$$

$\underbrace{}_{1}\ \underbrace{}_{n}$

Diffeo: ✓

$(F_1^* g_{eukl})(e_1,e_1) = g_{eukl}(p,p) = |p|^2$

$(F_1^* g_{eukl})(e_i,e_i) \overset{i\geq 2}{=} g_{eukl}(r\cdot e_{i-1}, r\cdot e_{i-1}) = r^2$

$(F_1^* g_{eukl})(e_1,e_i) \overset{i\geq 2}{=} g_{eukl}(p, r\cdot \underbrace{e_{i-1}}_{\in T_pS^{n-1}}) = 0$

$(F_1^* g_{eukl})(e_i,e_j) = 0$

$\Rightarrow F_1^* g_{eukl} = |p|^2\cdot g^1_{eukl} +$

$= g^1_{eukl} + r^2 g^n_{eukl}|_{S_1^{n-1}}$

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2. $(0,\pi)\times S^0 = S^1\setminus\{\pm e_2\}$

$r \times \{\pm 1\} \longmapsto (\sin(r), \cos(r))\cdot p$

$(0,\pi)\times S^1 \longrightarrow S^2\setminus\{\pm e_3\}$

$(r, \vartheta) \longmapsto$ ~~$(\sin r\cos\vartheta, \sin r\sin\vartheta)$~~ $\cos\vartheta)$ ~~$dF_2 = \begin{pmatrix}\cos r\cos\vartheta & \cos r\sin\vartheta\\ -\sin r\sin\vartheta & \sin r\cos\vartheta\end{pmatrix}$~~

~~$(F_2^* g_{S_1^2})(e_1,e_1) = \cos r^2\cos\vartheta^2 + \sin r^2\cdot\sin\vartheta^2$~~

~~$(e_2,e_2) = \cos r^2\sin\vartheta^2 + \sin r^2\cos\vartheta^2$~~

$(e_1,e_2) = 0$

$\sin r\cdot\cos\vartheta$
$\sin r\ \sin\vartheta$
$\cos r$

$F_2: (0,r\pi)\times S_1^{n-1} \longmapsto S_r^n$, $dF_2 = \begin{pmatrix}\cos(\frac{\vartheta}{r})\cdot p_1 & \sin(\frac{\vartheta}{r})\cdot r & \\ \cos(\frac{\vartheta}{r})\cdot p_2 & 0 & \ddots\\ \vdots & \vdots & \\ \cos(\frac{\vartheta}{r})\cdot p_n & 0 & \\ -\sin(\frac{\vartheta}{r}) & 0 & 0\dots\end{pmatrix}$

$\vartheta\quad p \longrightarrow (\sin(\frac{\vartheta}{r})\cdot r\cdot p, r\cdot\cos\frac{\vartheta}{r})$

$(F_2^* g_{S_r^n})^{(e_\vartheta,e_\vartheta)} = g_{S_r^n}(\dots,\dots) = \cos(\frac{\vartheta}{r})^2\cdot|p|^2_{eukl} + \sin(\frac{\vartheta}{r})^2 = 1$

$(F_2^* g_{S_r^n})(v_i,v_j) = r^2\sin(\frac{\vartheta}{r})^2\cdot\delta_{ij}$

$(F_2^* g_{S_r^n})(e_\vartheta,v_i) = \left(\begin{matrix}\cos(\frac{\vartheta}{r})\cdot p\\ -\sin(\frac{\vartheta}{r})\end{matrix}, \begin{matrix}\sin(\frac{\vartheta}{r})\cdot r\cdot v\\ 0\end{matrix}\right) = 0 \quad v\perp p.$

$\Rightarrow F_2^* g_{eukl} = g^1_{\mathbb{R}} + r^2\sin(\frac{\vartheta}{r})^2 g_{S_1^{n-1}}.$

$$(t,p) \longmapsto \left(r \sinh \tfrac{t}{r} \cdot p \,,\, r \cosh \tfrac{t}{r}\right)$$

$$dF_3 = \begin{pmatrix} \cosh \frac{t}{r}\, p_1 & r \sinh \frac{t}{r} & & 0 \\ \vdots & 0 & \ddots & \\ \cosh \frac{t}{r}\, p_n & 0 & & r \sinh \frac{t}{r} \\ \sinh \frac{t}{r} & 0 & \cdots & 0 \end{pmatrix}$$

$$(F_3^* g_{H_r})(e_t, e_t) = \cosh \tfrac{t}{r}^2 - \sinh \tfrac{t}{r}^2 = 1$$

$$(F_3^* g_{H_r})(v_i, v_j) = g_{H_r}\left(r \sinh \tfrac{t}{r}\, v_i, r \sinh \tfrac{t}{r}\, v_j\right) = r^2 \sinh \tfrac{t}{r}^2 \cdot$$

$$(F_3^* g_{H_r})(e_t, v_i) = 0 \quad \text{da } p \perp v_i$$

$$\Rightarrow \quad F_3^* g_{H_r} = g_{\mathbb{R}} + r^2 \sinh \tfrac{t}{r}^2 g_{S_1^{n-1}} .$$

Aufgabe 4

a) Die Projek. ist eind. def. durch $\operatorname{grad} f \in \ker \Pi$. und $\Pi|_{TN} = id.$ und $\operatorname{im}(\Pi) = TN$

- $\operatorname{grad} f \in \ker \Pi$. r.h.s. ✓
- $v \in TN \overset{B1A3}{=} \langle \operatorname{grad} f \rangle^\perp$ ✓
- $g_p(\operatorname{grad} f, \text{r.h.s}(v)) = g_p\left(\operatorname{grad} f, v - \frac{g(v, \operatorname{grad} f)}{|\operatorname{grad} f|^2} \operatorname{grad} f\right)$

$$= g(\operatorname{grad} f, v) - \frac{g(v, \operatorname{grad} f)}{|\operatorname{grad} f|^2} \cdot g(\operatorname{grad} f, \operatorname{grad} f)$$

$$= 0.$$

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b) $f(x,y,z) = z - S(x,y)$

$$df = dz - \frac{\partial S}{\partial x} dx - \frac{\partial S}{\partial y} dy \to \operatorname{grad} f = \left(-\frac{\partial S}{\partial x}, -\frac{\partial S}{\partial y}, 1\right)$$

$$\partial_u = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial S}{\partial x} \end{pmatrix}, \quad \partial_v = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial S}{\partial y} \end{pmatrix}$$

$$\Pi_p v = v - \frac{g_p(v, \operatorname{grad} f)}{|\operatorname{grad} f|^2} \operatorname{grad} f$$

$$= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - \underbrace{\frac{-v_1 S_x - v_2 S_y + v_3}{S_x^2 + S_y^2 + 1}}_{=: h(p,v)} \begin{pmatrix} -S_x \\ -S_y \\ 1 \end{pmatrix}$$

$$= [v_1 - h(p,v) \cdot (-S_x)] \cdot \partial_u + [v_2 - h(p,v)(-S_y)] \partial_v$$