Montag, 22. Juni 2020 11:12

Mathers graph thm. $\pi|_{\tilde{\mathcal{A}}_c} : \tilde{\mathcal{A}} \subseteq TM \longrightarrow M$ is inj, where π is can proj. and $\tilde{\mathcal{A}}$ is the Aubry set.

And $(\pi|_{\tilde{\mathcal{A}}})^{-1} : \mathcal{A} \longrightarrow \tilde{\mathcal{A}}$ is Lipschitz. $\mathcal{A}_c = \pi(\tilde{\mathcal{A}}_c)$

The Proof is based on Lemma 4.1.31

Lemma 4.1.31 Let $K>0$. $\exists\, \varepsilon, \delta, \eta > 0, C>0$ s.t. if α, β are sol of the EL eq with

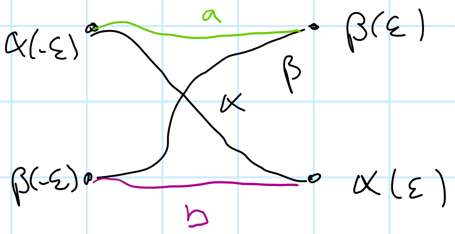
$\|(\alpha(0), \dot\alpha(0))\| \leq K$, $\|(\beta(0), \dot\beta(0))\| \leq K$ and $d(\alpha(0), \beta(0)) \leq \delta$

and $d((\alpha(0), \dot\alpha(0)), (\beta(0), \dot\beta(0))) > C \cdot d(\alpha(0), \beta(0))$

$\Rightarrow \exists$ C^1-curves $a, b : [-\varepsilon, \varepsilon] \longrightarrow M$ with endpt.

$a(-\varepsilon) = \alpha(-\varepsilon)$, $a(\varepsilon) = \beta(\varepsilon)$, $b(-\varepsilon) = \beta(-\varepsilon)$, $b(\varepsilon) = \alpha(\varepsilon)$ s.t.

$$A_L(\alpha) + A_L(\beta) - A_L(a) - A_L(b) \geq \eta\, d((\alpha(0), \dot\alpha(0)), (\beta(0), \dot\beta(0)))^2 > 0$$



Proof of graph thm

1) We want to prove that $(\pi|_{\tilde{\mathcal{A}}})^{-1}$ is Lipschitz.

$K := \max_{\tilde{\mathcal{A}}} \|(x,v)\|$, exist since $\tilde{\mathcal{A}}_c$ is compact.

Let $\varepsilon, \delta, \eta, C$ be as Lemma 4.1.31

Claim: $(x_1, v_1), (x_2, v_2) \in \tilde{\mathcal{A}}$ then

$$d(x_1, x_2) \leq \delta \Rightarrow \boxed{d((x_1,v_1),(x_2,v_2)) \leq C\, d(x_1, x_2)}$$

Proof of claim: We assume the conv. is true i.e.

$\exists (x_1, v_1), (x_2, v_2) \in \tilde{\mathcal{A}}_c$ s.t. $d(x_1, x_2) \leq \delta$ and

$d((x_1, v_1), (x_2, v_2)) > C \cdot d(x_1, x_2)$. Consider the flow this pt

$\alpha(t) := \Phi^L_t(x_1, v_1)$, $\beta(t) := \Phi^L_t(x_2, v_2)$

$d((x_1,v_1),(x_2,v_2)) > c\, d(x_1,x_2)$. Consider the flow this p.

$\alpha(t) := \Phi_t^L(x_1,v_1)$, $\beta(t) := \Phi_t^L(x_2,v_2)$

<u>Ex 1:</u> Prove they sat. hypth. of Lemma 4.1.31

$\Longrightarrow \exists\, a,b : [-\varepsilon,\varepsilon] \longrightarrow M$ with end pts.

$a(-\varepsilon) = \alpha(-\varepsilon)$, $\beta(-\varepsilon) = b(-\varepsilon)$, $a(\varepsilon) = \beta(\varepsilon)$, $b(\varepsilon) = \alpha(\varepsilon)$ s.t.

$A_L(\alpha) + A_L(\beta) > A_L(a) + A_L(b)$

<u>Ex.: 2:</u> $\phi_{n,c(L)}(a(-\varepsilon), a(\varepsilon)) + \phi_{n,c(L)}(b(-\varepsilon), b(\varepsilon))$

$\leq -\phi_{n,c(L)}(\alpha(\varepsilon), \alpha(-\varepsilon)) - \phi_{n,c(L)}(\beta(\varepsilon), \beta(-\varepsilon))$

<u>Ex.: 3:</u> use Δ-inq for $\phi_{n,c(L)}$ to prove

$\phi_{n,c(L)}(\alpha(-\varepsilon), \beta(\varepsilon)) < -\phi_{n,c(L)}(\beta(\varepsilon), \alpha(-\varepsilon))$

$\Longrightarrow \phi_{n,c(L)}(\alpha(-\varepsilon), \beta(\varepsilon)) + \phi_{n,c(L)}(\beta(\varepsilon), \alpha(-\varepsilon)) < 0$

Prop. 4.1.9 (5) states that this has be pos.

$\rightsquigarrow$ ↯ #

If $x_1, x_2 \in \pi(\tilde{\mathcal A}) = \mathcal A$ s.t. $\underbrace{\pi(x_1,v_1)}_{=x_1} = \underbrace{\pi(x_2,v_2)}_{=x_2}$

$\rightsquigarrow d(x_1,x_2) = 0 \Longrightarrow d((x_1,v_1),(x_2,v_2)) = 0$

$\Longrightarrow \pi|_{\tilde{\mathcal A}}$ is inj. $\square$

<u>Goal:</u>

$$\begin{tikzcd} \tilde{\mathcal M} \arrow[d] & \subseteq & \tilde{\mathcal A} \arrow[d, bend left, "\pi"] & \subseteq \tilde{\mathcal N} \subseteq \tilde{\mathcal E} = \{E(x,v) = c(L)\} \subseteq & TM \arrow[d] \\ \mathcal M \arrow[u, bend left, "(\pi|_{\tilde{\mathcal M}})^{-1}"] & \subseteq & \mathcal A \arrow[u, bend left, "(\pi|_{\tilde{\mathcal A}})^{-1}"] & & M \end{tikzcd}$$

<u>Q.:</u> Are Aubry set / Mañe set empty?

<u>A.:</u> 1) If $\tilde{\mathcal M} \neq \emptyset \Longrightarrow \tilde{\mathcal A} \neq \emptyset \Longrightarrow \tilde{\mathcal N} \neq \emptyset$ prob. measure measure theory

2) $\tilde{\mathcal N} \neq \emptyset$ we get from Max talk via α, ω-limit of $\tilde{\mathcal N}$ that $\tilde{\mathcal A} \neq \emptyset$. via weak KAM theory

<u>We will follow 1)</u>

<u>Prob. measures:</u>

- M compact smooth mfd. - $\mathcal B(M)$ is Borel σ-Algebra.

each measure on $\mathcal B(M)$ is called a Borel measure

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each measure on $\mathcal{B}(M)$ is called a Borel measure

<u>Def.:</u> μ is Borel measure on M, then

1) $\operatorname{supp} \mu = \{ x \in M \mid \mathcal{U}_x \in \mathcal{B}(M) \text{ open ngh. of } x \Rightarrow \mu(\mathcal{U}_x) > 0 \}$

2) $f : (M, \mathcal{B}(M)) \longrightarrow (N, \mathcal{B}(N))$ measur. The pushforw. of μ w.r.t f is def. through $(f_* \mu)(A) := \mu(f^{-1}(A)) \; \forall A \in \mathcal{B}(N)$

3) μ is called prob. meas $\Longleftrightarrow \mu(M) = 1$

4) $\mathcal{M} := \{ \text{Borel prob. meas. on } M \}$

<u>Q.:</u> How to get a „good" topology on $\mathcal{M}$?

<u>Rmk.:</u> μ fin. borel meas. on M

$\downarrow$?

$I_\mu = (f \mapsto \int_M f \, d\mu)$ $\longrightarrow$ bound. lin. pos. funct.

<u>Riesz – Markow repr. thm.</u>

X comp. Hausdorff space then

pos lin. bnd. ~~fct~~. on $C(X) \overset{1:1}{\Longleftrightarrow}$ fin. Borel meas. on X.

<u>Rmk.:</u> each fin. Borel meas μ. ind. a Borel prob. via

$$\mu \mapsto \frac{\mu}{\mu(M)}.$$

<u>Def.:</u> (weak* topology) let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$, then $\mu_n \to \mu \in \mathcal{M}$

$$:\Longleftrightarrow \lim_{n\to\infty} \int_M f \, d\mu_n = \int_M f \, d\mu \quad \forall f \in C(X)$$

<u>Rmk.:</u> Using Riesz-Markow we get for each sequ. $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ with $\mu_n \to \mu \in \mathcal{M}$ a sequ of pos lin funct. on $C(X)$ s.t

$$I_{\mu_n}(f) \longrightarrow I_\mu(f) \; \forall f \in C(X). \Rightarrow \mu \mapsto \int f \, d\mu \text{ is cont. } \forall f \in C(X)$$

The weak* top. is the weak. top on $\mathcal{M}$ s.t. $\mu \mapsto \int f \, d\mu$ is cont $\forall f \in C(X)$ and weak* top is metrizable.

<u>Thm:</u> Let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ and $\mu \in \mathcal{M}$: $\mu_n \to \mu$ then

$$\operatorname{supp} \mu \subseteq \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m \geq n} \operatorname{supp} \mu_m}$$

Def.: $f: M \to M$ diffeo, then we call a prob meas. μ f-inv iff $f_* \mu = \mu$. ($\Leftrightarrow \int \varphi \circ f \, d\mu = \int \varphi \, d\mu \quad \forall \varphi \in C(M)$)

Rmk.: If μ is f-inv $\Rightarrow$ $\operatorname{supp} \mu$ is a f-inv. closed set.

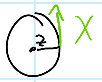
Since it's closed per definition and $\forall A \in \operatorname{supp} \mu$: $0 < \mu(A) \overset{\mu \ f\text{-inv.}}{=} \mu(f^{-1}(A))$

$\Rightarrow \forall A \in \operatorname{supp} \mu \leadsto f^{-1}(A) \in \operatorname{supp} \mu$ #

Def.: X is vf. with flow φ_t, then a prob meas. μ is inv by φ_t iff $(\varphi_t)_* \mu = \mu \quad \forall t \in \mathbb{R}$.

Example: Let μ be the Lebesgue measure on $[0,1]$.
$S^1 = [0,1]/0 \sim 1$ and $f: [0,1] \twoheadrightarrow S^1$, $t \mapsto e^{2\pi i t}$

Then $f_* \mu$ is Lebesgue measure on S^1.

Define $\varphi_t: S^1 \to S^1$, $z \mapsto e^{2\pi i t} \cdot z \quad \forall t \in \mathbb{R}$ which is the flow generated by the vf X which rotates z per $90°$ degree anti clockwise. 

Claim $(\varphi_t)_*(f_* \mu) = (f_* \mu) \quad \forall t \in \mathbb{R}$.

Proof φ_t corresponds to the map $x \mapsto x+t$ on $\mathbb{R}/\mathbb{Z}$.
This maps obs. preserves the transl. invariant Lebesgue measure. $\square$

Ergodicity

meas. pres. map.

Def.: • $(M, \mathcal{B}(M), \underset{m}{\mu}, f)$

• $A \subseteq M$ is f-inv $\Leftrightarrow f^{-1}(A) = A$

in the lit. this is sometimes called f-ergodic

$(M, \mathcal{B}(M), \mu, f)$ is ergodic $\Leftrightarrow$ each f-inv. set has meas. 0 or 1.

Lemma: 1) $(M, \mathcal{B}(M), \mu, f)$ is ergodic

$\Leftrightarrow$ 2) $\mu(f^{-1}(A) \underset{\text{sym difference}}{\Delta} A) = 0 \Rightarrow \mu(A) = 0$ or $\mu(A) = 1 \quad \forall A \in \mathcal{B}(M)$

$\Leftrightarrow$ 3) all meas inv. fct w.r.t f are const. μ.a.e.

$\Leftrightarrow$ 4) $\varphi \in L^1$ and $\varphi \circ f = \varphi$ a.e. $\Rightarrow \varphi$ const μ-a.e.

$\Longleftarrow$ 4) $\varphi \in \mathcal{L}^1$ and $\varphi \circ f = \varphi$ a.e. $\Rightarrow$ φ const μ-a.e.

Example irrat. rotation vector

- $R_\alpha : S^1 \to S^1 ; x \mapsto e^{2\pi i \alpha} \cdot x$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
- μ is norm Lebesque meas. on S^1
- $\varphi \in C^\infty(S^1, \mu)$ f-inv $\leadsto \varphi \in L^2(S^1, \mu)$

Char. of the comp group S^1 are given by $\chi_n(z) = z^n$

By abuse of not. we get $\varphi = \sum_{n \in \mathbb{Z}} a_n \chi_n$

$$\leadsto \varphi = \varphi \circ R_\alpha \Leftrightarrow \sum_{n \in \mathbb{N}} a_n z^n = \sum_{n \in \mathbb{Z}} a_n \left(z \cdot e^{2\pi i \alpha}\right)^n$$
$$= \sum_{n \in \mathbb{Z}} a_n \cdot e^{2\pi i \alpha \cdot n} \cdot z^n$$

$$\Rightarrow a_n = a_n \underbrace{e^{2\pi i \overbrace{\alpha \cdot n}^{\notin \mathbb{Z}}}}_{\neq 1 \,\forall n \neq 0} \quad \forall n \in \mathbb{Z}.$$

$$\Rightarrow a_n = 0 \quad \forall n \neq 0$$

$$\leadsto \varphi = a_0 z^0 = a_0 \Rightarrow \varphi \text{ is const.}$$

$\leadsto$ So this is a erg. system. $\square$

Thm.: (Birkhoff-Ergodic-thm.)

$f : (X, \mu) \to (X, \mu)$ is meas. pres. and $\varphi \in \mathcal{L}^1(X, \mu)$

$$\Rightarrow \underbrace{\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x))}_{\text{time average}} =: \varphi_f(x) \text{ exist } \mu\text{. a.e. } x \in X$$

$= \mathbb{E}(\varphi \mid C)$ C is σ-Algebra of f-inv. sets.

Cor.:
- $\int_M \varphi_f \, d\mu = \int_M \varphi \, d\mu$
- f is invertible $\Rightarrow \varphi_f = \varphi_{f^{-1}}$

Cor.: Let $(M, \mathcal{B}(M), \mu, f)$ ergodic, $\varphi \in \mathcal{L}^1(M, \mu)$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_M \varphi \, d\mu \quad \mu\text{-a.e. } x \in M$$

Proof.: $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \mathbb{E}(\varphi \mid C)$ f-inv. sets.

Since we erg. system C is triv.

$$\leadsto \mathbb{E}(\varphi \mid C) = \mathbb{E}(\varphi) = \int_M \varphi \, d\mu \qquad \square$$

Thm: Let μ a f-inv Borel prob. meas. on M

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$\Longrightarrow$ $\exists$ decomp. of M: $M = \bigcup_{\alpha \in A} M_\alpha$, A is Lebesgue set.

s.t. $\forall \alpha \in A$ there $\exists$ fin. f-inv. prob. measure μ_α on M_α

s.t. $(M_\alpha, \mathcal{B}(M_\alpha), \mu_\alpha, f)$ is ergodic and

$$\int_M \varphi \, d\mu = \int_A \left(\int_{M_\alpha} \varphi|_{M_\alpha} \, d\mu_\alpha \right) d\alpha \qquad \varphi \in L^1(M, \mu).$$

"Lebesgue measure".

Rmk: $\forall A' \subseteq A$ with $\mu\left(\bigcup_{\alpha \in A'} M_\alpha \right) = 1$ we have

$$\operatorname{supp} \mu \subseteq \overline{\bigcup_{\alpha \in A'} \operatorname{supp} \mu_\alpha}$$

Let $S \in \operatorname{supp} \mu \Longrightarrow 0 < \mu(S) = \mu\left(S \cap \left(\bigcup_{\alpha \in A'} M_\alpha\right)\right)$

erg. decomp. $\overset{\sim}{=} \int_{A'} \int \chi_{S \cap M_\alpha} \, d\mu_\alpha \, d\alpha$

$= \int_{A'} \mu_\alpha(S \cap M_\alpha) \, d\alpha$

$\Longrightarrow S \in \overline{\bigcup_{\alpha \in A'} \operatorname{supp} \mu_\alpha}$.

Rotation vector

Setting:
- M compact smooth mfld.
- μ Borel measure on M
- $X \in \mathfrak{X}(M)$ with flow φ_t, φ_t is meas pres.
- ω is closed 1-Form on M

Claim: $\int_M i_X \omega \, d\mu$ depends only on $[\omega] \in H^1(M, \mathbb{R})$.

Proof: Let $\omega_1, \omega_2 \in [\omega] \Rightarrow \omega_1 - \omega_2 = dF$, $F \in C^\infty(M)$

$$\Longrightarrow \int i_X \omega_1 \, d\mu - \int i_X \omega_2 \, d\mu = \int i_X dF \, d\mu$$
$$= \int \mathcal{L}_X F \, d\mu$$
$$= \frac{d}{dt}\Big|_{t=0} \int_M F \circ \varphi^t \, d\mu = 0$$

$\square$

$\Longrightarrow$ we get a lin map $\underbrace{H^1(M,\mathbb{R}) \to \mathbb{R} : [\omega] \mapsto \int_M i_X \omega\, d\mu}_{\in (H^1(M,\mathbb{R}))^* \cong H_1(M,\mathbb{R}).}$

$\rightsquigarrow \big(\underbrace{[\omega] \mapsto \int_M i_X \omega\, d\mu}_{=: \rho_\mu}\big) \in H_1(M,\mathbb{R})$

Def.: ρ_μ def as above is called the asymp. cycle of the flow φ_t w.r.t. the borel meas μ.

Lemma: If $\mu_n \to \mu$ in weak* top., then $\rho_{\mu_n} \to \rho_\mu$.

Suppose now $(M, \mathcal{B}(M), \mu, \varphi^t)$ is erg. $\forall t \in \mathbb{R}$. We can apply Birkhoff-Ergod. thm: (Cor.)

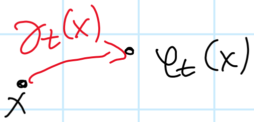
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \omega(X(\varphi_t^k(p))) = \int_M \omega(X)\, d\mu \quad \mu\text{.a.e } p \in M.$$

$$\rightsquigarrow \int_M i_X \omega\, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \omega(X(\varphi_t^k(p)))$$

$$= \lim_{t\to\infty} \frac{1}{t} \int_0^t \omega(X(\varphi_s(p)))\, ds$$

$$\Longrightarrow \int i_X \omega\, d\mu = \lim_{t\to\infty} \frac{1}{t} \int_0^t \omega(X(\varphi_s(p)))\, ds$$

Let $\gamma_t(x)$ the orbit segm. connecting for all $t \in \mathbb{R}$.



Let $\eta_{x,\varphi_t(x)}$ a family of bounded arcs connecting x and $\varphi_t(x)$ for all $t \in \mathbb{R}$.

$\rightsquigarrow \gamma_t(x) * (\eta_{x,\varphi_t(x)})^{-1}$ is closed loop. $\forall t \in \mathbb{R}$.

We have $$\lim_{t\to\infty} \frac{1}{t} \underbrace{\int_0^t \omega(X(\varphi^s(p)))\, ds}_{= \int_{\gamma_t(x)} \omega} = \int i_X \omega\, d\mu$$

And on the other hand we have

$$\lim_{t\to\infty} \frac{1}{t} \int_{\gamma_t(x) * u_{x,\varphi_t(x)}} \omega = \lim_{t\to\infty} \frac{1}{t} \int_{\gamma_t(x)} \omega + \lim_{t\to\infty} \frac{1}{t} \int_{u_{x,\varphi_t(x)}} \omega$$

Claim $\lim_{t\to\infty} \frac{1}{t} \cdot \int_{u_{x,\varphi_t(x)}} \omega = 0$, clear since $u_{x,\varphi_t(x)}$ is bnd. fam. of paths.

$$\Rightarrow \lim_{t\to\infty} \frac{1}{t} \int_{\underbrace{\gamma_t(x) * (u_{x,\varphi_t(x)})^{-1}}_{\tilde{\gamma}_t(x)}} \omega = \underbrace{\int_M i_X \omega \, d\mu}_{\cong \rho_\mu}$$

Since each hom. class depends uniq. on it's values on a basis of the cohomology we can deduce

$$\lim_{t\to\infty} \frac{1}{t} [\tilde{\gamma}_t(x)] \cong \rho_\mu$$

$\Rightarrow$ Thm.: $\rho_\mu = \lim_{t\to\infty} \frac{1}{t} [\tilde{\gamma}_t(x)]$ μ a.e. $x \in M$ if "$(M, \mathcal{B}(M), \mu, \varphi_t)$ is erg. $\forall t \in \mathbb{R}$.

Example We want to prove that, if μ is given by a per. orbit γ of X then $\rho_\mu = \frac{1}{T}[\gamma]$, T is period of γ.

$\Rightarrow \gamma : \mathbb{R}/T\cdot\mathbb{Z} \longrightarrow M$

Let λ be the norm. Lebesg. meas. on $\mathbb{R}/\mathbb{Z}$.

$\Rightarrow \frac{1}{T}\lambda$ is the norm. Lebesgue meas. on $\mathbb{R}/T\cdot\mathbb{Z}$.

$\Rightarrow \frac{1}{T}(\gamma_* \lambda)$ is a prob. meas. on M.

$$\begin{aligned}
\Rightarrow \rho_\mu &\cong \int_M i_X \omega \tfrac{1}{T} d(\gamma_* \lambda) \\
&= \int_M \omega(X(p)) \tfrac{1}{T} \underbrace{d(\gamma_*\lambda)(p)}_{0, \text{ if } p \notin \operatorname{im}\gamma} \\
&= \int_{\operatorname{im}\gamma} \omega(X(p)) \tfrac{1}{T} d(\gamma_*\lambda)(p) \\
&= \int_{\mathbb{R}/T\cdot\mathbb{Z}} \omega(X(\gamma(t))) \tfrac{1}{T} d\lambda(t) \\
&= \int_0^T \omega(X(\gamma(t))) \tfrac{1}{T} d\lambda(t)
\end{aligned}$$

$$= \int_0^T \omega(X(\gamma(t)))\,\frac{1}{T}\,d\lambda(t)$$

$$\overset{\gamma \text{ per. orbit of } X}{=} \int_0^T \omega(\dot{\gamma}(t))\,d\lambda(t)$$

$$= \int_\gamma \omega \cdot \frac{1}{T}$$

A sim. argument as before shows now that:

$$\frac{1}{T}[\gamma] \cong \rho_\mu$$

References:

- For the part of the graph thm see [Sor]
- For the part of the prob. meas and rot. vectors see 4.1 (a)-(f) in [KH]
- For the Lemma of ergodicity: http://math.huji.ac.il/~mhochman/courses/ergodic-theory-2012/notes.final.pdf