Aubry & Mañé Sets

M compact, $L : TM \to \mathbb{R}$ Tonelli Lag.
All curves in M abs. cont.

① Recap

- $\gamma : \mathbb{R} \to M$ is glob. min for L if
  $\forall a < b$: $\mathcal{A}_L(\gamma|_{[a,b]}) = \min\limits_{\substack{\sigma : [a,b] \to M \\ \sigma(a) = \gamma(a),\ \sigma(b) = \gamma(b)}} \mathcal{A}_L(\sigma)$.

- $\gamma : \mathbb{R} \to M$ is time-free min. for L if
  $\forall a < b$: $\mathcal{A}_L(\gamma|_{[a,b]}) = \min\limits_{\substack{\sigma : [a',b'] \to M \\ \sigma(a') = \gamma(a),\ \sigma(b') = \gamma(b)}} \mathcal{A}_L(\sigma)$.

γ time-free min. $\Rightarrow$ γ global min.

γ global min. for L $\Rightarrow$ γ global min. for $L + k$, $k \in \mathbb{R}$.

② Mañé Potential & Crit. Value

$x, y \in M$, $T > 0$ $\quad C_T(x,y) := \{\gamma : [0,T] \to M \mid \gamma(0) = x,\ \gamma(T) = y\}$

this is actually a min

Def: $\Phi_k(x,y,T) := \inf\limits_{\gamma \in C_T(x,y)} \mathcal{A}_{L+k}(\gamma)$

$$\Phi_k(x,y) := \inf_{T>0} \Phi_k(x,y,T) \quad \text{for } k \in \mathbb{R}.$$

is this also a min? (1)

When is $\Phi_k(x,y) > -\infty$?

If $\exists\, T>0,\ \gamma \in C_T(x,x)$ $\quad \mathcal{A}_{L+k}(\gamma) < 0$,

then $\mathcal{A}_{L+k}(\gamma^n) = n \cdot \mathcal{A}_{L+k}(\gamma) \to -\infty$

$\Rightarrow \Phi_k(x,x) = -\infty$

Prop: For all $x,y \in M$ have

$$\Phi_k(x,y) \begin{cases} = -\infty & \text{if } k < c(L) \\ \in \mathbb{R} & \text{if } k \geq c(L), \end{cases}$$

where $c(L) := \sup\{k \mid \exists\, x \in M,\ T>0,\ \gamma \in C_T(x,x)$ with $\mathcal{A}_{L+k}(\gamma) < 0\}$.

(Mañé crit. value)

Moreover, for $k \geq c(L)$ $\Phi_k : M \times M \to \mathbb{R}$ is Lipschitz.

$(x,y) \mapsto \Phi_k(x,y) + \Phi_k(y,x)$ is a (pseudo-) metric for $k > c(L)$ $(k \geq c(L))$.

③ Aubry & Mañé sets

Answer to (1) is yes if $k > c(L)$.

(For $x, y \in M$, have $\Phi_k(x,y,T) \to \infty$ if $T \to \infty$
if $T \to 0$)

$\Rightarrow$ We can connect any two pts in M by a time-free min. of energy $k > c(L)$ & there are no min. for $k < c(L)$.

<u>Q</u>: What happen at $k = c(L)$?

<u>Def</u>: • $\gamma : \mathbb{R} \to M$ is semi-static for L if $\forall\, a < b$: $\mathcal{A}_{L+c(L)}(\gamma|_{[a,b]}) = \Phi_{c(L)}(\gamma(a), \gamma(b))$.

(i.e. γ time-free min. for $L + c(L)$).

• The Mañé set is

$\tilde{\mathcal{N}} := \{ (x,v) \in TM \mid \exists \gamma$ semi-static, $t \in \mathbb{R}$ s.th. $x = \gamma(t)$, $v = \dot\gamma(t) \}$

Note: - $\tilde{\mathcal{N}}$ is invariant under the EL-flow

- $\tilde{\mathcal{N}}$ is closed

Recall: $\Phi_{c(L)}(\gamma(a), \gamma(b)) \neq \Phi_{c(L)}(\gamma(b), \gamma(a))$

(Have $A_{L+k}(\gamma^{-1}) = - A_{L+k}(\gamma)$)

$\Rightarrow \Phi_k(x,y) \geq - \Phi_k(y,x)$ (for $k \geq c(L)$)

<u>Def</u>: • $\gamma : \mathbb{R} \to M$ is static for L if

$\forall a < b$: $A_{L+c(L)}(\gamma|_{[a,b]}) = - \Phi_{c(L)}(\gamma(b), \gamma(a))$

(i.e. γ is semi-static and

$\Phi_{c(L)}(\gamma(a), \gamma(b)) = - \Phi_{c(L)}(\gamma(b), \gamma(a))$)

• The Aubry set is

$\tilde{\mathcal{A}} := \{ (x,v) \in TM \mid \exists \gamma \text{ static}, t \in \mathbb{R}: x = \gamma(t), v = \dot\gamma(t) \}$

Note: - $\tilde{\mathcal{A}}$ is invariant under EL-flow
- $\tilde{\mathcal{A}}$ is closed
- $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{N}}$

④ α- and ω-limits

For $(x,v) \in TM$, set

$$\alpha(x,v) := \{ (y,w) \in TM \mid \exists (t_n) \text{ with } t_n \to -\infty \text{ and } \varphi^L_{t_n}(x,v) \to (y,w) \}$$

$$\omega(x,v) := \{ (y,w) \in TM \mid \exists (t_n) \text{ with } t_n \to \infty \text{ and } \varphi^L_{t_n}(x,v) \to (y,w) \}.$$

Prop: Let $(x,v) \in \tilde{\mathcal{N}}$. Then
(1) $\alpha(x,v) \subseteq \tilde{\mathcal{A}}$
(2) $\omega(x,v) \subseteq \tilde{\mathcal{A}}$.

Cor: $\tilde{\mathcal{A}} \neq \emptyset$ iff $\tilde{\mathcal{N}} \neq \emptyset$.

Pf of Prop: (only (2))

Let $(y,w) \in \omega(x,v)$ with (t_n) s.th. $t_n \to \infty$ and $\varphi^L_{t_n}(x,v) \to (y,w)$.

$\tilde{\gamma}(t) := \varphi_t(x,v)$, $\tilde{\eta}(t) := \varphi^L_t(y,w)$

$\gamma := \pi \circ \tilde{\gamma}$, $\eta := \pi \circ \tilde{\eta}$.

$(x,v) \in \tilde{\mathcal{N}} \Rightarrow \gamma$ semi-static.

<u>To show:</u> η static.

Enough to check $\forall s > 0$:

$$A_{L+c(L)}\left(\eta|_{[-s,s]}\right) + \Phi_{c(L)}(\eta(s), \eta(-s)) = 0.$$

Denote $\tilde{\gamma}_n(t) := \tilde{\gamma}(t_n + t)$, $\gamma_n := \pi \circ \tilde{\gamma}_n$.

$\tilde{\gamma}_n(0) \to (y,w)$, so uniqueness of EL-sol. implies $\tilde{\gamma}_n|_{[-s,s]} \xrightarrow{C^1} \tilde{\eta}|_{[-s,s]}$, same without $\sim$.

$$A_{L+c(L)}\left(\eta|_{[-s,s]}\right) + \Phi_{c(L)}(\eta(s), \eta(-s))$$

$$= \lim_n A_{L+c(L)}\left(\gamma_n|_{[-s,s]}\right) + \lim_n \lim_{m \gg 0} \Phi_{c(L)}(\gamma_n(s), \gamma_n(t_n - t_m - s))$$

$$= \lim_n \mathcal{A}_{L+c(L)}\left(\gamma_n|_{[-s,s]}\right)$$

$$+ \lim_n \lim_{m \gg 0} \mathcal{A}_{L+c(L)}\left(\gamma_n|_{[s,\, t_m - t_n - s]}\right)$$

$$= \lim_n \lim_m \mathcal{A}_{L+c(L)}\left(\gamma_n|_{[-s,\, t_m - t_n - s]}\right)$$

$$= \lim_n \lim_m \phi_{c(L)}\left(\gamma_n(-s), \gamma_n(t_m - t_n - s)\right)$$

$$= \lim_n \lim_m \phi_{c(L)}\left(\gamma_n(-s), \gamma_m(-s)\right)$$

$$= \phi_{c(L)}\left(\gamma(-s), \gamma(-s)\right) = 0. \qquad \square$$

5 Non-wandering points

<u>Def</u>: $\varphi_t : X \to X$ flow. $x \in X$ non-wandering if $\forall\, U \ni x$ nbh. $n > 0$ $\exists\, t > n$ s.th.

$$\varphi_t(U) \cap U \neq \emptyset.$$

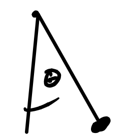
Write $\Omega(\varphi) := \{x \in X \mid x \text{ non-wandering}\}$

Facts: • $\alpha(x), \omega(x) \overset{(*)}{\subseteq} \Omega(\varphi) \quad \forall x \in X$

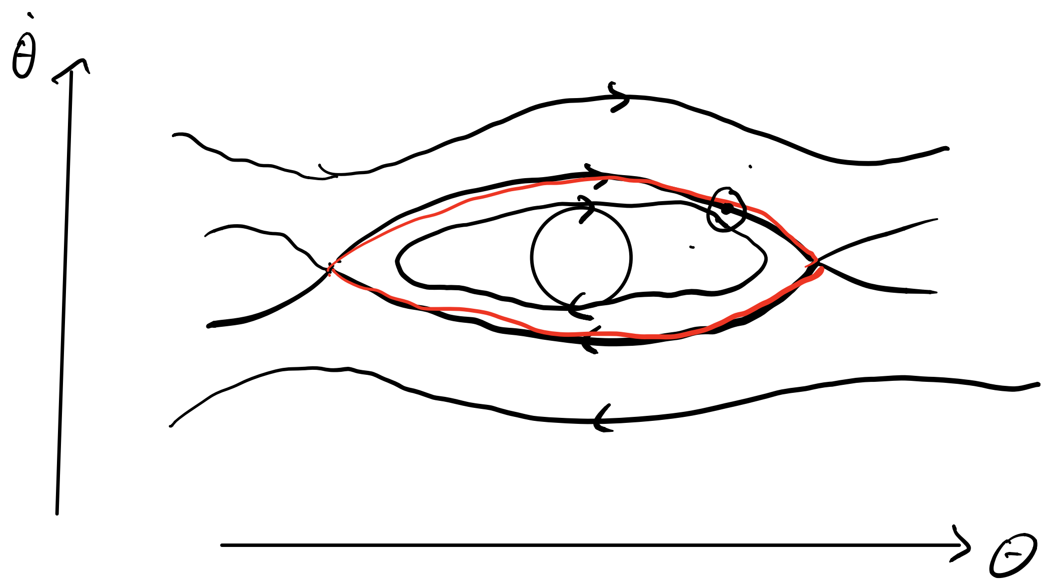
$\cdot\ \Omega(\varphi^L|_{\tilde{\mathcal{N}}}) \subseteq \tilde{\mathcal{A}}$ (need weak KAM)

Inclusion (*) can be proper:

Consider the pendulum



$$\ddot\theta + \sin\theta = 0$$



• : non-wandering pt., which are neither α- nor ω-limits.

(b) Mañé Lagrangians

Let $X \in \Gamma^r(TM)$ vector field, $r \geq 2$

Define a Lag. $L_X : TM \to \mathbb{R}$ $(x,v) \mapsto \frac{1}{2}\|v - X(x)\|^2$,

where $\|\cdot\|$ comes from a Riem. metric.

$$\mathrm{Graph}(X) = X(M) = \{(x,v) \in TM \mid v = X(x)\}$$

$$L_X\big|_{\mathrm{Graph}(X)} \equiv 0 \;, \quad L_X\big|_{\mathrm{Graph}(X)^c} > 0$$

$$\Rightarrow A_{L_X}(\gamma|_{[a,b]}) \begin{cases} = 0 & \text{if } \dot\gamma(t) = X(\gamma(t)) \ \forall t \\ > 0 & \text{else} \end{cases}$$

$\Rightarrow$ integral curves of X are time-free min. for L_X.

Also:

$$\begin{tikzcd} \mathrm{Graph}(X) \arrow[r, "\varphi^{L_X}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Graph}(X) \arrow[d, "\pi"] \\ M \arrow[r, "\varphi^X"'] & M \end{tikzcd}$$

It turns out that $\mathrm{Graph}(X) \subseteq \tilde{\mathcal{N}}(L_X)$.

This is true because integral curves are semi-static.