Last time:

- $\tilde{G}_+ := \{ f : \mathbb{R} \to \mathbb{R} \text{ homeomorphisms}, \ f(y+1) = f(y) + 1 \ \forall y \in \mathbb{R} \}$ group
  [$f \in \tilde{G}_+$ yields a $\mathbb{Z}^2$-action T^f on $\mathbb{R}$: $T^f_{(q,p)} y = f^q(y) + p$]
- $\emptyset \neq S \subset \mathbb{R}^{\mathbb{Z}}$ closed, T-invariant, totally ordered $\Rightarrow \exists f \in \tilde{G}_+$: $\begin{tikzcd} S \arrow[r, "T"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[d, "p_0"] \\ \mathbb{R} \arrow[r, "T^f"'] & \mathbb{R} \end{tikzcd}$ $\left[\begin{array}{l} \forall x \in S \\ \forall i \in \mathbb{Z} \\ f(x_i) = x_{i+1} \end{array} \right.$
  [If ℓ is C^2 and $D_1 D_2 \ell < 0$, then f and f^{-1} Lipschitz].
- $\forall x \in M$, $\exists f \in \tilde{G}_+ : f(x_i) = x_{i+1} \ \forall i \in \mathbb{Z}$. $\left[|x_{i+1} - x_i| = |f(x_i) - x_i| \leq \max(f - \mathrm{id}_{\mathbb{R}}) \right]$

Rmk 1 (i) $\tilde{G}_+$ are lifts of elements of $G_+ := \{ \varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \text{ homeo}, \ \deg \varphi = 1 \}$

(ii) $\forall \rho \in \mathbb{R}$, $T_\rho : \mathbb{R} \to \mathbb{R}$ translation by ρ is in $\tilde{G}_+$

(iii) $f \in \tilde{G}_+ \Rightarrow \exists g_f : \mathbb{R} \to \mathbb{R}$ 1-periodic s.t. $f(y) - y =: g_f(y)$. f homeo $\Rightarrow \max g_f - \min g_f < 1$.

(iv) $\forall k > 0$: $g_{f^k} = \sum_{j=0}^{k-1} g_f \circ f^j$ (telescopic sum) ⊛

Thm 1 If $f \in \tilde{G}$, then $\forall y \in \mathbb{R}$, $\exists \rho(f) := \lim_{|i| \to \infty} \frac{f^i(y)}{i} = \lim_{|i| \to \infty} \frac{g_{f^i}(y)}{i}$ and is independent of y.
[Katok-Hasselblatt] ☒

Rmk 2 (i) $\rho(f)$ is called the rotation number of f and measures the average rate at which f rotates.

(ii) $\rho(T_\rho) = \rho$

(iii) $\forall i \in \mathbb{Z}$ define $r_i : \mathbb{R} \to \mathbb{R}$, $r_i(y) := g_{f^i}(y) - i\rho(f)$. Then:

(a) $\exists y_i$ s.t. $r_i(y_i) = 0$ and (b) $|r_i| < 1$.

Proof: If (a) does not hold, then WLOG $\exists \delta > 0$: $g_{f^i} \geq i\rho(f) + \delta \overset{⊛}{\Longrightarrow} g_{f^{ki}} \geq ki\left(\rho(f) + \frac{\delta}{i}\right)$

Hence $\rho(f) = \lim_{k \to \infty} \frac{g_{f^{ki}}(y)}{ki} \geq \rho(f) + \delta/i$ ↯

If (b) does not hold, then WLOG $\exists y : r_i(y) \geq 1$. Then:

$g_{f^i}(y) - g_{f^i}(y_i) = r_i(y) - r_i(y_i) \geq 1$ ↯ by Rmk 1. (iii).

(iv) $\rho(f) = \frac{p}{q} \in \mathbb{Q} \Leftrightarrow \exists y$ with period (q,p): $f^q(y) = y + p$.

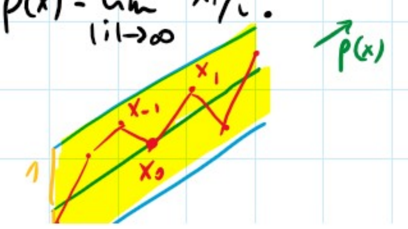
(v) The map $f \mapsto \rho(f)$ is continuous in the C^0-topology.

Corollary 1 $\exists$ continuous, T-invariant function $\rho : M \to \mathbb{R}$ such that

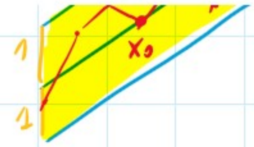
(a) $\forall x \in M$, $\forall i \in \mathbb{Z}$: $|x_i - x_0 - i\rho(x)| < 1$, thus $\rho(x) = \lim_{|i| \to \infty} x_i / i$.

(b) If $x \in M_{q,p}$, then $\rho(x) = p/q$.

Thus, $\forall x, x^* \in M$, $\rho(x) \neq \rho(x^*) \Rightarrow x$ and x^* cross.



Thus, $\forall x, x' \in M$, $p(x) \neq p(x') \Rightarrow x$ and x' cross.



Corollary 2 For all $C>0$ and $C_1, C_2 > 0$, the sets
$$M^C := \{x \in M \mid |x_0| \le C, |p(x)| \le C\},\ M^{C_1,C_2} := \{x \in M \mid \begin{smallmatrix} x_0 \geq -C_1 \\ x_{-1} \le C_1 \end{smallmatrix},\ |p(x)| \le C_2\} \text{ are compact.}$$

Proof By Corollary 1(a): $x_0 < x_{-1} + 1 + p(x)$. Hence: $M^{C_1,C_2} \subseteq M^{C_1+C_2+1}$. Again by Corollary 1(a): M^C is contained in the compact set $\{x \in M \mid \forall i \in \mathbb{Z} : |x_i| \le 1 + (1+|i|)C\}$. ▨

Corollary 3 For all $p \in \mathbb{R}$, $M_p := \{x \in M \mid p(x) = p\} \neq \emptyset$.

Proof Take $\frac{p_n}{q_n} \to p$ $\quad \left|\frac{p_n}{q_n}\right| \le |p|+1$. Consider $x^{(n)} \in M^{|p|+1} \cap M_{q_n,p_n}$ (why does it exist?). Corollary 2 yields $x^{(n_k)} \to x \in M_p$. ▨

Dynamics of $f \in \tilde{G}_+$

We know that if $x \in M_p$, $\exists f \in \tilde{G}_+$ with $p(f) = p$ and $x_i = f^i(x_0)$ $\forall i \in \mathbb{Z}$.
Question: if $y \in \mathbb{R}$ arbitrary, how does $x^y \in \mathbb{R}^{\mathbb{Z}}$, $(x^y_i)_{i\in\mathbb{Z}} := \{f^i(y) \mid i \in \mathbb{Z}\}$ look like?

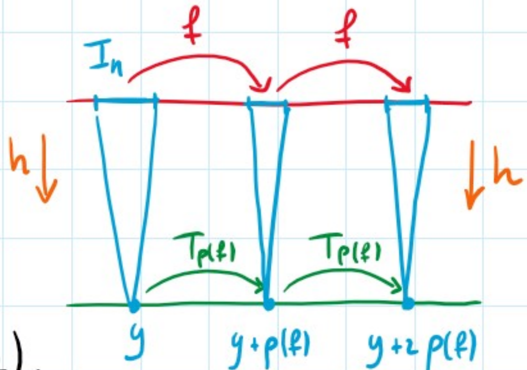
Case $p(f) = \frac{p}{q} \in \mathbb{Q}$ The set $\mathrm{Per}_f := \{y \in \mathbb{R} \mid T^f_{(q,p)} y = y\}$ is closed and f-invariant. If $y \notin \mathrm{Per}_f$, then x^y is α- (resp. ω-) asymptotic to x^{y^α}, x^{y^β} with $y^\alpha, y^\omega \in \mathrm{Per}_f$ s.t. $\nexists\, \tilde{y} \in \mathrm{Per}_f$ between y^α and y^ω (both $y^\alpha < y^\omega$ and $y^\omega < y^\alpha$ can happen). y^α and y^ω are neighboring (see example with $p=0, q=1$).

Case $p(f) \notin \mathbb{Q}$ The set $\mathrm{Rec}_f := \{y \in \mathbb{R} \mid \exists (q_i, p_i) : y = \lim_i T^f_{(q_i,p_i)} y\}$ is the smallest closed T^f-invariant set. Then: $h \circ f = T_{p(f)} \circ h$, $h : \mathbb{R} \to \mathbb{R}$, $h(y+1) = h(y) + 1$, non-decreasing.

Two possibilities:

(a) $\mathrm{Rec}_f = \mathbb{R}$, then h is homeo;

(b) Rec_f has no interior and no isolated points (is a Cantor set)
$\mathbb{R} \setminus \mathrm{Rec}_f$ is a disjoint union of open intervals I_n such that $I_n \cap T^f_{(q,p)}(I_n) = \emptyset$ if $(q,p) \in \mathbb{Z}\setminus\{0\} \times \mathbb{Z}$



The map h collapses each I_n to a point (see picture).

Lemma 1 • x^{y_1} and x^{y_2} are α/ω-asymptotic iff $y_1, y_2 \in \bar{I}_n$,

Lemma 1'
- x^{y_1} and x^{y_2} are α/ω-asymptotic iff $y_1, y_2 \in \bar{I}_n$,
- for all $y \in \mathbb{R}$ and all (q_n, p_n) s.t. $\exists \lim_n T^f_{(q_n,p_n)} y =: y^*$, then $y^* \in Rec_f$. ▨

Lemma 2 Let $f_1, f_2 \in \tilde{G}_+$ with $\rho(f_1) = \rho(f_2) \notin \mathbb{Q}$. Then:
- either $Rec_{f_1} = Rec_{f_2}$ and $f_1|Rec_{f_1} = f_2|Rec_{f_2}$,
- or $\exists\, y_1 \in Rec_{f_1}, y_2 \in Rec_{f_2}$ s.t. x^{y_1} and x^{y_2} cross infinitely many times. ▨

Structure of M_ρ for $\rho \notin \mathbb{Q}$

Theorem 2 If $\rho \notin \mathbb{Q}$, then M_ρ is totally ordered.

Proof Let $x^{(1)}, x^{(2)} \in M_\rho$ and let $f_1, f_2 \in \tilde{G}_+$ be the corresponding maps $\rho(f_1) = \rho = \rho(f_2)$.
Since $p_0(\overline{Tx^{(1)}})$ is closed and f_1-invariant, $p_0(\overline{Tx^{(1)}}) \supseteq Rec_{f_1}$. Analogously $p_0(\overline{Tx^{(2)}}) \supseteq Rec_{f_2}$.
By the crossing lemma, the first alternative in Lemma 2 must hold: $Rec_{f_1} = Rec_{f_2}$ and $f_1|Rec_{f_1} = f_2|Rec_{f_2}$. Thus f_1 and f_2 have the same wandering intervals I_n.
If $x_0^{(1)}, x_0^{(2)}$ do not belong to the closure of the same wandering interval $\exists\, y \in Rec_{f_1} = Rec_{f_2}$ with $x_0^{(1)} < y < x_0^{(2)}$ (WLOG), then $x_i^{(1)} = f_1^i(x_0^{(1)}) < f_1^i(y) = f_2^i(y) < f_2^i(x_0^{(2)}) = x_i^{(2)}\ \forall i \in \mathbb{Z}$.
If $x_0^{(1)}, x_0^{(2)} \in \bar{I}_n = [y^-, y^+]$, then $x^{(1)}$ is asymptotic to $(f_1^i(y_-))_{i\in\mathbb{Z}}$ and $x^{(2)}$ is asymptotic to $(f_2^i(y_-))_{i\in\mathbb{Z}}$ by Lemma 1. $f_1^i(y_-) = f_2^i(y_-)\ \forall i \in \mathbb{Z} \Rightarrow x^{(1)}$ and $x^{(2)}$ are asymptotic. By the crossing lemma, they do not cross. ▨

Corollary 4 If $\rho \notin \mathbb{Q}$ $\exists\, f_\rho \in \tilde{G}_+$, $\rho(f_\rho) = \rho$ s.t. $\forall x \in M_\rho\ \forall i \in \mathbb{Z}: f_\rho(x_i) = x_{i+1}$. Hence: $p_0(M_\rho)$ is closed and f_ρ-invariant. Moreover: $Rec_{f_\rho} = p_0(M_\rho^{rec})$, where
$M_\rho^{rec} := \{x \in M_\rho \mid \exists (q_i, p_i) : x = \lim_i T_{(q_i,p_i)} x\}$. ▨

Remark 3 If $Rec_{f_\rho} = \mathbb{R}$, then $\forall y \in \mathbb{R}\ \exists x \in M_\rho^{rec}$ with $x_0 = y$.
If $Rec_{f_\rho} \neq \mathbb{R}$, then M_ρ^{rec} is a Cantor set and there are cases where $M_\rho = M_\rho^{rec}$.

Structure of M_ρ for $\rho = p/q \in \mathbb{Q}$

Recall that $M_{q,p}$ is a totally ordered subset of M_p. $x^- < x^+$ are neighboring in $M_{q,p}$ if no $x^* \in M_{q,p}$ exists with $x^- < x^* < x^+$. For neighboring elements we define:

$$M_p^{\pm}(x^-, x^+) := \{ x \in M_p \mid x \text{ is } \overset{\omega}{\alpha}\text{-asympt. to } x^- \text{ and } \overset{\alpha}{\omega}\text{-asympt. to } x^+ \}.$$

Remark 5 Crossing lemma $\Rightarrow$
- $x \in M_p^{\pm}(x^-, x^+) \Rightarrow x^- < x < x^+$;
- $\tilde{x}^{\pm} \in M_p^{\pm}(x^-, x^+) \Rightarrow \tilde{x}^+$ and $\tilde{x}^-$ cross
- $x, x^* \in M_p^{+}(x^-, x^+) \Rightarrow x, x^*$ don't cross (Same for $M_p^-(x^-, x^+)$).

Theorem 3 Define $M_p^{\pm} := \bigcup_{\substack{x^- < x^+ \\ \text{neigh.}}} M_p^{\pm}(x^-, x^+)$. Then: $M_p = M_{q,p} \sqcup M_p^+ \sqcup M_p^-$.

Proof Take $x \in M_p \setminus M_{q,p}$. Consider $f \in \widehat{G}_+$ s.t. $f(x_i) = x_{i+1}$. Then x is α-asymptotic to $x^- := x^{y_-}$ and ω-asymptotic to $x^+ := x^{y_+}$ with $y_\pm \in \mathrm{Perf}$. Assume $y_- < y_+$ (other case is similar). $\forall j \in \mathbb{Z}$: $\lim\limits_{k \to +\infty} \left(T_{(-kq, kp)} x - x^+ \right)_j = \lim\limits_{k \to +\infty} \left[x_{j+kq} - (x_j^+ - kp) \right] \overset{x^- \in \mathrm{Perf}}{=} \lim\limits_{k \to +\infty} \left[x_{j+kq} - x_{j+kq}^+ \right] = 0.$

M is closed $\Rightarrow x^+ \in M_{q,p}$ (analog. $x^- \in M_{q,p}$).

Left to prove: x^- and x^+ neighboring. By contradiction $\exists\, x^* \in M_{q,p}$ $x^- < x^* < x^+$.
$\Rightarrow$ x and x^* cross, say between 0 and 1 (Same for other cases).

Claim: $\exists j$ big enough so that $(x_0, \ldots, x_{j+q+1})$ is not minimal.

Consider $(z_0, \ldots, z_{j+q+1})$ with $z_0 = x_0$ and $z_{j+q+1} = x_{j+q+1}$ as follows:
- $z_i = x_i^*$ for $1 \le i \le q$;
- $z_i = (T_{(q,p)} x)_i$ for $q+1 \le i \le j+q+1$.

We have: $a(x_0, \ldots, x_{j+q+1}) = \ell(x_0, x_1) + a(x_1, \ldots, x_j) + \overbrace{\ell(x_j, x_{j+1})}^{= \ell(x_j + p,\, x_{j+1} + p)} + \overbrace{a(x_{j+1}, \ldots, x_{j+q+1})}^{\xrightarrow{j \to \infty} a(x_1^+, \ldots, x_{q+1}^+) = \inf a_{q,p}}$.

Moreover:

$$a(z_0, \ldots, z_{j+q+1}) = \ell(x_0, x_1^*) + a(x_1^*, \ldots, x_q^*) + \ell(\underset{x_0^* + p}{x_q^*}, \underset{x_1 + p}{(T_{(q,p)}x)_{q+1}})$$
$$+ a((T_{(q,p)}x)_{q+1}, \ldots, (T_{(q,p)}x)_{j+q}) + \ell((T_{(q,p)}x)_{j+q}, x_{j+q+1}).$$

$$= \ell(x_0, x_1^*) + a(x_1^*, \ldots, x_q^*) + \ell(x_0^*, x_1) + a(x_1, \ldots, x_j) + \ell(x_j + p, x_{j+q+1})$$

$= a(x_0^*, \ldots, x_q^*) = \inf a_{q,p}$

$\overset{(\ell 3)}{=} -\varepsilon + \ell(x_0^*, x_1^*) + a(x_1^*, \ldots, x_q^*) + \ell(x_0, x_1) + a(x_1, \ldots, x_j) + \ell(x_j + p, x_{j+q+1})$

$$= \ell(x_0, x_1^*) + a(x_1^*, \ldots, x_q^*) + \ell(x_0^*, x_1) + a(x_1, \ldots, x_J) + \ell(x_J + p, x_{J+q+1})$$

$$\underset{(\ell 3)}{=} -\varepsilon + \underbrace{\ell(x_0^*, x_1^*) + a(x_1^*, \ldots, x_q^*)}_{= \inf a_{q,p}} + \ell(x_0, x_1) + a(x_1, \ldots, x_J) + \ell(x_J + p, x_{J+q+1})$$

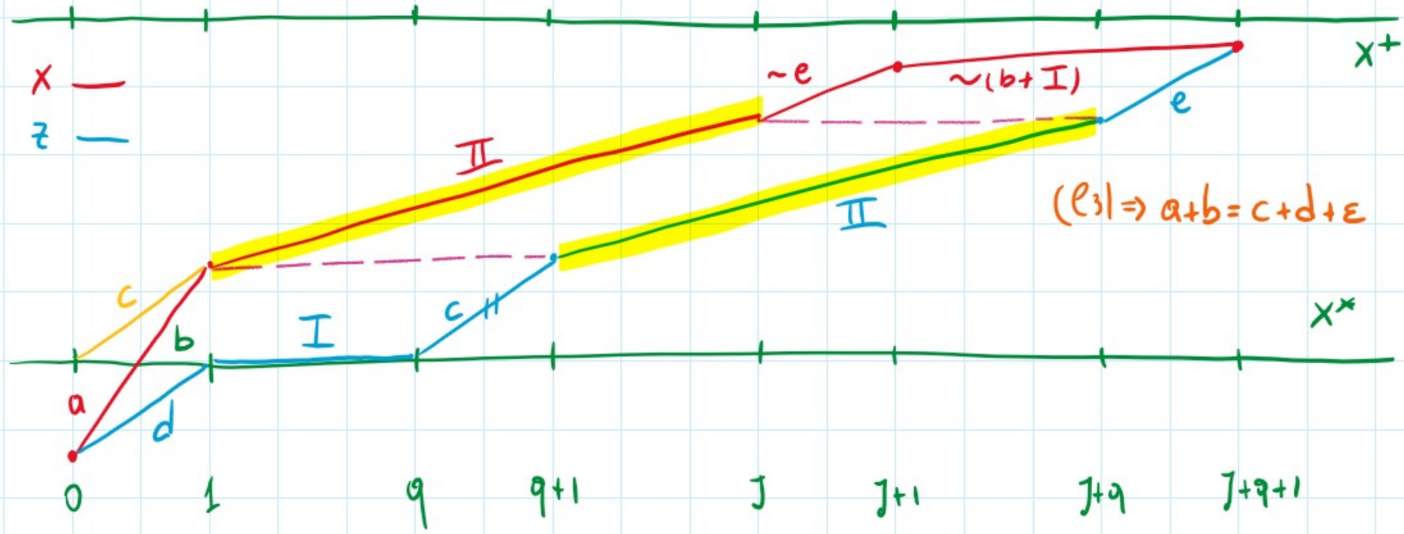
Thus: $\lim\limits_{J\to\infty} \left[a(x_0, \ldots, x_{J+q+1}) - a(z_0, \ldots, z_{J+q+1})\right] \geq \cancel{\inf a_{q,p}} + \varepsilon - \cancel{\inf a_{q,p}} + \lim\limits_{J\to\infty} \left[\begin{matrix}\ell(x_J+p, x_{J+1}+p) \\ -\ell(x_J+p, x_{J+q+1})\end{matrix}\right]$

The last limit is zero since $|x_{J+1} + p - x_{J+q+1}| = |x_{J+1} - x_{J+1}^* + (x_{J+q+1} - x_{J+q+1}^*)|$

$$\leq |x_{J+1} - x_{J+1}^*| + |x_{J+q+1} - x_{J+q+1}^*| \xrightarrow{J\to\infty} 0.$$

Thus $\lim\limits_{J\to\infty} \left[a(x_0, \ldots, x_{J+q+1}) - a(z_0, \ldots, z_{J+q+1})\right] \geq \varepsilon$ and the claim (and the theorem) is proved. ▨

The following picture gives the idea of the proof for $p=0$, $q=1$.



Corollary 5 $M_p^{\pm}(x_-, x_+)$ are non-empty for all neighboring $x^-, x^+ \in M_{q,p}$.

Proof Take $p_n \to p$ with $p < p_n < p+1$. Take $x^{(n)} \in M_{p_n}$. Let $\varepsilon := \min\limits_{i\in\mathbb{Z}} |x_i^+ - x_i^-|$.

$\exists\, i(n) \in \mathbb{Z}$ s.t. • $x_i^{(n)} \leq x_i^- + \frac{\varepsilon}{2}$ $\forall i < i(n)$ • $x_{i(n)}^{(n)} > x_{i(n)}^- + \frac{\varepsilon}{2}$.

Write $i(n) = k(n)q + j(n)$ for $j(n) \in [0, \ldots, q-1]$, define $\tilde{x}^{(n)} := T_{(-k(n)q, k(n)p)} x^{(n)}$.

Since $T_{(q,p)} x^- = x^-$ we get • $\tilde{x}_i^{(n)} \leq x_i^- + \frac{\varepsilon}{2}$ $\forall i < j(n)$ • $\tilde{x}_{j(n)}^{(n)} > x_{j(n)}^- + \frac{\varepsilon}{2}$

Up to subsequence $j(n) \equiv j$. By Corollary 2: $\tilde{x}^{(n)} \to x \in M_p$ (up to sub).

Then: (i) $x_i \leq x_i^- + \varepsilon/2 < x_i^+$ $\forall i < j$, (ii) $x_j \geq x_j^- + \varepsilon/2$.

Hence: $x \notin M_{q,p}$ ⇒ $\exists\, \tilde{x}^-, \tilde{x}^+ \in M_{q,p}$ neighboring s.t. $x \in M_p^+(\tilde{x}^-, \tilde{x}^+)$.

By (i) $\tilde{x}^- < x^+$. By (ii) $\tilde{x}^+ > x^-$. Since $\tilde{x}^-, \tilde{x}^+$ and x^-, x^+ are both neighboring pairs the only possibility is $\tilde{x}^- = x^-$, $\tilde{x}^+ = x^+$. ▨

Remark 6 Two cases for M_p, $p = \frac{p}{q} \in \mathbb{Q}$: (i) $p_0(M_{q,p}) = \mathbb{R}$: $\forall y \in \mathbb{R}\ \exists!\ x \in M_{q,p}$, $y = x_0$,
$\Rightarrow M_p = M_{q,p}$

(ii) $p_0(M_{q,p}) \neq \mathbb{R}$: $M_p^+ \neq \phi$, $M_p^- \neq \phi$ and $M_{q,p} \sqcup M_p^+$, $M_{q,p} \sqcup M_p^-$ totally ordered.