Our goal: M closed, $L: TM \to \mathbb{R}$ Tonelli, ϑ closed 1-form on M. Find global time-free minimizers $\gamma: \mathbb{R} \to M$ for $L_\vartheta + k$ for some $k \in \mathbb{R}$.

Lift to universal cover $\pi: \tilde{M} \to M$, $\tilde{L} := L \circ d\pi : T\tilde{M} \to \mathbb{R}$

$\gamma: \mathbb{R} \to M$ global time-free minimizer for $L_\vartheta + k$ $\Rightarrow$ lift $\tilde{\gamma}: \mathbb{R} \to \tilde{M}$ global time-free minimizer for $\tilde{L}_{\tilde{\vartheta}} + k$ $\Leftrightarrow$ $\tilde{\gamma}$ global time-free minimizer for $\tilde{L} + k$ (since $\tilde{\vartheta}$ is exact).

Question: Is the converse of "$\Leftarrow$" true for some ϑ? In general, I don't know. We look at an example where it does.

Length-minimizing geodesics

Take $L(x,v) = \frac{1}{2}|v|_x^2$ for some Riem. metric on M. $\tilde{L}(\tilde{x},\tilde{v}) = \frac{1}{2}|\tilde{v}|_x^2$ for lifted metric on $\tilde{M}$

Lemma $\tilde{\gamma}: \mathbb{R} \to \tilde{M}$ global time-free min. for $\tilde{L} + \frac{1}{2}$ $\Leftrightarrow$ $\tilde{\gamma}$ global length-minimizing geodesic with unit speed: $d(\tilde{\gamma}(a), \tilde{\gamma}(b)) = \text{length}(\tilde{\gamma}|_{[a,b]})$ $\forall a,b \in \mathbb{R}$. ☑

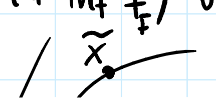
Thm 1 $\tilde{M}$ not compact $\Leftrightarrow$ $\exists$ global length-minimizing unit-speed geodesic (at least $\dim H_1(M;\mathbb{R})$ of them)

Proof Take $\tilde{\gamma}_n: [-n,n] \to \tilde{M}$ unit-speed min. geod. Acting by deck $\Rightarrow$ $\{\tilde{\gamma}_n(0)\}_{n \in \mathbb{N}} \subset K$ compact $\Rightarrow$ $\gamma_{n_k} \to \gamma$ min. ☑

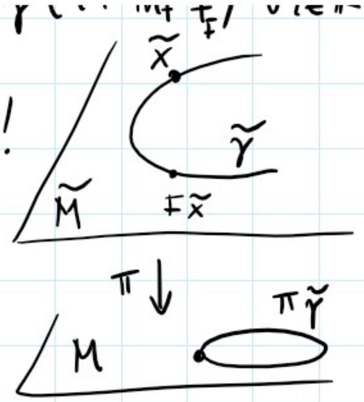
Let $F: \tilde{M} \to \tilde{M}$ be a deck transformation ($\pi \circ F = \pi$), $F \neq \mathrm{id}_{\tilde{M}}$. Define

$$f_F: \tilde{M} \to \mathbb{R}, \qquad f_F(\tilde{x}) := d(\tilde{x}, F\tilde{x}) \quad \forall \tilde{x} \in \tilde{M}$$

Thm 2 The set $S_F := \{\tilde{x} \in \tilde{M} \mid f_F(\tilde{x}) = \inf f_F\}$ is non-empty. Geodesics $\tilde{\gamma}: \mathbb{R} \to \tilde{M}$ extending a minimal segment $\tilde{\gamma}_0$ from $\tilde{x}$ to $F\tilde{x}$ for some $\tilde{x} \in S_F$ are exactly lifts of periodic geodesics on M minimizing the length in the free homotopy class of loops associated with F. Moreover: $F(\tilde{\gamma}(t)) = \tilde{\gamma}(t + \inf f_F)$ $\forall t \in \mathbb{R}$.



class of loops associated with F. Moreover: $F(\tilde\gamma(t)) = \tilde\gamma(t + \inf f_F)$ $\forall t \in \mathbb{R}$.

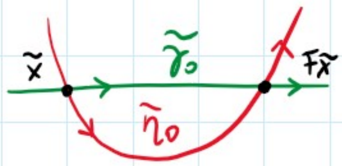


<u>Question</u>: Is $\tilde\gamma$ as above a minimizing geodesic? In general, No!
$\exists\, g$ on $\mathbb{T}^3$ with only 3 min. geodesics as above (Hedlund).

<u>Thm 3</u> (Morse, Hedlund) YES, if M orientable surface.

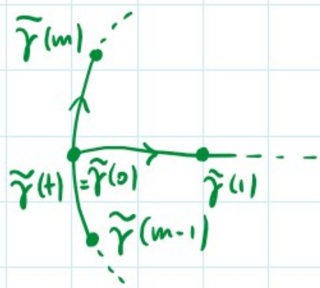
<u>Proof</u> Three steps (WLOG $\inf f_F = 1$). Each proven by contradiction.

<u>Step 1</u> The segment $\tilde\gamma_0$ connecting $\tilde x$ to $F\tilde x$ is unique.



If $\tilde\eta_0 \neq \tilde\gamma_0$ another segment, then $\tilde\gamma$ and $\tilde\eta$ cross at $\tilde x$ and $F\tilde x$ $\Rightarrow$ $d_{\tilde x}F$ not orientation preserving $\lightning$.

<u>Step 2</u> $\tilde\gamma$ has no self-intersection.



Suppose $\exists\, t > 0 : \tilde\gamma(0) = \tilde\gamma(t)$. Then $t \neq n$, $\forall n \in \mathbb{N}$ (otherwise F^n has fixed point $\Rightarrow F^n = \mathrm{id}_{\tilde M}$ $\lightning$ M orient. surface).
$\exists\, m \in \mathbb{N}$, $t \in (m-1, m)$. Then:

$$\begin{aligned} d(\tilde\gamma(0), \tilde\gamma(1)) = d(\tilde\gamma(m-1), \tilde\gamma(m)) &= d(\tilde\gamma(m-1), \tilde\gamma(t)) + d(\tilde\gamma(t), \tilde\gamma(m)) \\ &= d(F\tilde\gamma(m-1), F\tilde\gamma(t)) + d(\tilde\gamma(m), \tilde\gamma(t)) \\ &= d(\tilde\gamma(1), \tilde\gamma(m)) + d(\tilde\gamma(m), \tilde\gamma(t)) \end{aligned}$$

$\Rightarrow \exists$ segment from $\tilde\gamma(t) = \tilde\gamma(0)$ to $\tilde\gamma(1)$ extending the segment $\tilde\gamma|_{[t,m]}$ $\lightning$ Step 1.

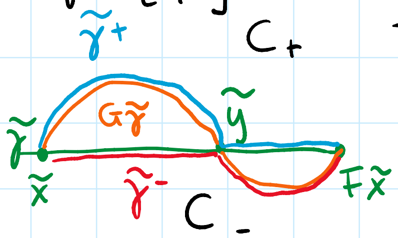
<u>Step 3</u> If $F = G^n$, then $\tilde\gamma$ is G-invariant

Let $\tilde M \setminus \tilde\gamma = C_- \sqcup C_+$. <u>Suppose</u> $\underline{G\cdot\tilde\gamma \subset C_+}$. Then $\tilde\gamma$ and $G\cdot\tilde\gamma$ bound a strip and point in the same direction. Indeed, if $M = \mathbb{R}^2/\mathbb{Z}^2$, then G is a translation. If M has higher genus G is a Möbius Transformation of the Poincaré disc with the same two fixed points p_1, p_2 on the boundary as F. Hence, $\tilde\gamma$ and $G\cdot\tilde\gamma$ both start from p_1 and end in p_2.
Thus $G(\overline{C_+}) \subset C_+$, so that $\tilde\gamma = G^n\tilde\gamma \subset C_+$ $\lightning$
Suppose $\tilde\gamma$ and $G\tilde\gamma$ cross at $\tilde\gamma(0) = \tilde x = G\tilde\gamma(t_0)$. Then they cross at $\tilde\gamma(1) = F\tilde x = G\tilde\gamma(t_0+1)$

Thus $G(C_+) \subset C_+$, so that $\tilde\gamma = G\tilde\gamma \subset C_+$ ↯

Suppose $\tilde\gamma$ and $G\tilde\gamma$ cross at $\tilde\gamma(0) = \tilde x = G\tilde\gamma(t_0)$. Then they cross at $\tilde\gamma(1) = F\tilde x = G\tilde\gamma(t_0+1)$ with same orientation. Hence they cross once again at $\tilde\gamma(s) = \tilde y = G\tilde\gamma(t)$.

Let $\tilde\gamma^{\pm} : [0,1] \to \tilde M$ be the two non-smooth curves connecting $\tilde x$ and $F\tilde x$ as in the picture.



Then $\mathrm{length}(\tilde\gamma^+) + \mathrm{length}(\tilde\gamma^-) = 2\inf f_F \Rightarrow \tilde\gamma^+, \tilde\gamma^-$ minimize and therefore are smooth. ↯ The only possibility is that $\tilde\gamma = G\tilde\gamma$.

Step 4 $\tilde\gamma$ is minimizing

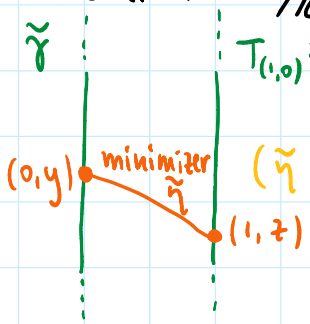
Enough to show: $\mathrm{length}(\tilde\gamma|_{[0,n]}) \le d(\tilde\gamma(0), F^n\tilde\gamma(0))$ $\forall n \in \mathbb{N}$.

Let $\delta := \inf f_{F^n}$ and take $\tilde\eta : \mathbb{R} \to \tilde M$ corresponding to F^n as in Theorem 2. By Step 3 $F\tilde\eta(t) = \tilde\eta(t + \delta/n)$. Hence $d(\tilde\gamma(0), F^n\tilde\gamma(0)) \geq \delta \geq n \cdot d(\tilde\eta(0), F\tilde\eta(0)) \geq n \cdot \mathrm{length}(\tilde\gamma|_{[0,1]})$
$= \mathrm{length}(\tilde\gamma|_{[0,n]})$.

$\boxtimes$

Minimizing geodesics on the two-torus

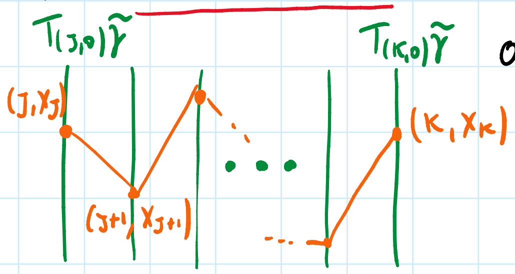
Consider $M = \mathbb{R}^2/\mathbb{Z}^2$, $\tilde M = \mathbb{R}^2$, $F = T_{(q,p)}$ translation by $(q,p) \in \mathbb{Z}^2$. Take $\tilde\gamma : \mathbb{R} \to \mathbb{R}^2$ minimizing geodesic associated with $T_{(0,1)}$. Up to diffeo of M: $\tilde\gamma(t) = (0,t)$.



Define the discrete Lagrangian ($\tilde\eta$ contained in the region between $\tilde\gamma$ and $T_{(1,0)}\tilde\gamma$; why?)

$$\ell : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \quad \ell(y,z) := d((0,y),(1,z)) \quad \left(\text{length of minim. geodesic } \tilde\eta\right)$$

For all finite sequences $(x_j, \ldots, x_k)$ of real numbers with $j<k$ integers, we define the discrete action



$$a(x_j,\ldots,x_k) := \sum_{i=j}^{k-1} \ell(x_i, x_{i+1}) \quad \left(\text{length of broken geodesic through } (j,x_j),\ldots,(k,x_k)\right)$$

Def A finite sequence $(x_j, \ldots, x_k)$ is minimal if $a(x_j, \ldots, x_k) \le a(x_j^*, \ldots, x_k^*)$ for all $(x_j^*, \ldots, x_k^*)$ with $x_j = x_j^*$, $x_k = x_k^*$

$a(x_J, \dots, x_K) \leq a(x_J^*, \dots, x_K^*)$ for all $(x_J^*, \dots, x_K^*)$ with $x_J = x_J^*$, $x_K = x_K^*$.

Rmk Minimal sequences correspond to minimizing segments connecting (J, x_J) and (K, x_K).

Dfn Let $\mathbb{R}^{\mathbb{Z}}$ be the space of bi-infinite real sequences $(x_i)_{i\in\mathbb{Z}}$. $(x_i)_{i\in\mathbb{Z}}$ is called minimal if all its finite segments $(x_J, \dots, x_K)$ are minimal.
We write $\mathcal{M} \subset \mathbb{R}^{\mathbb{Z}}$ for the set of minimal elements.

Rmk Elements of $\mathcal{M}$ correspond to minimizing geodesics intersecting $T_{(i,0)}\tilde{\gamma}$ $\forall i \in \mathbb{Z}$.

Next time: 4 properties of $\ell : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$; describe $\mathcal{M}$ for all discrete Lagrangians satisfying these properties.