TALK 4A: THE HAMILTON-JACOBI METHOD

1. Time-dependent subsolutions of the HJ-equation

The Hamilton-Jacobi method is very useful to find action minimizing orbits. It comes into two flavours that are adapted to the case of Tonelli minimizers and of time-free minimizers. In the following discussion, $L : TM \to \mathbb{R}$ will be a Tonelli Lagrangian on a manifold M endowed with a complete metric and $H : T^*M \to \mathbb{R}$ will be the associated Hamiltonian.

Definition 1.1. *Let $[a, b]$ be an interval and $S : M \times [a, b] \to \mathbb{R}$ be a C^1-function. We write $S_t := S(\cdot, t)$ for every $t \in [a, b]$. The L-gradient of S is the time-dependent vector field on M given by*

$$\mathrm{grad}_L S_t(x) = \mathrm{Leg}^{-1}(\mathrm{d}_x S_t), \qquad \forall\, (x, t) \in M \times [a, b].$$

Definition 1.2. *Let $[a, b]$ be an interval. A C^1-function $S : M\times[a, b] \to \mathbb{R}$ is a time-dependent subsolution of the Hamilton-Jacobi equation if*

$$H(x, \mathrm{d}_x S_t) + \partial_t S_t(x) \le 0, \qquad \forall\, (x, t) \in M \times [a, b].$$

We denote by $N_S \subset M\times[a, b]$ the set of pairs (x, t), where equality holds. We call S a solution if $N_S = M \times [a, b]$.

Theorem 1.3. *Let S be a time-dependent subsolution in the interval $[a, b]$. Let x_0 and x_1 be two points in M. Then, for all $\gamma \in C^{ac}_{x_0,x_1}([a, b], M)$, there holds*

$$A_L(\gamma) \ge S_b(x_1) - S_a(x_0)$$

with equality if and only if γ is a flow line of the L-gradient of S such that $(t, \gamma(t)) \in N_S$ for all $t \in [a, b]$. Therefore, each such flow line would be a Tonelli minimizer in $C^{ac}_{x_0,x_1}([a, b], M)$.

Proof. By the Fenchel inequality we have

$$L(x, v) + H(x, \mathrm{d}_x S_t) \ge \mathrm{d}_x S_t \cdot v$$

with equality if and only if $v = \mathrm{grad}_L S_t(x)$. Thus,

$$L(x, v) \ge \mathrm{d}_x S_t \cdot v - H(x, \mathrm{d}_x S_t) \ge \mathrm{d}_x S_t \cdot v + \partial_t S_t(x) = \mathrm{d}_{(x,t)} S \cdot (v + \partial_t)$$

with equality if and only if $v = \mathrm{grad}_L S_t(x)$ and $(x, t) \in M_S$. Therefore,

$$A_L(\gamma) \ge \int_a^b \mathrm{d}_{(\gamma(t),t)} S \cdot (\dot\gamma(t), 1)\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\Big[S(\gamma(t), t)\Big]\mathrm{d}t = S_b(x_1) - S_a(x_0). \qquad \square$$

We want to use Theorem 1.3 to prove the first version of Weierstrass Theorem. The second version follows actually from the first one and an implicit function theorem. Indeed, what one needs is that if $K \subset M$ is compact and $C > 0$ is a constant, then there exists a δ_0 such that for all $\delta < \delta_0$ and all $x \in K$ the map

$$\{v \in T_x M \mid |v|_x < C\} \to M, \qquad v \mapsto \gamma_{x,v}(\delta)$$

is an embedding containing the ball $\bar B_{C\delta}(x) = \{y \in M \mid d(x, y) \le C\delta\}$.

To use Theorem 1.3 we need the existence of local solutions of the HJ-equation with arbitrary initial condition.

Theorem 1.4. *Let $\tilde{K}$ be a compact set of TM. There are $\delta, \epsilon > 0$ such that for all $(x, v) \in \tilde{K}$ there exists a time-dependent HJ-solution $S : B_\epsilon(x) \times [0, \delta] \to M$ such that $v = \mathrm{grad}_L S_0(x)$.* □

The proof of Theorem 1.4 is a nice exercise in symplectic geometry and can be found in Section 2.8 of Fathi's book.

Theorem 1.5 (Weierstrass (Part 1))**.** *Let L be bounded from below on TM and let $\tilde{K}$ be a compact set in TM. There exists $\delta > 0$ such that the EL-solution $\gamma_{(x,v)} : [0, \delta] \to M$ is well-defined and the unique Tonelli minimizer in $C^{ac}_{x,\gamma_{(x,v)}(\delta)}([0, \delta], M)$.*

Proof of Weierstrass (Part 1). By Theorem 1.4 we can find δ and ϵ like in the statement. For $(x, v) \in \tilde{K}$ consider now the HJ-solution S on $B_\epsilon(x) \times [0, \delta]$ with $v = \mathrm{grad}_L S_0(x)$ given above. By Theorem 1.3 the flow line of $\mathrm{grad}_L S_t$ passing through x is a Tonelli minimizer among curves contained in $B_\epsilon(x)$ and therefore coincides with $\gamma_{(x,v)} : [0, \delta] \to B_\epsilon(x)$ (up to shrinking δ so that $\gamma_{(x,v)}([0, \delta]) \subset B_\epsilon(x)$). It is left to show that $\gamma_{(x,v)}$ minimizes the action also among curves $\gamma : [0, \delta] \to M$ with $\gamma(0) = x$ and $\gamma(\delta) = \gamma_{(x,v)}(\delta)$ whose image is not contained in $B_\epsilon(x)$. Up to adding a constant we can suppose $L \geq 0$ since L is bounded from below. Therefore, if δ_1 is the first time in $[0, \delta]$ with $\gamma(\delta_1) \in \partial B_\epsilon(x)$, we have for some constant B

$$A_L(\gamma) \geq \int_0^{\delta_1} L(\gamma, \dot{\gamma})\mathrm{d}t \geq d(\gamma(0), \gamma(\delta_1)) + B\delta_1 \geq \epsilon - |B|\delta \geq \epsilon/2,$$

where the second inequality follows by the superlinearity of L and the third by taking a smaller δ. However, since $\tilde{K}$ is compact, L is bounded from above by a constant C on $(\gamma_{(x,v)}, \dot{\gamma}_{(x,v)})$. Thus, $A_L(\gamma_{(x,v)}) \leq C\delta < \epsilon/2$ where the second inequality follows by taking a smaller δ. □

Remark 1.6. *As you can see, the question of existence of global (meaning defined on $M \times \mathbb{R}$) (sub)solutions to the HJ-equation is very important and represent an interesting problem in the theory of PDE. In general, only (sub)solutions in some weak sense will exist and there will be no classical solutions as the one considered here.*

2. k-subsolutions of the HJ-equation

Definition 2.1. *Let k be a real number. A C^1-function $u : M \to \mathbb{R}$ is a k-subsolution of the Hamilton-Jacobi equation if*

$$H(x, d_x u) \leq k, \qquad \forall\, x \in M.$$

We denote by $M_u \subset M$ the set of points x, where equality holds. We call u a solution if $M_u = M$.

Remark 2.2. *If u is a k-subsolution, then $S(x, t) = u(x) - kt$ for all $(x, t) \in M \times \mathbb{R}$ is a time-dependent subsolution.*

Remark 2.3. *If M is closed, there exists at most one value of k for which solutions might exist. Indeed, if u_1 is a k_1-solution and u_2 is a k_2-solution, there exists a point $x \in M$ for which $\mathrm{d}_x(u_2 - u_1) = 0$ since M is closed. Then $k_1 = H(x, \mathrm{d}_x u_1) = H(x, \mathrm{d}_x u_2) = k_2$. We will see later that weak solutions always exists and the corresponding unique energy value k is exactly the Mañé critical value $c(L)$.*

Remark 2.4. *If $L(x,v) = \frac{1}{2}|v|_x^2$ for some Riemannian metrics then the radial coordinate $r : B_\epsilon(x) \to (0,\infty)$ in normal coordinates is a $\frac{1}{2}$-solution of the HJ-equation since by Gauss Lemma $|\mathrm{d}r| = 1$. Indeed, this function was essential to prove that geodesics locally minimize the length which is the Riemannian analogue of Weierstrass theorem.*

A k-subsolution u will give us (global) time-free minimizers under certain conditions between M_u and $\mathrm{grad}_L u$. The proof of the result is analogous to the one of Theorem 1.3 and is left to the reader.

Theorem 2.5. *Let u be a k-subsolution. For all $\gamma : [0,T] \to M$ there holds*

$$A_{L+k}(\gamma) \geq u(\gamma(T)) - u(\gamma(0))$$

with equality if and only if γ is a flow line of $\mathrm{grad}_L u$ contained in the set M_u. Therefore, each such flow line would be a time-free minimizer for $L + k$ in

$$\bigcup_{T'>0} C^{ac}_{\gamma(0),\gamma(T)}([0,T'],M).$$

If the flow is define on the whole $\mathbb{R}$, then γ is a global time-free minimizer. □

Remark 2.6. *Let us consider $L_\theta := L + \theta$ where θ is a closed 1-form. Then, the new Hamiltonian is $H_\theta(x,p) = H(x, p - \theta_x)$. Therefore, a k-subsolution u_θ for L_θ satisfies*

$$H(x, \mathrm{d}_x u - \theta_x) \leq k, \qquad \forall\, x \in M.$$

Finding such a function u is equivalent to finding a closed 1-form $\tilde\theta$ on M satisfying

$$H(x, \tilde\theta_x) \leq k, \quad \forall\, x \in M, \qquad [\tilde\theta] = -[\theta] \in H^1(M;\mathbb{R}).$$

Moreover, $M_u = \{x \in M \mid H(x,\tilde\theta_x)\}$ and $\mathrm{grad}_{L_\theta} u = \mathrm{Leg}_L^{-1}(\tilde\theta)$, so we can read off M_u and $\mathrm{grad}_{L_\theta} u$ directly from $\tilde\theta$.

Thanks to the above remark, we understand better what we did in the case of the pendulum. There we showed that for $k \geq 0$

$$H(x, -\theta_k^\pm) = k, \quad \forall\, x \in S^1$$

so that we obtain that the orbits with energy k are time-free minimizer for $L + \theta_k^\pm + k$. Actually $\theta_0^\pm$ is not C^2 but only continuous. However, we can write $\theta_0^\pm = \eta_0^\pm - \mathrm{d}u^\pm$ for some smooth 1-form $\eta_0^\pm$ and some C^1-function $u_0^\pm$. Then, $u_0^\pm$ is a HJ-solution for $L + \eta_0^\pm$. For $r \in [0,1]$ we have

$$H(x, -r\theta_0^\pm) \leq 0$$

and the equality holds exactly at $x = 0$. Therefore, the unstable equilibrium $x = 0$ is the only time-free minimizer for $L + r\theta_0^\pm$. As before, one can substitute the forms $r\theta_0^\pm$ with the smooth $r\eta_0^\pm$ for $r \in (0,1)$. Actually, in this case one can also smoothen $r\theta_0^\pm$ to some 1-form β_r with same cohomology class as $r\theta_0^\pm$ and satisfying $H(x, -\beta_r) \leq 0$ with equality exactly at $x = 0$.