Weierstrass Theorem and global minimizers

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Overview

We are going to take a look at:

1. Regularity of minimizers
2. Global minimizers and (global) time-free minimizers
3. Energy level $e_0(L)$
4. Example of pendulum.

Recap

We will consider a Tonelli Lagrangian $L : TM \to \mathbb{R}$ on (Riemannian) manifold M (Projection $\pi : TM \to M$) with Action functional $A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))dt$ as before. We denote the Euler-Lagrange flow by ϕ_t.
A curve $\gamma : [a, b] \to \mathbb{R}^n$ is absolutely continuous (C^{ac}), if $\forall \varepsilon > 0 : \exists \delta > 0$ s.t. for each family $(a_i, b_i)_{i\in\mathbb{N}}$ of disjoint intervals in $[a, b]$ with $\sum_{i\in\mathbb{N}}(b_i - a_i) < \delta$, we have $\sum_{i\in\mathbb{N}} d(\gamma(b_i), \gamma(a_i)) < \varepsilon$. An important property is, that the derivative of an absolutely continuous curve exists almost everywhere on $[a, b]$. Also we will write $C_{x,y}([a, b], M)$ for the curves connecting x and y, i.e. $\gamma(a) = x, \gamma(b) = y$.

Weierstrass Theorem

We consider L Tonelli-Lagrangian on manifold M with $\inf_{(x,v)\in TM} L(x,v)$ finite.

Theorem (Part 1)

Then for each compact subset $K \subset TM$ there exists a constant $\delta_0 > 0$ such that

- *for $(x,v) \in K$ the local flow $\phi_t(x,v)$ is defined for $|t| \leq \delta_0$.*
- *for each $(x,v) \in K$ and $\delta \in (0,\delta_0]$, the extremal curve*

$$\gamma_{(x,v,\delta)} : [0,\delta] \to M,\ t \mapsto \pi\phi_t(x,v)$$

is such that for any absolutely continuous curve $\gamma_1 : [0,\delta] \to M$, with $\gamma_1(0) = x, \gamma_1(\delta) = \pi\phi_\delta(x,v), \gamma_1 \neq \gamma_{(x,v,\delta)}$ it holds $A_L(\gamma_1) > A_L(\gamma_{(x,v,\delta)})$.

Weierstrass Theorem

We consider L Tonelli-Lagrangian on manifold M with $\inf_{(x,v)\in TM} L(x,v)$ finite.

Theorem (Part 2)

Let d be a distance on M given by a Riemannian metric. If $K \subset M$ is compact and $C > 0$ a constant, then there exists a constant $\delta_0 > 0$ such that, if $x \in K, y \in M$ and $\delta \in (0, \delta_0]$ satisfy $d(x,y) \leq C\delta$, then there exists an extremal curve $\gamma_{(x,y,\delta)} : [0,\delta] \to M$ with $\gamma_{(x,y,\delta)}(0) = x, \gamma_{(x,y,\delta)}(\delta) = y$ and for every absolutely continuous curve $\gamma : [0,\delta] \to M$ which satisfies $\gamma(0) = x, \gamma(\delta) = y, \gamma \neq \gamma_{(x,y,\delta)}$ it holds that $A_L(\gamma) > A_L(\gamma_{(x,y,\delta)})$.

Regularity of Tonelli minimizers

Let $L \in C^r(TM)$ be a Tonelli Lagrangian on a Riemannian manifold M.

Theorem

For an absolutely continuous curve $\gamma : [a, b] \to M$ that minimizes the action $A_L(\gamma) \leq A_L(\gamma_1)$ in the space of absolutely continuous curves $\gamma_1 : [a, b] \to M$ with $\gamma_1(a) = \gamma(a), \gamma_1(b) = \gamma(b)$. Then the curve γ is an extremal curve and therefore C^r.

Keep in mind, that extremizers are C^2 by definition.

Proof of regularity

Proof.

Sufficient to consider $M = U \subset \mathbb{R}^n$ an open subset of euclidean space.
First step is to show:
If $\dot{\gamma}(t_0)$ exists, then γ coincides with an extremal curve in neighborhood of t_0.
To show this we choose $C > \|\dot{\gamma}(t_0)\|$ and by the existence of $\dot{\gamma}(t_0)$ we find $\eta > 0$ such that $0 < |t - t_0| \leq \eta \Rightarrow \|\gamma(t) - \gamma(t_0)\| < C|t - t_0|$. By the uniform superlinearity of L on compact subsets we know that the Lagrangian is bounded from below on a compact neighborhood of $\gamma([a, b])$.

Proof of regularity

Proof.

Now C is a constant and $\gamma([a,b])$ is a compact subset, so we can use Weierstrass theorem to find $\delta_0 > 0$ (Wlog. $\delta_0 \leq \eta$) and an extremal curve $\gamma_1 : [0, \delta_0] \to M, s \mapsto \gamma(s + t_0 - \frac{\delta_0}{2})$, which minimizes the action among curves connecting $\gamma(t_0 - \frac{\delta_0}{2}), \gamma(t_0 + \frac{\delta_0}{2})$. Indeed,

$$\left\| \gamma(t_0 - \frac{\delta_0}{2}) - \gamma(t_0 + \frac{\delta_0}{2}) \right\| \leq C|t_0 - \frac{\delta_0}{2} - (t_0 + \frac{\delta_0}{2})| = C\delta_0$$

That means, the restriction of γ to $[t_0 - \frac{\delta_0}{2}, t_0 + \frac{\delta_0}{2}]$ is extremal as it is the unique minimizer on this subinterval. For $t_0 \in \{a, b\}$ don't substract/add the fraction.

Proof of regularity

Proof.

The second step is to show that the open subset $O \subset [a, b]$ formed by the points, such that γ coincides with an extremal curve in their neighborhood is $O = [a, b]$.

We notice that for every connected component I of O the restriction to $\overline{I}$ solves the Euler-Lagrange equation.

Now assume $O \neq [a, b]$ and consider the case $I = (\alpha, \beta)$ (other possible cases: $[a, \beta), (\alpha, b]$). $\alpha, \beta \notin O, [\alpha, \beta]$ compact and $\gamma|_{[\alpha,\beta]}$ extremal (C^2), hence $\dot{\gamma}$ is bounded. Apply Weierstrass to compact subset $\gamma([a, b])$ and constant $C > \|\dot{\gamma}\|_{\infty,(\alpha,\beta)}$ to find $\delta_0 > 0$. There holds:

$$\|\gamma(\beta) - \gamma(\beta - \frac{\delta_0}{2})\| \leq \int_{\beta-\frac{\delta_0}{2}}^{\beta} \|\dot{\gamma}(s)\| ds < C\frac{\delta_0}{2}$$

Proof of regularity

Proof.

Hence for $\varepsilon > 0$ sufficiently small we get that $\|\gamma(\beta+\varepsilon) - \gamma(\beta - \frac{\delta_0}{2})\| < C(\varepsilon + \frac{\delta_0}{2}) < C\delta_0$. Therefore γ coincides with the Weierstrass minimizer on $[\beta - \frac{\delta_0}{2}, \beta + \varepsilon] \ni \beta$. Hence $\beta \in O$ which is a contradiction. Therefore γ is an extremal curve and with result from talk 1 in C^r. □

Global Tonelli minimizers

Problem

Are there curves $\gamma : \mathbb{R} \to M$ *such that* $\gamma|_{[a,b]}$ *is Tonelli minimizer in* $C^{ac}_{\gamma(a),\gamma(b)}([a,b],M)$ *for all* $a < b$*. What is their energy?*

This leads us to the definition of global minimizers.

Definition (global minimizer)

An absolutely continuous curve $\gamma : \mathbb{R} \to M$ is a global (Tonelli) minimizer for L if, for any given $a < b \in \mathbb{R}$

$$A_L(\gamma|_{[a,b]}) = \min_{\sigma \in C^{ac}_{\gamma(a),\gamma(b)}([a,b],M)} A_L(\sigma).$$

Time-free minimizer

There is also a related Problem which will help us answer the problem above:

Problem

Find time-free minimizers of L connecting two points x and y. This means to minimize the action on the set $\bigcup_{T>0} C^{ac}_{x,y}([0,T],M)$.

The time-free minimizers for L and for $L+k$ might be different:

Example L vs. $L + k$

Example

Let $\gamma : [0, T] \to M, \sigma : [0, T'] \to M$ with $T' < T, \gamma(0) = \sigma(0), \gamma(T) = \sigma(T')$, such that $\int_0^T L(\gamma(t), \dot{\gamma}(t))dt = A_L(\gamma) < A_L(\sigma) = \int_0^{T'} L(\sigma(t), \dot{\sigma}(t))dt$. Now we choose $k > \frac{1}{T-T'}(A_L(\sigma) - A_L(\gamma))$. Hence

$$A_{L+k}(\gamma) = A_L(\gamma) + kT = A_L(\gamma) + k(T - T') + kT' >$$
$$> A_L(\gamma) + (A_L(\sigma) - A_L(\gamma)) + kT' = A_L(\sigma) + kT' = A_{L+k}(\sigma).$$

This inverts the inequality.

Energy of time-free minimizers

There is a relation between k and the Energy $E = \frac{\partial L}{\partial v} \cdot v - L$ for time-free minimizers of $L + k$.

Theorem

Time-free minimizers of $L + k$ connecting x and y have energy $E = k$.

Proof.

Consider $\mathcal{A}(\lambda) := A_{L+k}(\gamma_\lambda)$ where $\gamma_\lambda(t) := \gamma(\lambda t)$ and calculate the first derivative of $\mathcal{A}$ with the help of a clever substitution $\lambda t = s$.

$$\mathcal{A}(\lambda) = \int_0^{\frac{T}{\lambda}} [L(\gamma(\lambda t), \lambda\dot{\gamma}(\lambda t)) + k]dt = \int_0^T [L(\gamma(s), \lambda\dot{\gamma}(s)) + k]\frac{1}{\lambda}ds$$

Energy of time-free minimizers

Proof.

Hence $\mathcal{A}'(1) = \int_0^T [\frac{\partial L}{\partial v}(\gamma(s), \dot{\gamma}(s)) \cdot \dot{\gamma}(s) - L(\gamma(s), \dot{\gamma}(s)) - k]ds = \int_0^T [E(\gamma(t), \dot{\gamma}(t)) - k]dt$. Now γ is a time-free minimizer and therefore a solution of the Euler-Lagrange equation and hence the Energy is constant, which implies, that $0 = \mathcal{A}'(1) = T(E - k)$. □

This means, that finding a time-free minimizer yields orbits connecting two points with given energy.

Global time-free minimizers

Definition (Global time-free minimizers)

An absolutely continuous curve $\gamma : \mathbb{R} \to M$ is a global time-free minimizer for L if, for any $a, b \in \mathbb{R}$, $a < b$

$$A_L(\gamma|_{[a,b]}) = \min_{\sigma \in \left[\bigcup_{a' < b'} C^{ac}_{\gamma(a),\gamma(b)}([a',b'],M)\right]} A_L(\sigma)$$

Relation time-free minimizers, global minimizers

Obviously a global time-free minimizer for $L + k$ is a global minimizer (for L). In general the converse is not true, but for a special k it holds and the time-free minimizers does exist.

Theorem

There exists $c(L) \in \mathbb{R}$ called Mañé critical value of L such that

- *for all $k > c(L), x, y \in M$ there exists a time-free minimizer of $L + k$ connecting x, y*
- *there exists global time-free minimizers of $L + c(L)$*
- *global Tonelli minimizers are global time-free minimizers of $L + c(L)$ (in particular, all global time-free minimizers have energy $c(L)$).*

Global Tonelli minimizers

With this result we get the answers to our problem:

- there exists a global minimizer. We have to find c(L) and the corresponding global time-free minimizer, which then is a global minimizer.
- the energy of a global Tonelli minimizer is $E = c(L)$.

We will take a closer look to the energy:

Energy level

Recall that the energy, given by
$E : TM \to \mathbb{R}, E(x,v) = H \circ Leg(x,v) = \frac{\partial L}{\partial v}(x,v) \cdot v - L(x,v)$, is constant for solutions of the Euler-Lagrange equation:

$$\frac{d}{dt}E = \frac{d}{dt}\frac{\partial L}{\partial v} \cdot \dot{\gamma} + \frac{\partial L}{\partial v} \cdot \ddot{\gamma} - \frac{\partial L}{\partial x} \cdot \dot{\gamma} - \frac{\partial L}{\partial v} \cdot \ddot{\gamma} = 0$$

We want to look at the maximal critical value of the Energy. So let us calculate the critical point of E for a fixed $x \in M$.

Energy level

Therefore define $f_x : T_x M \cong \mathbb{R}^n \to \mathbb{R}$, $f_x(v) := E(x, v)$ and look at $\nabla f_x(v) = \frac{\partial^2 L}{\partial v^2}(x, v) \cdot v + \frac{\partial L}{\partial v}(x, v) - \frac{\partial L}{\partial v}(x, v) = \frac{\partial^2 L}{\partial v^2}(x, v) \cdot v$. The convexity of L implies, that $v \cdot \frac{\partial^2 L}{\partial v^2} \cdot v > 0$ for $v \neq 0$. This yields $v = 0$ as the only possible critical point. In fact this is a minimum for f_x because of the same inequality from above:

$$U(x) := \min_{v \in T_x M} E(x, v) = E(x, 0) = -L(x, 0).$$

Now we can define the energy level

$$e_0(L) := \max_{x \in M} U(x) = -\min_{x \in M} L(x, 0).$$

Energy level

We notice, that x is a critical point of U if and only if the constant curve $x(t) = x$ solves the Euler-Lagrange equation. The idea behind is: $\frac{d}{dt}\frac{\partial L}{\partial v}(x, 0) = 0$ for x constant.
Now we can define the level sets

$$S_k := \{(x, v) \in TM | E(x, v) = k\}$$

and claim the following properties:

- $S_k = \emptyset$ for $k < \min_{x \in M} U(x)$.
- $\pi(S_k) \neq M$ for $\min_{x \in M} U(x) \leq k < e_0(L)$.
- $\pi(S_k) = M$ for $k \geq e_0(L)$.

Energy level

Let $k < \min_{x\in M} U(x)$ and assume there exists $(x, v) \in TM$ such that $E(x, v) = k$. Hence $E(x, v) = k < \min_{x\in M} \min_{v\in T_xM} E(x, v)$, what yields a contradiction. This means there is no motion with that energy.
Now let $\min_{x\in M} U(x) \leq k < e_0(L)$ and assume that for all $x \in M$ exists a $v_x \in T_xM$ such that $E(x, v_x) = k$. This implies

$$e_0(L) = \max_{x\in M} \min_{v\in T_xM} E(x, v) \leq \max_{x\in M} E(x, v_x) = k.$$

Therefore $\pi|_{S_k}$ can't be surjective. Physical this means, there are regions, which can't be accessed with this energy.

Energy level

For the last case $k \geq e_0(L)$ we first notice, that for a Tonelli Lagrangian L the energy E is also Tonelli. Therefore in particular the energy is superlinear. Hence

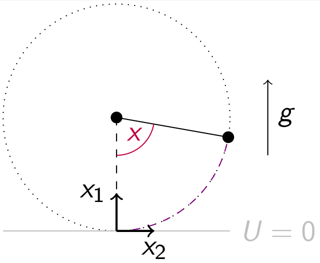
$$\lim_{|v| \to \infty} E(x, v) = +\infty.$$

If a $x \in M$ wouldn't be in $\pi(S_k)$, by continuity of E this would imply that $E(x, v) < k$ for all $v \in T_x M$. This is a contradiction to the limit above. Therefore $\pi|_{S_k}$ is surjective.
This means $e_0(L) = \min\{k \in \mathbb{R} \,|\, \pi : S_k \to M \text{ is surjective}\}$ and hence there are solutions passing through every point.
Now we will find time-free minimizers of the simple pendulum:

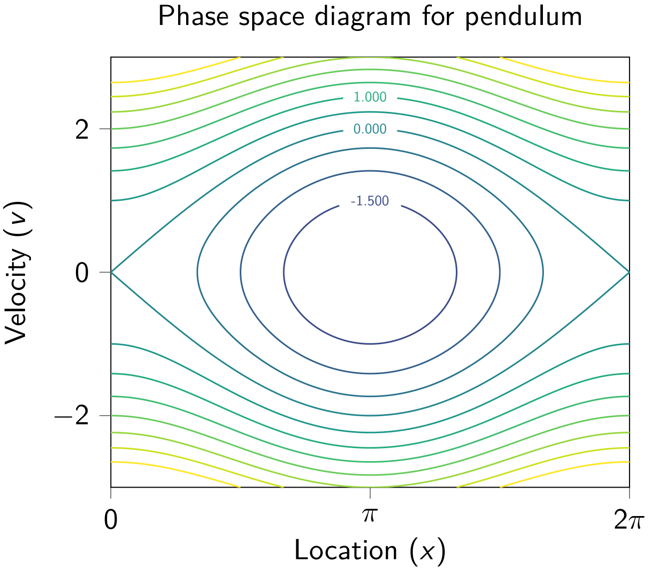
Simple Pendulum

Example

We consider pendulum with $L : TS^1 \to \mathbb{R}, L(x, v) = \frac{1}{2}v^2 - U(x)$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $U(x) := -(1 - \cos(x))$. This obviously defines a Tonelli Lagrangian. First we want to draw the trajectories in coordinates (x, v) and then interpret the results with the energy level $e_0(L)$. The trajectories are exactly the level sets for the energy $E(x, v) = \frac{\partial L}{\partial v}(x, v)v - L(x, v) = v^2 - \frac{1}{2}v^2 + U(x) = \frac{1}{2}v^2 + U(x)$. Hence for $E = k$ this means, that $v_{1/2}(x) = \pm\sqrt{2(k - U(x))}$.



Simple Pendulum



Simple Pendulum

Example

For the energy level we have $e_0(L) = \max_{x \in S^1} E(x, 0) = \max_{x \in S^1} U(x) = 0$ at $x = 0$ and $\min_{x \in S^1} U(x) = -2$ at $x = \pi$. Comparing this to our plot we have that for energy $E < -2$ we have no motion and therefore no level sets in the plot. For the energy between -2 and 0 we have a motion, but can't reach every point. Of course especially not $x = 0$. If the energy is $E \geq 0$ we have enough energy to reach every point (even $x = 0$). In the plot we can see this aswell in the region outside $E = 0$.

Simple Pendulum

Example

Now define $\theta_k^\pm := \pm\sqrt{2(k - U(x))}dx$. We observe, that we get all cohomology classes in $H^1(S^1; \mathbb{R})$ with the 1-forms $r\theta_0^\pm, \theta_k^\pm$, with $r \in [0, 1]$ and $k \geq 0$. Therefore if we find the global minimizers for $L + r\theta_0^\pm, L + \theta_k^\pm + k$ we have all possible global time-free minimizers. Note that we have to add k to the second type of Lagrangian, which corresponds to the Energy (Theorem from above).

Simple Pendulum

Example

For a fixed $x \in M$ and $r \in [0,1]$ we have

$$
\begin{aligned}
(L + r\theta_0^{\pm})(x,v) &= \frac{1}{2}v^2 - U(x) \pm r\sqrt{2(-U(x))}dx(x,v) = \\
&= \frac{1}{2}v^2 \pm r\sqrt{2(-U(x))}v - U(x) = \\
&= \frac{1}{2}\left(v \pm r\sqrt{-2U(x)}\right)^2 - (1-r^2)U(x).
\end{aligned}
$$

If we want to calculate the zero points, we observe:

- for $U(x) = 0$ there is only one solution $v = 0$
- for $U(x) \neq 0, r < 1$ there is no solution
- for $U(x) \neq 0, r = 1$ there is only one solution $v = \mp\sqrt{-2U(x)}$.

Where the $\mp$ gives the sign corresponding to θ. In summary we can say $L + r\theta_0^{\pm} \geq 0$ for all $r \in [0,1]$

Simple Pendulum

Example

Analogous

$$\begin{aligned}(L+\theta_k^{\pm})(x,v)+k &= \frac{1}{2}v^2 \pm \sqrt{2(k-U(x))}v - U(x) + k = \\ &= \frac{1}{2}\left(v \pm \sqrt{2(k-U(x))}\right)^2\end{aligned}$$

There is only one zero point: $v = \mp\sqrt{2(k-U(x))}$. Thus $L+\theta_k^{\pm}+k \geq 0$ for all $k \geq 0$.

Simple Pendulum

Example

So in summary we know $L + r\theta_0^{\pm}, L + \theta_k^{\pm} + k \geq 0$ which implies, that $A_{L+r\theta_0^{\pm}}, A_{L+\theta_k^{\pm}+k} \geq 0$. The equality is achieved for the trajectories $v(x) = \mp\sqrt{-2U(x)}, v(x) = \mp\sqrt{2(k - U(x))}$ and the constant trajectory $v(x) = 0, x = 0$ depending on the Lagrangian we are looking at. Hence these describe the global minimizers for the Energy k and therefore are all possible time-free minimizers. As we can see the time-free minimizers depend on adding a 1-form to L.

Nice side note:

$Leg : TS^1 \to T^*S^1, Leg(x, v) = (x, \frac{\partial L}{\partial v}(x, v)) = (x, v) \in T_x^*S^1$ is trivial.

Summary

Let us summarize (not formal), we have:

- *Weierstrass Theorem I*: The local flow exists and defines a unique minimizer.
- *Weierstrass Theorem II*: For any two points close enough together there exists a unique minimizer connecting those points.
- If $L \in C^r$ a minimizer is in C^r as well.
- Time-free minimizers depend on adding a constant to L.
- There exists a time-free minimizer between any two points for $L + k$ for k big enough.
- Global minimizers correspond to global time-free minimizers by adding a special constant to L.
- Global time-free minimizers have a fixed Energy.
- We can find $e_0(L) = \min\{k|\pi(S_k) = M\}$.
- Adding a closed one-form to L does change stuff.

References

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