For this presentation we will consider M to be a closed (= compact and without boundary) smooth manifold of dimension $dim\ M = n$. Let TM be the tangent bundle M and $\pi : TM \to M$ the projection. We will denote a point in TM by (x, v), where $x \in M$ and $v \in T_xM := \pi^{-1}(x)$. Thus π is just the projection onto the first factor. Similarly we will consider the cotangent bundle T^*M. A point in T^*M is denoted by (x, p), where $p \in T_x^*M$. Furthermore let g be a Riemannian metric on M and we will denote the induced norms on T_xM and T_x^*M both by $|| \cdot ||_x$ for each $x \in M$.

1 Lagrangian-Mechanics

Definition 1. A C^2 function $L:\ TM \to \mathbb{R}$ is called a Lagrangian on M.

Definition 2. For $x_0, x_1 \in M$, $a \leq b$ we set

$$C^2_{x_0,x_1}([a,b], M) := \{\gamma \in C^2([a,b], M)|\ \gamma(a) = x_0, \gamma(b) = x_1\}.$$

Additionally given a homotopy class α we define

$$C^2_{x_0,x_1}([a,b], M; \alpha) := C^2_{x_0,x_1}([a,b], M) \cap \alpha.$$

Then the action $\mathscr{A} : C^2_{x_0,x_1}([a,b], M; \alpha) \to \mathbb{R}$ is given by:

$$\mathscr{A}(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t))\ dt$$

Remark 1. Of course, at this point we could have defined the action $\mathscr{A}$ for a greater class of curves $\gamma : [a,b] \to M$, for example any curve, that would have made the function $t \mapsto L(\gamma(t), \dot{\gamma}(t))$ integrable. Later we'll look into absolutely continuous curves as our domain for $\mathscr{A}$.

Definition 3. A C^2 variation $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ of a C^2 curve $\gamma : [a,b] \to M$ is a C^2 mapping with $\varepsilon > 0$, s.t.

- $\Gamma(0,t) = \gamma(t)$, $\forall t \in [a,b]$ and
- $\Gamma(s,a) = \gamma(a)$ and $\Gamma(s,b) = \gamma(b)$, $\forall s \in (-\varepsilon, \varepsilon)$.

Definition 4. A C^2 curve is called an extremizer (or motion) of the Lagrangian L, if

$$\left.\frac{d}{ds}\right|_{s=0} \mathscr{A}(\Gamma(s,\cdot)) = 0$$

for each C^2 variation $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to M$

It turns out, that the extremizers γ of L are the C^2 curves, that satisfy the Euler-Lagrange equation in local coordinates:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t)) - \frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t)) = 0 \tag{1}$$

To see this, we just calculate the derivative of $\mathscr{A}$ with respect to s directly. Let $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be the variation of a C^2 extremal curve $\gamma : [a, b] \to M$. Then we consider a subdivision $a = r_0 < r_1 < \cdots < r_n = b$, such that the support $\gamma([r_k, r_k + 1])$ is contained in some coordinate chart (U_k, ϕ_k) for each $k = 0, \ldots, n-1$. For convenience we are going to set

$$\sigma(t) := \frac{\partial \Gamma}{\partial s}(0, t),\ \forall t \in [a, b].$$

and

$$\sigma_k(t) := d_{\gamma(t)}\phi_k \cdot \sigma(t).$$

$$\begin{aligned}
0 &\overset{!}{=} \frac{d}{ds}\bigg|_{s=0} \mathscr{A}(\Gamma(s,\cdot)) \\
&= \sum_{k=0}^{n-1} \int_{r_k}^{r_{k+1}} \left(\frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t))\sigma_k(t) + \frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t))\dot\sigma_k(t) \right) dt \\
&= \sum_{k=0}^{n-1} \int_{r_k}^{r_{k+1}} \left(\frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t)) \right) \sigma_k(t)\ dt \\
&\quad + \underbrace{\sum_{k=0}^{n-1} \left(\frac{\partial L}{\partial v}(\gamma(r_{k+1}),\dot\gamma(r_{k+1}))\sigma(r_{k+1}) - \frac{\partial L}{\partial v}(\gamma(r_k),\dot\gamma(r_k))\sigma(r_k) \right)}_{\overset{\text{telescope sum}}{=} \frac{\partial L}{\partial v}(\gamma(b),\dot\gamma(b))\sigma(b) - \frac{\partial L}{\partial v}(\gamma(a),\dot\gamma(a))\sigma(a) = 0} \\
&= \sum_{k=0}^{n-1} \int_{r_k}^{r_{k+1}} \underbrace{\left(\frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t)) \right)}_{(*)} \sigma_k(t)\ dt
\end{aligned}$$

The sum in the second to last equation vanishes completely, because the variation fixes the endpoints, meaning its derivatives in (s,a) and (s,b) vanish $\forall s \in (-\varepsilon, \varepsilon)$. In order for the integral above to equal 0 for any variation Γ, by the fundamental lemma of calculus of variations, $(*)$ must be equal to zero.

Theorem 1. *Let L be a C^2 Lagrangian on M and let $\gamma : [a,b] \to M$ be a C^2 curve. Then:*

(i) γ is extremal $\Rightarrow \forall\ [a',b'] \subseteq [a,b]$, s.t. $\gamma([a',b'])$ is contained in a chart (U,ϕ), then $\gamma|_{[a',b']}$ solves the Euler-Lagrange equation.

(ii) If for every $t \in [a,b]$ there exists an $[a',b'] \subseteq [a,b]$ containing t, s.t. $\gamma([a',b'])$ lies in an coordinate chart (U,ϕ) and $\gamma|_{[a',b']}$ solves the Euler-Lagrange equation, then γ is an extremal curve.

For now we want to consider what happens to the Euler-Lagrange equation and the action functional, when we add a function $f : TM \to \mathbb{R}$ to our Lagrangian L. How does this change look like?

Euler-Lagrange:

$$\frac{\partial(L+f)}{\partial x}(\gamma,\dot\gamma) - \frac{d}{dt}\frac{\partial(L+f)}{\partial v}(\gamma,\dot\gamma) = \frac{\partial L}{\partial x}(\gamma,\dot\gamma) - \frac{d}{dt}\frac{\partial L}{\partial v}(\gamma,\dot\gamma) + \underbrace{\frac{\partial f}{\partial x}(\gamma,\dot\gamma) - \frac{d}{dt}\frac{\partial f}{\partial v}(\gamma,\dot\gamma)}_{(1)}$$

the action $\mathscr{A}$:

$$\tilde{\mathscr{A}}(\gamma) = \int_a^b (L+f)(\gamma,\dot\gamma)dt = \mathscr{A}(\gamma) + \underbrace{\int_a^b f(\gamma,\dot\gamma)dt}_{(2)}$$

First let $f : TM \to \mathbb{R}$ be a constant function with $f = C$ for some $C \in \mathbb{R}$. Obviously (1) equal to 0 and (2) is equal to $C(b-a)$, which is a constant dependent on the length of the interval.
For some 1-form $\theta \in \Omega^1(M)$ we will set the function $\tilde\theta : TM \to \mathbb{R}$, $\tilde\theta(x,v) := \theta_x(v)$. In this case, (1) is equal to the exterior derivative $-d\theta_x(\dot x,\cdot)$. Hence (1) vanishes if θ is closed. In this case, (2) only depends on the homotopy class α (and thus it constant if α is fixed). We can verify this directly in two different ways:

1. Notice $\int_a^b \theta_{\gamma(t)}\dot\gamma(t)dt = \int_\gamma \theta = \int_{\tilde\gamma}\theta$ for some other C^2 curve $\tilde\gamma : [a,b] \to M$ homotopic to γ. This result is derived from Stoke's theorem; this approach works even if the curves are just C^1.

2. Another way is to look at some variation Γ of a curve γ and calculate the derivative of the action term directly:

$$\frac{d}{ds}\int_a^b \tilde\theta(\gamma(t)),\dot\gamma(t)\ dt = \sum_{k=0}^{n-1}\int_{r_k}^{r_{k+1}} \underbrace{\left(\frac{\partial\tilde\theta}{\partial x}(\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\frac{\partial\tilde\theta}{\partial v}(\gamma(t),\dot\gamma(t))\right)}_{= -d\theta_{\gamma(t)}(\dot\gamma(t),\cdot) = 0}\sigma_k(t)dt = 0$$

In the case of θ being exact, that is $\theta = du$ for some function $u \in C^\infty(M)$, then the integral is equal to $u(x_1) - u(x_0)$ which depends only on the end points x_0, x_1 but not on the homotopy class α.

We will now study the Euler-Lagrange equation itself a little further. Using chain rule, we can expand the EL to:

$$\frac{\partial^2 L}{\partial v^2}(\gamma(t),\dot{\gamma}(t))\ddot{\gamma}(t) = \frac{\partial L}{\partial x}(\gamma(t),\dot{\gamma}(t)) - \frac{\partial^2 L}{\partial v \partial x}(\gamma(t),\dot{\gamma}(t))\dot{\gamma}(t)$$

If $\frac{\partial^2 L}{\partial v^2}(x,v)$ is non-degenerate at every point $(x,v) \in TM$, which is equivalent to

$$\det \frac{\partial^2 L}{\partial v^2}(x,v) \neq 0,$$

which we will refer to as the Legendre-condition. If it is met, then we can solve for $\ddot{\gamma}(t)$. Thus we can define a vector field X_L on TM

$$X_L(x,v) = (x,v,v,\tilde{X_L}) \in T_{(x,v)}TM,$$

where $\tilde{X_L}$ satisfies the equation above (corresponding to $\ddot{\gamma}(t)$). Thus its solutions are exactly the solutions to the Euler-Lagrange equation. Appropriately we call this vector field X_L the *Euler-Lagrange vector field* and its flow ϕ_t^L (if it exists) is called the *Euler-Lagrange flow* associated with L. Since L is C^2, X_L is only C^0 and we cannot apply the theorem on existence and uniqnesss of solutions of ordinary differential equations (for that you need X_L locally Lipschitz). To solve this problem, we consider the *Legendre transform.*

Definition 5. Let L be a Lagrangian on M. We will define the (global) Legendre transform as:

$$\text{Leg}: TM \to T^*M,\ (x,v) \mapsto \frac{\partial L}{\partial v}(x,v)$$

When we introduce the Hamiltonian later in this presentation, we get a lot of nice properties and a duality between certain Lagrangians and their corresponding Hamiltonians H, if the (global) Legendre-transform is a diffeomorphism. In the following we define a class of Lagrangians, for which this will be the case:

Definition 6. Let (M,g) be a Riemannian manifold. We will call $L : TM \to \mathbb{R}$ a Tonelli-Lagrangian if:

(1) L is C^2

(2) $\forall (x,v) \in TM$: $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite

(3) L is superlinear:

$$\forall x \in M: \lim_{||v||_x \to \infty} \frac{L(x,v)}{||v||_x} = +\infty$$

or equivalently:

$$\forall x \in M, A \in \mathbb{R}, \exists B \in \mathbb{R}: \ L(x,v) \geq A||v||_x - B$$

Remark 2. Instead of (2) we could also require $\frac{\partial^2 L}{\partial v^2}(x, v)$ to be non-degenerate for all $(x, v) \in TM$ and L to be convex.

Remark 3. The fiberwise superlinearity is in fact uniform over compact subsets of M (and thus M itself), this means we can modify (3) and the constant B can be chosen to be independent from $x \in M$.

Remark 4. Since M is compact, the superlinearity is independent of the metric g.

Theorem 2. *The (global) Legendre transform $Leg : TM \to T^*M$ is diffeomorphism if L is a Tonelli-Lagrangian.*

Proof. Since Leg is fiber-preserving, we only need to consider the restriction $\mathrm{Leg}\big|_{T_xM} : T_xM \to T_x^*M$. Applying the following lemma will finish the proof.

Lemma 1. *Let V be a (finite dimensional) vector space. For $F : V \to \mathbb{R}$, C^2 and strictly convex ($HessF > 0$) we have:*

$$F \text{ superlinear} \Leftrightarrow dF : V \to V^* \text{ is a diffeomorphism}$$

"$\Rightarrow$": define $F^{p_0} : \mathbb{R}^n \to \mathbb{R}$ by $F^{p_0}(v) = F(v) - p_0(v)$ for $p_0 \in (\mathbb{R}^n)^*$ arbitrary but fixed. This function is superlinear, thus it reaches its minimum for some $v_0 \in V$ $\Rightarrow dF^{p_0}(v_0) = 0 \Rightarrow dF(v_0) = p_0$ (surjectivity). Since the Hessian of F^{p_0} is positive definite, it can at most have one critical point, thus dF is injective. Thus F is bijective and due to the Hessian being positive definite, we apply the inverse function theorem and see that dF is bijective local diffeomorphism.

"$\Leftarrow$": For every $k > 0$, we define the compact set

$$S_k := \{v \in V \setminus \{0\} | \, |dF(v)| = k\}.$$

Since dF is a diffeomorphism, there exists a unique $v_0 \in V$, such that

$$dF(v_0) = \frac{k}{|v|}\langle v, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle$ is some inner product on V. We have $v_0 \in S_k$ and $dF(v_0)v = k|v|$. Thus by convexity of F we have:

$$\begin{aligned} F(v) &\geq dF(v_0)v + F(v_0) - dF(v_0)v_0 \\ &\geq k|v| + \inf_{w \in S_k} \{F(w) - dF(w)w\} \end{aligned}$$

which shows, that F is superlinear.

□

Theorem 3. *Let L be Tonelli. Then every extremizing curve $\gamma : [a, b] \to M$ is 'just as smooth as its Lagrangian L'. That means if L is C^r, $r \geq 2$, then γ will be C^r as well.*

After this rather long discussion lets look at an example for a Tonelli-Lagrangian: the electromagnetic Lagrangian:

$$L(x,v) = \frac{1}{2}g_x(v,v) + \theta_x(v) - U(x)$$

where g is the Riemannian metric, $U : M \to \mathbb{R}$ and θ is a 1-form. In physics the first term would correspond to the kinetic energy of particle, the U-term is its potential energy (in this specific case the electric potential of the electric field $\vec{E}$) and the 1-form corresponds to the 'vector potential' of the magnetic field $\vec{B}$. Its solutions satisfy the Newton's equation:

$$^{x}\nabla_{\partial_t}\dot{x} = -\nabla U(x) - Y_x \cdot \dot{x},$$

where ∇U is the gradient of U with respect to g and Y is the Lorentz force defined by:

$$g_x(Y_x \cdot u, v) = d\theta_x(u,v),\ \forall x \in M,\ u,v \in T_xM$$

2 Hamiltonian Mechanics

Definition 7. Let L be Tonelli and $\text{Leg} : TM \to T^*M$ the Legendre transform. We define the *Hamiltonian* H by

$$H : T^*M \to \mathbb{R},\ H(x,p) := \langle p, \text{Leg}^{-1}(x,p)\rangle_x - L(\text{Leg}^{-1}(x,p))$$

where $\langle \cdot,\cdot\rangle_x$ is the canonical pairing between the tangent and cotangent bundles. We say that H is the Legendre dual of L.

Remark 5. For now, all we know is that H is C^1 since Leg is C^1.

Definition 8. We say that a Hamiltonian $H : T^*M \to \mathbb{R}$ is a Tonelli-Hamiltonian if:

(1) H is C^2

(2) $\frac{\partial^2 H}{\partial p^2}(\cdot,\cdot) > 0$

(3) H is superlinear:

$$\forall x \in M : \lim_{||p||_x \to \infty} \frac{H(x,p)}{||p||_x} = +\infty$$

or equivalently:

$$\forall x \in M, A \in \mathbb{R}, \exists B \in \mathbb{R} :\ H(x,p) \geq A||p||_x - B.$$

Remark 6. As before, since M is compact, the superlinearity is uniform and independent of the metric g.

A few properties of the Hamiltonian H are given in this lemma:

Lemma 2. *Let L be Tonelli and H its Legendre dual. Let $x \in M$, $v \in T_xM$, $p \in T_x^*M$, s.t. $p = Leg(x, v)$. Then:*

(i) $\frac{\partial H}{\partial p}(x, p) = v$

(ii) $\frac{\partial H}{\partial x}(x, p) = -\frac{\partial L}{\partial x}(x, v)$

(iii) *H is Tonelli*

(iv) *(Fenchel inequality): $\forall p' \in T_x^*M, v' \in T_xM$:*

$$\langle p', v' \rangle \leq L(x, v') + H(x, p')$$

with equality if and only if $p' = Leg(x, v')$

(v) $H(x, p) = \sup_{v' \in T_xM} [\langle p, v' \rangle - L(x, v')]$

Like we did for our Lagrangian L, we can also define a vector field X_H for the Hamiltonian. First we equip the cotangent bundle T^*M with its canonical symplectic structure ω, which can be defined through the tautological 1-form or Liouville form λ of T^*M, which is given by

$$\lambda = \sum_{i=1}^{n} p_i dx^i$$

in local coordinates. (This definition is independent from the used coordinates.) The canonical symplectic structure is defined by $\omega = -d\lambda$, given in local coordinates by

$$\omega = \sum_{i=1}^{n} dx^i \wedge dp_i.$$

This 2-form is closed and non-degenerate. Thus we can define the *Hamiltonian vector field* X_H by:

$$\omega(X_H(x, p), \cdot) = d_{(x,p)}H$$

This means that the Hamiltonian vector field X_H is in local coordinates given by:

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i}$$

Since H is C^2 this vector field is C^1 and can be locally integrated. Its integral curves satisfy the *Hamiltonian equations*:

$$\frac{\partial H}{\partial p}(x, p_x) = \dot{x}$$
$$\frac{\partial H}{\partial x}(x, p_x) = -\dot{p}_x$$

We can then define the *Hamiltonian flow* ϕ_t^H. It turns out that H is an integral of motion, meaning it is constant along its integral curves, because

$$\frac{d}{dt}H(\phi_t^H) = dH(X_H(\phi_t^H)) = \omega(X_H(\phi_t^H), X_H(\phi_t^H)) \overset{\omega \text{ antisymm.}}{=} 0.$$

For every $K \in \mathbb{R}$ the sets $\{(x,p) \in T^*M |\ H(x,p) = K\}$ are compact (by superlinearity of H) and invariant by ϕ_t^H. Thus ϕ_t^H is complete if M is compact.

Given the definition of the Hamiltonian H, we might ask, whether the projection onto M of the solutions (γ, p_γ) of the Hamilton equations solve the Euler-Lagrange equation as well. This turns out to be the case. In fact, from (i) and (ii) in Lemma 2 we obtain the relation:

$$d_{(x,v)}\text{Leg} \cdot X_L(x,v) = X_H \circ \text{Leg}(x,v),\ \forall (x,v) \in TM$$

In other words: The Lagrangian and Hamiltonian flows are conjugated by the Legendre transform:

$$\begin{aligned} \text{Leg} \circ \phi_t^L &= \phi_t^H \circ \text{Leg} \\ \phi_t^L &= \text{Leg}^{-1} \circ \phi_t^H \circ \text{Leg} \end{aligned}$$

Since the Hamiltonian flow is well defined, the Lagrangian flow is also well defined making the solutions to the Euler-Lagrange equations unique as well. Since the Hamiltonian flow ϕ_t^H is complete, the Euler-Lagrange flow ϕ_t^L is then also complete.

Lastly, we'd like to mention that for Tonelli-Lagrangians L one can find extremizers, that minimize the action in $C^2([a,b], M; \alpha)$, which are called minimizers. Their existence will be shown in the next talk.