Lax-Oleinik semi-group and weak KAM solutions

The goal of this talk is to prove the existence of weak KAM solutions. Indeed, we will show:

Theorem. *(Existence of negative weak KAM solutions). Let L be a Tonelli Lagrangian on a compact connected manifold M. Then there exists a continuous weak KAM solution of negative type with the Mañé critical value as the constant.*

The key to the proof of this theorem will be provided by the Lax-Oleinik semi-group.

0 Recap

Let us at first recall some assumptions, notions and definitions. In the following we assume that M is a compact connected manifold. Let $C^1_{pcw}([a,b],M)$ denote the set of piecewise continuous curves from an interval $[a,b]$ to M and $\mathcal{A}_L(\gamma)$ the action of a curve $\gamma \in C^1_{pcw}([a,b],M)$ with respect to a Tonelli Lagrangian L on M; the set of arbitrary, respectively bounded, functions from M to $X \subset (\mathbb{R} \cup \{\pm\infty\})$ is denoted by $\mathcal{F}(M,X)$, respectively $\mathcal{B}(M,X)$. We also need the notions of the minimal action, dominated functions and calibrated curves.

Definition 0.1. (Minimal action, dominated functions and calibrated curves).

(i) (Minimal action). Let L be a Tonelli Lagrangian on M. For $t > 0$ we define the minimal action $h_t : M \times M \to \mathbb{R}$ as

$$h_t(x,y) := \inf_{\gamma \in C^1_{\text{pcw}}} \int_0^t L(\gamma(s),\dot\gamma(s))ds = \inf_{\gamma \in C^1_{\text{pcw}}} \mathcal{A}_L(\gamma) \tag{1}$$

with the infimum taken over all curves $\gamma \in C^1_{\text{pcw}}([0,t],M)$ with $\gamma(0) = x$ and $\gamma(t) = y$.

(ii) (Dominated functions). Let $u : M \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that u is dominated by $L + c$ and write $u \prec L + c$ if

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s),\dot\gamma(s))ds + c(b-a) = \mathcal{A}_{L+c}(\gamma)$$

for all curves $\gamma \in C^1_{pcw}([a,b],M)$.

(iii) (Calibrated curves). We say that a curve $\gamma \in C^1_{pcw}(I,M)$ is (u,L,c)-calibrated in regard to $u : M \to \mathbb{R}$ and $c \in \mathbb{R}$ if

$$u(\gamma(t')) - u(\gamma(t)) = \int_t^{t'} L(\gamma(s),\dot\gamma(s))ds + c(t'-t) = \mathcal{A}_{L+c}(\gamma|_{[t,t']}) \quad \forall t \leq t' \in I.$$

Remark. (i) We recall that h_t is well defined since M is connected; it is finite valued by the superlinearity of L and the compactness of M. Furthermore, the infimum on the right side of (1) is indeed a minimum by Tonelli's theorem. We also note that h_t is continuous on $M \times M$ by the following lemma of Fleming.

(ii) Let $u : M \to \mathbb{R}$. Then it suffices for u to be Lipschitz that $u \prec L + c$ for a $c \in \mathbb{R}$ (see [**Fat08**, proposition 4.2.1]).

(iii) We know that for every extremal curve $\gamma : I \to M$ of the Tonelli Lagrangian all restrictions $\gamma|_{I'}$ for any subinterval $I' \subset I$ are still extremal. Thus, under the assumption that $u \prec L + c$, with theorem 0.3 it suffices to show that a curve $\gamma \in C^1_{\text{pcw}}([t_0, t_1], M)$ fulfils

$$u(\gamma(t_1)) - u(\gamma(t_0)) = \int_{t_0}^{t_1} L(\gamma(s), \dot{\gamma}(s))ds + c(t_1 - t_0) \tag{2}$$

to verify that it is (u, L, c)- calibrated. Therefore it is important to observe from the proof of theorem 0.3 that the statement given there only requires the calibration of the curve with respect to its endpoints as in (2).

It is also worth mentioning Fleming's Lemma which states the following very important property of the minimal action (for the proof see [**Fat08**, theorem 4.4.3]):

Theorem 0.2. *(Fleming's Lemma). For each $t_0 > 0$ there exists a constant $\kappa_{t_0} \in [0, \infty)$ such that, for each $t > t_0$ the minimal action is Lipschitzian with a Lipschitz constant $\leq \kappa_{t_0}$.* □

The concept of dominated functions and calibrated curves is important because together they provide extremal curves of the Tonelli Lagrangian (compare [**Fat08**, theorem 4.1.9]).

Theorem 0.3. *(Calibrated curves are minimizers). Let $u : M \to \mathbb{R}$ be a function and $c \in \mathbb{R}$ such that $u \prec L + c$. If $\gamma \in C^1_{\text{pcw}}(I, M)$ is (u, L, c)-calibrated, then it is a minimizer of the Tonelli Lagrangian L.* □

At last we state again what a (negative) weak KAM solution is since our goal is to prove their existence:

Definition 0.4. (Negative weak KAM solutions). A weak KAM solution of negative type for a constant $c \in \mathbb{R}$ is a function $u : M \to \mathbb{R}$ with the following properties:

(i) $u \prec L + c$,

(ii) For every $x \in M$ there exists a (u, L, c)-calibrated C^1-curve $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$.

Remark. A weak KAM solution can only have the Mañé critical value, which we denote by C_L, as a constant (see [**Fat08**, corollary 4.3.7]). Therefore it will suffice to show the existence of a weak KAM solution for an arbitrary constant.

1 Main

Our first step will be defining semi-groups. We will then introduce the Lax-Oleinik semi-group and state some important properties of it. Furthermore we will find an equivalent formulation of weak KAM solutions in terms of the Lax-Oleinik semi-group and fixed points. Using this reformulation, we will give two proofs of the existence of negative weak KAM solutions.

1.1 Semi-groups

Definition 1.1. (Semi-group). A (continuous) semi-group (S, T) is a set S with an operation $* : S \times S \to S$ and a neutral element Id, together with an associative map $T : [0, \infty) \to S$ that is compatible with $*$ in the following sense:

$$T(0) = Id \qquad \textit{and} \qquad T(s+t) = T(s) * T(t) \ \forall s, t \in [0, \infty).$$

In the most general definition a semi-group is just a set with a binary associative operation; but definition 1.1 is more what we need here since it stresses the continuous structure of the Lax-Oleinik semi-group. We will now set $S := \mathcal{O}(X)$ which is the set of operators on a Banach space X.

Definition 1.2. (Strong continuity). We call a semi-group $(\mathcal{O}(X), T)$ strongly continuous if

$$\lim_{t \to 0} T(t)x = x \quad \forall x \in X.$$

Proposition 1.3. *(Continuity). Let $(\mathcal{O}(X), T)$ be a strongly continuous semi-group. Then the map $t \mapsto T(t)x$ is continuous for all $x \in X$ and all $t > 0$.* □

1.2 The Lax-Oleinik semi-group

Definition and proposition 1.4. (Lax-Oleinik semi-group). The Lax-Oleinik semi-group (LOS) is the semi-group $(\mathcal{O}(\mathcal{F}(M, [-\infty, +\infty])), T^-)$ with the operator $T_t^- = T^-(t)$ defined as

$$\begin{aligned} T_t^- u(x) := T^-(t)(u(x)) &:= \inf_{\gamma \in C^1_{\text{pcw}}([0,t],M), \gamma(t) = x} [u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s)) ds] \qquad (3) \\ &= \inf_{\gamma \in C^1_{\text{pcw}}([0,t],M), \gamma(t) = x} [u(\gamma(0)) + \mathcal{A}_L(\gamma)] \\ &= \inf_{y \in M} [u(y) + h_t(y, x)] \end{aligned}$$

for $t > 0$. For $t = 0$ we set $T_0^- u := u$. □

Remark. The LOS indeed fulfils the semigroup properties since $T_{s+t}^- = T_s^- \circ T_t^-$.

The first property of the LOS we want to show is its strong continuity.

Proposition 1.5. *(Strong continuity of the LOS). The LOS is strongly continuous for $u \in C^0(M, \mathbb{R})$ with respect to the supremum-norm.*

Proof. Since $C^1(M, \mathbb{R}) \supset C^\infty(M, \mathbb{R})$ is dense in $(C^0(M, \mathbb{R}), \| \cdot \|_\infty)$ and the LOS is non-expansive by proposition 1.8, it is enough to show the statement for Lipschitz u. Let K be the Lipschitz constant of u. Since M is compact and L is superlinear there exists a constant C_K such that

$$L(x, v) \geq K \|v\| + C_K \quad \forall (x, v) \in TM.$$

It follows that for every curve $\gamma : [0, t] \to M$ we have

$$\int_0^t L(\gamma(s, \dot\gamma(s))) ds \geq K d(\gamma(0), \gamma(t)) + C_K t.$$

Since the Lipschitz constant of u is K, we conclude that

$$\int_0^t L(\gamma(s,\dot{\gamma}(s)))ds + u(\gamma(0)) \geq u(\gamma(t)) + C_K t$$

which yields

$$T_t^- u(x) \geq u(x) + C_K t. \tag{4}$$

On the other hand, by choosing the constant curve $\gamma_x : [0,t] \to M$, $s \mapsto x$, we obtain

$$T_t^- u(x) \leq u(x) + L(x,0)t. \tag{5}$$

From (4) and (5) it follows with $A_0 := \max_{x\in M} L(x,0)$

$$\|T_t^- u - u\|_\infty \leq t \max(C_K, A_0) \xrightarrow{t\to 0} 0.$$

□

Remark. For the LOS in respect to $C^0(M,\mathbb{R})$ the map $t \mapsto T_t^- u$ is even uniformly continuous for all $u \in C^0(M,\mathbb{R})$ and $t > 0$ by the non-expansiveness in proposition 1.8.

It follows directly by Fleming's lemma 0.2, taking into account the non-expansiveness of the LOS by proposition 1.8, that the operators T_t^- are equi-Lipschitzian on the space of continuous functions (see [**Fat08**, proposition 4.6.6 (3)] for an even stronger result):

Corollary 1.6. *(Equi-Lipschitzianity for fixed time). The family of functions* $(T_t^- u)_{u\in C^0(M,\mathbb{R})}$ *is equi-Lipschitzian for any* $t > 0$. □

Let us collect the following important properties of the LOS which we will not fully prove here but are fairly simple to show (compare [**Fat08**, proposition 4.6.2]):

Proposition 1.7. *(General properties of the LOS). The LOS has the following properties:*

(i) (Estimates)

$$\inf_M u + t \inf_{TM} L \leq T_t^- u(x) \leq \inf_M u + \max_{M\times M} h_t.$$

(ii) (Finiteness) If $\inf_M u$ *is finite, then it already follows that the function* $T_t^- u$ *is finite for all* $t > 0$.

(iii) (Some linearity)

$$T_t^-(c+u) = c + T_t^- u \quad \forall c \in \mathbb{R}.$$

(iv) (Inf commutativity) Let $(u_i)_{i\in I} \subset \mathcal{F}(M,[-\infty,+\infty])$ *be a family of functions. Then we have*

$$T_t^-(\inf_{i\in I} u_i) = \inf_{i\in I} T_t^- u_i.$$

(v) (Monotony) For all $u,v \in \mathcal{F}(M,[-\infty,+\infty])$ *it holds*

$$u \leq v \quad \Longrightarrow \quad T_t^- u \leq T_t^- v \quad \forall t \geq 0.$$

(vi) (Dominance I) For every $c \in \mathbb{R}$ *and* $u \in \mathcal{F}(M,[-\infty,+\infty])$ *with* $u \neq \pm\infty$ *it holds*

$$u \leq T_t^- u + ct \quad \forall t \geq 0 \quad \Longleftrightarrow \quad u \prec L + c.$$

(vii) (Dominance II)

$$u \prec L + c \quad \Longrightarrow \quad T_t^- u \prec L + c \quad \forall t \geq 0.$$

Proof. Because the properties (vi) and (vii) take central positions in the proof of the existence of weak KAM solutions I want to give their proof here, illustrating the schema of the other ones.
(vi). We assume that u is not identically $-\infty$ or $+\infty$.
$\Longrightarrow$. We first show that from our assumption follows: $\inf_M u \in \mathbb{R}$. If $\inf_M u = -\infty$, then we obtain from (i) and $u \leq T_t^- u + ct$ that $u \equiv -\infty$. On the other hand, $\inf_M u = +\infty$ immediately yields $u \equiv +\infty$. Consequently, we can indeed assume that $\inf_M u \in \mathbb{R}$. In this case it follows again from (i) that $T_t^- u$ is finite everywhere and therefore u, which satisfies $u \leq T_t^- u + ct$ and $u \geq \inf_M u > -\infty$, is also finite valued on the whole of M. The condition $u \leq T_t^- u + ct$ in particular implicates that $u(x) \leq u(y) + h_t(y, x) + ct$ for all $x, y \in M$ and $t \geq 0$. Since u is finite valued, this is equivalent to $u(x) - u(y) \leq h_t(y, x) + ct$ which itself is equivalent to $u \prec L + c$ by definition and Tonelli's theorem.
$\Longleftarrow$. As above we can again assume that $\inf_M u \in \mathbb{R}$ and reversing all steps yields the desired statement.
(vii). We know from remark (ii) after definition 0.1 that $u \prec L + c$ implies that u is Lipschitz and hence continuous. Accordingly it takes its minimum on the compact manifold M and by (ii) the function $T_t^- u$ is finite everywhere for each $t \geq 0$. It follows from (vi) that

$$u \leq T_{t'}^- u + ct' \quad \forall t' \geq 0$$

and applying assertions (iii) and (v) as well as the semi-group property yields

$$T_t^- u \leq T_t^- [T_{t'}^- + ct'] = T_{t'}^- [T_t^- u] + ct' \quad \forall t' \geq 0.$$

Therewith, in addition to the finiteness of the $T_t^- u$, we can use (vi) again which completes the proof. □

The next feature of the LOS will be focal in the proof of the existence of weak KAM solutions in section 1.4.

Proposition 1.8. *(Non-expansiveness of the LOS). The operators T_t^- are non-expansive on the space of functions $\mathcal{B}(M, \mathbb{R})$ with respect to the supremum-norm for all $t \geq 0$.*

Proof. If $u, v \in \mathcal{B}(M, \mathbb{R})$, we have

$$-\|u - v\|_\infty + v \leq u \leq \|u - v\|_\infty + v.$$

By the monotony of the LOS (proposition 1.7 (v)) and the linearity under addition of a constant (proposition 1.7 (iii)) we obtain

$$-\|u - v\|_\infty + T_t^- v \leq T_t^- u \leq \|u - v\|_\infty + T_t^- v.$$

This implies $\|T_t^- u - T_t^- v\|_\infty \leq \|u - v\|_\infty$ which is exactly what we wanted to show. □

1.3 Connection between the LOS and negative weak KAM solutions

We will show that the existence of negative weak KAM solutions is equivalent to finding a fixed point of the LOS up to a linear function in time. The main ingredient is the Tonelli theorem which enters via the following Lemma.

Lemma 1.9. *(T_t^- is realised). For each $u \in C^0(M, \mathbb{R})$, each $x \in M$, and each $t > 0$ we can find a curve $\gamma_{x,t} \in C^1_{\text{pcw}}([0,t], M)$ with $\gamma_{x,t}(t) = x$ that realises $T_t^- u(x)$, i.e.*

$$T_t^- u(x) = u(\gamma_{x,t}(0)) + \int_0^t L(\gamma_{x,t}(s), \dot{\gamma}_{x,t}(s)) ds.$$

Proof. Since $u \in C^0(M, \mathbb{R})$, the function $y \mapsto u(y) + h_t(y, x)$ is also continuous on the compact space M and consequently takes its minimum on M; we denote this minimum by y_x, i.e. $T_t^- u(x) = \inf_{y \in M}(u(y) + h_t(y, x)) = u(y_x) + h_t(y_x, x)$. Now we can apply Tonelli's theorem to find a curve $\gamma_{x,t} \in C^1_{\text{pcw}}([0,t], M)$ with $\gamma_{x,t}(0) = y_x$ and $\gamma_{x,t}(t) = x$ such that $h_t(y_x, x) = \int_0^t L(\gamma_{x,t}(s), \dot{\gamma}_{x,t}(s)) ds$. This directly yields the assertion. □

Theorem 1.10. *(The LOS and negative weak KAM solutions). For a function $u : M \to \mathbb{R}$ the following two statements are equivalent:*

(i) $T_t^- u + ct = u$ for each $t \geq 0$,

(ii) u is a negative weak KAM solution for the constant $c \in \mathbb{R}$.

Proof. $\implies$. From proposition 1.7 (vi) it directly follows that $u \prec L + c$. This also guarantees the continuity of u by remark (ii) after definition 0.1.
Now it remains to find, for each $x \in M$, a suiting C^1-curve that is (u, L, c)-calibrated. We already know by lemma 1.9 that for each $t > 0$ there exists $\gamma_{x,t} \in C^1_{\text{pcw}}([0,t], M)$ with $\gamma_{x,t}(t) = x$ and

$$u(x) - ct = T_t^- u(x) = u(\gamma_{x,t}(0)) + \int_0^t L(\gamma_{x,t}(s), \dot{\gamma_{x,t}}(s)) ds.$$

By shifting the interval, i.e. defining $\bar{\gamma}_{x,t}(s) := \gamma_{x,t}(s + t)$, we get a curve ending in x that is calibrated for its endpoints:

$$u(\bar{\gamma}_{x,t}(0)) - u(\bar{\gamma}_{x,t}(-t)) = \int_{-t}^0 L(\bar{\gamma}_{x,t}(s), \dot{\bar{\gamma}}_{x,t}(s)) ds + ct.$$

We know from remark (iii) after definition 0.1 that this already yields that $\bar{\gamma}_{x,t}$ is (u, L, c)-calibrated. Since in particular the $\bar{\gamma}_{x,t}$ are extremal curves, by the a priori compactness (see [**Fat08**, corollary 4.4.5]) there exists a compact subset $K_1 \subset TM$ such that

$$\forall t \geq 1, \forall s \in [-t, 0] : (\bar{\gamma}_{x,t}(s), \dot{\bar{\gamma}}_{x,t}(s)) \in K_1.$$

Accordingly, the sequence $((\bar{\gamma}_{x,t}(0), \dot{\bar{\gamma}}_{x,t}(0)))_{t \geq 1}$ has a convergent subsequence $((\bar{\gamma}_{x,t_n}(0), \dot{\bar{\gamma}}_{x,t_n}(0)))_{t_n}$ tending to $(x, v) \in TM$, where $t_n \to +\infty$ for $n \to +\infty$. The negative orbit

$$\gamma_-^x(s) := \phi_s(x, v) \quad \forall s \leq 0,$$

with ϕ_s being the Euler-Lagrange flow, is our candidate for the desired (u, L, c)-calibrated curve. Therefore, let us fix $t' \in [0, +\infty)$. Then, because the $\bar{\gamma}_{x,t_n}$ are al extremal curves, for n big enough such that $t_n > t'$ the equality $(\bar{\gamma}_{x,t_n}(s), \dot{\bar{\gamma}}_{x,t_n}(s)) = \phi_s(\bar{\gamma}_{x,t_n}(0), \dot{\bar{\gamma}}_{x,t_n}(0))$ holds for all $s \in [-t', 0]$. By the continuity of the Euler-Lagrange flow, the right hand side converges uniformly to the map $s \mapsto \phi_s(x, v) = \gamma_-^x(s)$ on the compact set $[-t', 0]$. Since all the $\bar{\gamma}_{x,t_n}$ are (u, L, c)-calibrated curves on the interval $[-t', 0]$, we arrive at the equation

$$u(x) - u(\gamma_-^x(t')) = \int_{-t'}^0 L(\gamma_-^x(s), \dot{\gamma}_-^x(s)) ds + ct',$$

which as above yields that the curve $\gamma_-^x|_{[-t',0]}$ is (u, L, c)-calibrated for all $t' \geq 0$ by remark (iii) after definition 0.1. This shows the desired property of γ_-^x.
$\Longleftarrow$. Conversely, let us suppose that $u \prec L + c$ and that, for each $x \in M$, there exists a C^1-curve $\gamma_-^x : (-\infty, 0] \to M$ with $\gamma_-^x(0) = x$ such that

$$u(x) - u(\gamma_-^x(t)) = \int_{-t}^{0} L(\gamma_-^x(s), \dot{\gamma}_-^x(s))ds + ct \quad \forall t \in [0, \infty).$$

If $x \in M$ and $t > 0$ we define the curve $\gamma : [0, t] \to M$ by $\gamma(s) := \gamma_-^x(s-t)$. It immediately follows $\gamma(t) = x$ and

$$u(x) = u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma(s)})ds + ct.$$

Hence, we have $T_t^- u(x) + ct \leq u(x)$ and thus $T_t^- u + ct \leq u$. The converse inequality $u \leq T_t^- u + ct$ results from $u \prec L + c$. □

1.4 Existence of negative weak KAM solutions

According to theorem 1.10 we need to find a fixed point of the operator $u \mapsto T_t^- u + ct$. Therefore we need the following lemma.

Lemma 1.11. *(Equi-boundedness). Let $u : M \to \mathbb{R}$ with $u \prec L + C_L$. Then the family $(T_t^- u + tC_L)_{t \geq 0}$ is equi-bounded with respect to the supremum-norm, i.e.*

$$\sup_{t \geq 0} \|T_t^- u + tC_L\|_\infty < +\infty.$$

□

I do not give the proof of this lemma here, since, in my opinion, it does not really allow new important insight into the theory of the LOS (see [**Fat08**, lemma 4.7.5]). Nevertheless, it provides the existence of the negative weak KAM solution. We are giving a first proof:

Theorem 1.12. *(Existence of negative weak KAM solutions I). Let $u : M \to \mathbb{R}$ with $u \prec L + C_L$. Then, as $t \to +\infty$, $T_t^- u + tC_L$ converges uniformly to a continuous function $u_- \in C^0(M, \mathbb{R})$ which is a negative weak KAM solution.*

Proof. At first we show that the family $(T_t^- u + tC_L)_{t \geq 0}$ is increasing in $t \geq 0$: It follows from $u \prec L + C_L$ that $u \leq T_t^- u + tC_L$ by dominance I of the LOS (proposition 1.7 (vi)). Together with the monotony of the LOS (proposition 1.7 (v)) and the linearity under addition of a constant (proposition 1.7 (iii)) we obtain

$$\begin{aligned} T_t^- u &\leq T_t^- [T_s^- u + sC_L] \\ &= T_{t+s}^- u + sC_L \quad \forall t, s \geq 0. \end{aligned}$$

Consequently, $T_t^- u + tC_L \leq T_{t+s}^- + (t+s)C_L$ for all $t, s \geq 0$ and thus $T_t^- u + tC_L \leq T_{t'}^- + t'C_L$ for all $t, t' \geq 0$ with $t \leq t'$.
Now we know from lemma 1.11 that the family $(T_t^- u + tC_L)_{t \geq 0}$ is also equi-bounded. Altogether it follows that the point-wise limit

$$u_-(x) := \lim_{t \to +\infty} T_t^- u(x) + tC_L$$

exists everywhere on M and is finite. We also know from remark (ii) after definition 0.1 and dominance II of the LOS (proposition 1.7 (vii)) that the family $(T_t^- u + tC_L)_{t\geq 0}$ is equi-Lipschitzian, in particular equi-continuous. This implies that the limit $u_- : M \to \mathbb{R}$ is also continuous because the convergence $T_t^- u + tC_L \to u_-$ is uniform for $t \to +\infty$.
All that remains is to prove that u_- is a negative weak KAM solution, i.e. that $T_s^- u_- + sC_L = u_-$ for each $s \geq 0$ by theorem 1.10. Thanks to the non-expansiveness of the LOS (proposition 1.8) we can swap the application of T_s^- with taking the limit $t \to +\infty$ and get by the definition of u_-:

$$\begin{aligned} T_s^- u_- + sC_L &= \lim_{t\to+\infty} T_s^- [T_t^- u + tC_L] + sC_L = \lim_{t\to+\infty} (T_{t+s}^- u + (t+s)C_L) \\ &= u_-. \end{aligned}$$

□

For the second proof, rather than directly constructing a fixed point, we want to make an approach involving some general theory of fixed points. Therefore, additionally, we need the following lemma.

Lemma 1.13. *(Fixpoint lemma). Let X be a Banach space and $(\varphi_t : X \to X)_{t\geq 0}$ be a family of maps with the following properties:*

(i) $\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}$ for all $t, t' \geq 0$,

(ii) φ_t is non-expansive for each $t \geq 0$,

(iii) $\varphi_t(X)$ is relatively compact in X for each $t > 0$,

(iv) the map $t \mapsto \varphi_t(x)$ is continuous on $[0, +\infty)$ for all $x \in X$.

Then the family $(\varphi_t)_{t\geq 0}$ has a common fixed point.

Proof. I will give the proof in two steps.
1. Let us first show that every φ_t has a fixed point. We denote by $E_t := \varphi_t(X) \subset X$ the image of φ_t which is relatively compact in X by condition (iii). We can assume that E_t is convex; otherwise take the convex envelope of E_t which is still relatively compact in X. We also note that $\overline{E_t}$ is in particular complete. Now we want to argue as in the proof of Banach's fixed point theorem; but the map φ_t is just non-expansive and not contractive. We fix this by looking at the family of functions $(\varphi_t^\lambda := \lambda\varphi_t)_{0<\lambda<1}$. Let us also assume that $0 \in E_t$; elsewise we translate the set E_t, respectively the functions $\lambda\varphi_t$, and look at the family of maps $(\lambda\varphi_t - \varphi_t(x_0))_{0<\lambda<1}$ for a $x_0 \in E_t$. Now these functions are contractive and since E_t is convex, they only take values in E_t. Thus we can argue as follows:
We claim that the sequence $(x_n^\lambda := (\varphi_t^\lambda)^n(x_0))_{n\in\mathbb{N}}$, for arbitrary $x_0 \in E_t$, converges to a fixed point of the map φ_t^λ. This can be shown as in the proof of Banach's fixed point theorem by observing that $(x_n^\lambda)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence it converges to a x_∞^λ in $\overline{E_t} \subset X$. It is easy to validate that x_∞^λ is a fixed point of φ_t^λ. Now, since $\overline{E_t}$ is compact, there exists a subsequence $\lambda_n \to 1$ as $n \to \infty$ such that $x_\infty^{\lambda_n} \to x_\infty$ for a $x_\infty \in \overline{E_t}$. It holds $\varphi_t^{\lambda_n}(x_\infty^{\lambda_n}) = x_\infty^{\lambda_n}$ and passing to the limit $n \to \infty$ yields

$$\begin{aligned} \varphi_t(x_\infty) &= \lim_{n\to\infty} \left(\varphi_t^{\lambda_n}\left(\lim_{n\to\infty} x_\infty^{\lambda_n}\right)\right) = \lim_{n\to\infty} \left(\varphi_t^{\lambda_n}(x_\infty^{\lambda_n})\right) = \lim_{n\to\infty} x_\infty^{\lambda_n} \\ &= x_\infty, \end{aligned}$$

where we used that the $\varphi_t^{\lambda_n}$ are continuous as they are contractive.
2. It remains to show that there exists a common fixed point of the φ_t, $t \geq 0$. It follows from property (i) that a fixed point of φ_t is a fixed point of φ_{kt} for any integer $k \geq 0$. Accordingly it is enough to show that for $t \in [0,1)$ all φ_t admit the same fixed point. Since the set $((1/2)^n)_{n\in\mathbb{N}}$ is dense in $[0,1)$ under addition and the map $t \mapsto \varphi_t(x)$ is continuous by condition (vi), it suffices to show that all $\varphi_{(1/2)^n}$ have a common fixed point. Therefore we define the sets $F_t := \{x \in X | \varphi_t(x) = x\}$. Those sets are non empty by part one of the proof and they are compact because $F_t \subset \overline{E_t}$. It follows by $\varphi_{(1/2)^n} = \varphi_{(1/2)^{n+1}} \circ \varphi_{(1/2)^{n+1}}$ that $F_{(1/2)^{n+1}} \subset F_{(1/2)^n}$. Therewith we can complete the proof by taking the decreasing intersection $F := \cap_{n\in\mathbb{N}} F_{(1/2)^n}$ of compact non empty sets which is non empty. Every $x \in F$ is a desired fixed point for all φ_t, $t \geq 0$. □

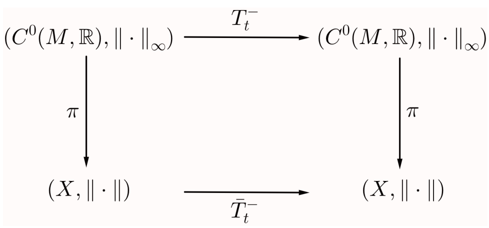
Now we are ready to give a second proof of the existence of negative weak KAM solutions.

Theorem 1.14. *(Existence of negative weak KAM solutions II). There exists a negative weak KAM solution $u_- \in C^0(M,\mathbb{R})$.*

Proof. We define the quotient $X := C^0(M,\mathbb{R})/\mathbb{R}$ where two functions are equivalent if they only differ by a constant $c \in \mathbb{R}$. We endow this space with the topology that is induced by the supremum-norm on $C^0(M,\mathbb{R})$, i.e.

$$\|[u]\| := \inf_{c\in\mathbb{R}} \|u + c\|_\infty.$$

We also know that $T_t^-(C^0(M,\mathbb{R})) \subset C^0(M,\mathbb{R})$, i.e. $T_t^- \in \mathcal{O}(C^0(M,\mathbb{R}))$, for each $t \geq 0$ (this follows either from corollary 1.6 or alternatively by Fleming's lemma 0.2). By the linearity under addition of a constant (proposition 1.7 (iii)), the projection $\pi : C^0(M,\mathbb{R}) \to X$ induces a well defined semi-group of operators $(\mathcal{O}(X), \bar{T}^-)$ such that the diagram



commutes. We know from theorem 1.10 that a negative weak KAM solution will in particular be a fixed point of $\bar{T}_t^-$ in X for each $t \geq 0$. We will see that this is also a sufficient condition. We know from corollary 1.6 that the families $(T_t^- u)_{u\in C^0(M,\mathbb{R})}$ are equi-Lipschitzian. We also want to show that they are equi-bounded. For that reason we normalise the functions $T_t^- u$ by fixing an arbitrary $x_0 \in M$ and introducing the operator

$$\tilde{T}_t^- u := T_t^- u - T_t^- u(x_0).$$

Then, by the estimates of proposition 1.7 (i), we observe that the family $(\tilde{T}_t^- u)_{u\in C^0(M,\mathbb{R})}$ is equi-bounded by the constant $\max_{M\times M} h_t - t \inf_{TM} L$ for every $t > 0$ - and it inherits the equi-Lipschitzianity from $(T_t^- u)_{u\in C^0(M,\mathbb{R})}$. Consequently, the conditions of the theorem of Arzelà-Ascoli are fulfilled (see for example [**Alt12**, theorem 2.12, page 110]) and applying it yields that $\tilde{T}_t^-(C^0(M,\mathbb{R}))$ is relatively compact in $C^0(M,\mathbb{R})$. The same is true for $\pi(\tilde{T}_t^-(C^0(M,\mathbb{R})) = \pi(T_t^-(C^0(M,\mathbb{R})) = \bar{T}_t^-(X) \subset X$ by construction. This enables us to apply lemma 1.13 on the family of operators $(\bar{T}_t^-)_{t\geq 0}$ which give us a common fixed point $[u_-] \in X$. It follows that $T_t^- u_- = u_- + f(t)$ where $f : [0, \infty) \to \mathbb{R}$. The semi-group property of the LOS and the continuity of the map $t \mapsto T_t^- u_-$ (proposition 1.3 and proposition 1.5) together with $T_0^- u_- = u_-$ yield that $f(t) = tf(1)$. It follows $T_t^- u_- + (-f(1))t = u_-$ which had to be shown. □

2 Conclusion

We have seen that a key ingredient to the existence of weak KAM solutions is the compactness of M. Among others, one important implication of this premise is that the values $T_t^- u(x)$ are realised by curves (lemma 1.9). With this property we were able to deduce a reformulation of weak KAM solutions in terms of the LOS (theorem 1.10). It turned out that weak KAM solutions are special fixed points of the LOS up to a linear function in time. This fixed points can be either directly constructed (theorem 1.12) or their existence can be proven by applying more general methods of the theory of fixed points (theorem 1.14). In either case not only the semi-group properties of the LOS have been crucial but also its very well behaviour, i.e. its equi-boundedness, equi-Lipschitzianity - for a fixed time as well as for a fixed function and varying time - and non-expansiveness.
It is also worth mentioning that the same techniques implemented above can be used - mutatis mutandis - to show the existence of positive weak KAM solutions.

References

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