Minimizing Orbits for Tonelli Lagrangians – Weak KAM Theory

compact

We study solutions $u: M \to \mathbb{R}$ of Hamilton-Jacobi equation $H(x, d_x u) = k$, $k \in \mathbb{R}$.

<u>Recall:</u> $H(x,p) = \sup_{v \in T_xM} \{ \langle p, v \rangle - L(x,v) \}$, where we always assume that $L: TM \to \mathbb{R}$ is Tonelli

<u>Rmk:</u> There ex. at most one value of k s.t. C^1 sol. of H-J eq. exist. We will later see, that this value is $c(L)$.

proof: If u, v where C^1 sol.

$\Rightarrow \exists x \in M$ s.t. $d_x u = d_x v$ as M is cpt.

$\Rightarrow \quad k = H(x, d_x u) = H(x, d_x v) = k'$

Action-minimizing Methods in Hamiltonian Dynamics, A. Sorrentino

<u>Def. 5.1.3</u> (Dominated functions)

Let $u: M \to \mathbb{R}$ cont.; u is dominated by $L+k$ $(u \prec L+k)$ if for all $a<b$ and $\gamma: [a,b] \to M$:

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(t), \dot\gamma(t))\, dt + k(b-a) = A_{L+k}(\gamma)$$

<u>Rmk</u> $u \prec L+k \Rightarrow u$ Lipschitz

To see this take a unit speed geod. and take into acc. that M is cpt.

Prop. 5.1.4 $u \prec L+k \iff d_x u$ ex. a.e. and where it ex. $H(x, d_x u) \leq k$

proof: „$\Rightarrow$" $u \prec L+k \Rightarrow u$ Lip. $\Rightarrow d_x u$ ex. a.e.

We look at

$$H(x, d_x u) := \sup_{v \in T_x M} \{ \underbrace{\langle d_x u, v \rangle} - L(x,v) \}$$

$$= \lim_{t \to 0} \frac{u(\gamma(t)) - u(\gamma(0))}{t} \leq L(x,v) + k$$

$$\leq \sup_{v \in T_x M} \{ L(x,v) + k - L(x,v) \} = k$$

„$\Leftarrow$" $k \geq H(x, d_x u) \geq \langle d_x u, v \rangle - L(x,v)$ a.e. $\forall v \in T_x M$

$\int_a^b dt \Rightarrow$ $k(b-a) \geq u(\gamma(b)) - u(\gamma(a)) - \int_a^b L(\gamma(t), \dot\gamma(t))\, dt$

Def. 5.1.7 (Critical subsolutions)

A fct. $u \prec L + c(L)$ is called critically dominated or critical subsolution

Rmk: There is no subsol. if $k < c(L)$, as

$u \prec L+k \Rightarrow u(x) - u(y) \leq A_{L+k}(\gamma$ conn. x and $y)$

$\Rightarrow u(x) - u(y) \leq \Phi_k(x,y) = -\infty$ for $k < c(L)$

see Lemma 1 below

On the other hand $u^y(x) := -\Phi_k(y,x)$ is dom. by $L+k$ if $k \geq c(L)$, we could have thus defined

$$c(L) = \inf_{\underbrace{u \in C^\infty(M)}} \max_{x \in M} H(x, d_x u)$$

not a min in C^∞
but in Lip.

Prop. 5.1.9 Let $u: M \to \mathbb{R}$ be a C^1 fct. and $k \in \mathbb{R}$

$H(x, d_x u) = k \iff u \prec L+k$ and $\forall x \in M \, \exists \gamma_x : \mathbb{R}_{\substack{\leq 0 \\ \geq 0}} \to M$

s.t. $\gamma_x(0) = x$ and for any $[a,b] \subseteq \mathbb{R}_{\substack{\geq 0 \\ \leq 0}}$

$u(\gamma_x(b) - \gamma_x(a)) = A_{L+k}(\gamma_x|_{[a,b]})$

Gabriele showed a very similar thm 1.3

proof: (in the case $\mathbb{R}_{\leq 0}$)

„$\Leftarrow$" $u(\gamma_x(0)) - u(\gamma_x(-t)) = \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + kt \quad \forall t > 0$

$\Rightarrow \quad d_x u(\dot\gamma_x(0)) = L(x, \dot\gamma(0)) + k$

Fenchel.
$\Rightarrow \quad H(x, d_x u) \geq d_x u(\dot\gamma_x(0)) - L(x, \dot\gamma(0)) = k$

$H(x, d_x u) \leq k$ follows by Prop. 5.1.4

„$\Rightarrow$" $u \prec L+k$ follows by 5.1.4

The Idea is to rewrite (*) as an ODE

s.t. a sol. ex. by Cauchy-Peano thm.

*'s are taken from A. Fathi's book

Def 3.4.5* (Lagrangien Gradient)

Let $L: TM \to \mathbb{R}$ be a C^r Tonelli Lagr., $r \geq 2$

and $u: M \to \mathbb{R}$ C^k, $k \geq 1$. We define

$\mathrm{grad}_L u(x)$ via

$$\frac{\partial L}{\partial v}(x, \mathrm{grad}_L u(x)) = d_x u$$

Obs: $\mathrm{Leg}: TM \to T^*M$ is C^{r-1} diffeo and

$$(x, \mathrm{grad}_L u) = \mathrm{Leg}^{-1}(x, d_x u)$$

$\Rightarrow$ $\mathrm{grad}_L u$ well-def and of class $C^{\min(k,r)-1}$

Prop 4.1.5* (Gabriele already proved it!)

Let $L: TM \to \mathbb{R}$ Tonelli, $u: M \to \mathbb{R}$ C^1 sol.

$$H(x, d_x u) = k$$

Then $\gamma: I \to M$ is a sol. of $\mathrm{grad}_L u$ iff

$$(*) \quad u(\gamma(b)) - u(\gamma(a)) = A_{L+k}(\gamma) \quad \forall\, a<b \in I$$

proof: $(x, d_x u) = \mathrm{Leg}(x, \mathrm{grad}_L u(x)) \quad \forall x$

$$\overset{\text{Fenchel}}{\Longleftrightarrow} \quad d_x u(\underbrace{\mathrm{grad}_L u(x)}_{\dot\gamma(t)}) = L(\underbrace{x}_{\gamma(t)}, \underbrace{\mathrm{grad}_L u(x)}_{\dot\gamma(t)}) + \underbrace{H(x, d_x u)}_{k} \quad \forall x$$

$$\Longleftrightarrow \quad \frac{d}{dt} u(\gamma(t)) = L(\gamma(t), \dot\gamma(t)) + k \quad \forall t$$

$$\Longleftrightarrow \quad u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(t), \dot\gamma(t)) + k(b-a) \quad \forall a<b$$

Cont. proof of 5.1.9:

By prop. 4.1.5* we know that γ sat $(*)$ if γ is a sol. of $\mathrm{grad}_L u$. But as L, u are C^1

$\Rightarrow \mathrm{grad}_L u$ is C^0

$\Rightarrow$ find sol. of ODE $\frac{d}{dt}\gamma(t) = \mathrm{grad}_L u(\gamma(t))$ by Cauchy-Peano.

Def. 5.1.10 (Calibrated curves)

Let $u \prec L+k$. A curve $\gamma: I \to M$ is (u,L,k)-calibrated if, for any $[a,b] \subset I$

$$u(\gamma(a)) - u(\gamma(b)) = A_{L+k}(\gamma|_{[a,b]})$$

Prop. 5.1.12 Let $u \prec L+k$ and $\gamma: [a,b] \to M$ (u,L,k)-cal.

i) If $d_{\gamma(t)}u$ ex for some $t \in [a,b]$

$$\Rightarrow H(\gamma(t), d_{\gamma(t)}u) = k \;\&\; d_{\gamma(t)}u = \frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t))$$

i.e. γ is a flow line of $\mathrm{grad}_L u$.

ii) If $t \in (a,b) \Rightarrow d_{\gamma(t)}u$ ex.

proof: By Prop. 5.1.9 we have $H(\gamma(t), d_{\gamma(t)}u) = k$ and if $d_{\gamma(t)}u$ ex. we can take the limit

$$d_{\gamma(t)}u(\dot\gamma(t)) = \lim_{\varepsilon \to 0} \frac{u(\gamma(t)) - u(\gamma(t+\varepsilon))}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} L(\gamma(s), \dot\gamma(s))\,ds + k \qquad (*)$$

$$= L(\gamma(t), \dot\gamma(t)) + H(x, d_x u)$$

Fenchel $\Rightarrow$ $$d_{\gamma(t)}u = \frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t))$$

ii) follows as for $t \in (a,b)$ left & right limit of $(*)$ can be taken and agree.

Prop. 5.1.11

If $u \prec L_\eta + k$ and $\gamma : [a,b] \to M$ is (u, L_η, k)-cal.
the γ is a Tonelli min. i.e.

$$A_{L+k}(\gamma) \leq A_{L+k}(\sigma) \quad \forall \sigma : [a,b] \to M \quad \begin{array}{l} \sigma(a) = \gamma(a) \\ \sigma(b) = \gamma(b) \end{array}$$

proof: By prop. 5.1.2 γ is a flow line of $\mathrm{grad}_L u$ on (a,b) and Gabriele showed that these are Tonelli min.

Def 5.1.14 (weak KAM sol.)

Let $u \prec L+k$, u is a weak KAM sol. of
positive
negative type if

$[0,\infty)$

$$\forall x \in M \ \exists \gamma_x : (-\infty, 0] \to M \text{ s.t. } \gamma_x(0) = x$$

and γ_x is $(u, L, c(L))$-cal.

We call $S^{\overset{+}{-}}$ the set of pos. neg. weak KAM sol.

Thm 5.1.6 / 5.1.2* (Weak KAM theorem)

There is only one value of k for which weak KAM sol. of pos. and neg. type of $H(x, d_x u) = k$ ex., $k = c(L)$.
In part. for any $u \prec L + c(L)$ there ex a unique weak KAM sol. of neg. / pos. type u_- / u_+ s.t.

$u_- = u_+$ on the proj. Aubry set $\mathcal{A}$ and

(1) $u_+ \leq u \leq u_-$

(2) if $\tilde{u}_\pm \in S^\pm$ sat. $\begin{cases} \tilde{u}_+ \leq u \\ \tilde{u}_- \geq u \end{cases}$ then $\begin{cases} \tilde{u}_+ \leq u_+ \\ \tilde{u}_- \geq u_- \end{cases}$

(3) $u_\pm = \lim_{t \to \infty} T^\pm u + c(L)t$ (uniform convergence)

where $T_t^- u(x) := \inf_{\substack{\gamma \in C^{ac}[0,t] \\ \gamma(t)=x}} \{ u(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))ds \}$

$$T_t^+ u(x) := \sup_{\substack{\gamma:[0,t]\to M \text{ piecew. } C^1 \\ \gamma(0)=x}} \{ u(\gamma(t)) - \int_0^t L(\gamma(s),\dot\gamma(s))ds \}$$

we will see more about these things in Leon's talk next week.

Def 5.1.4* (Conjugate pairs)

A pair (u_+, u_-) is said to be conj. if $u_\pm \in S^\pm$ and $u_- = u_+$ on $\mathcal{M}$

Def 5.2.1* Let (u_-, u_+) be a conj. pair.

$$\tilde{\mathcal{I}}_{(u_-,u_+)} := \{ x \in M \mid u_-(x) = u_+(x) \}$$

Thm 5.2.2* $\forall x \in \tilde{\mathcal{I}}_{(u_-,u_+)}\ \exists \gamma^x : \mathbb{R} \to M$ that is $(u_\pm, L, c(L))$-cal. and $\gamma^x(0)=x$, $\dot\gamma^x(0)=v$

proof: $\forall x \in \tilde{\mathcal{I}}_{(u_-,u_+)}\ \exists \gamma_+^x : \mathbb{R}_{\geq 0} \to M$ s.t.

$$A_{L+c(L)}(\gamma_+|_{[0,t]}) \overset{(u_+,L,c(L))\text{-cal.}}{=} u_+(\gamma_+^x(t)) - u_+(x)$$

$$\overset{u_+ \leq u_-,\ u_+(x)=u_-(x)}{\leq} u_-(\gamma_+^x(t)) - u_-(x)$$

$$\overset{u_- \prec L+c(L)}{\leq} A_{L+c(L)}(\gamma_+|_{[0,t]})$$

$\Rightarrow$ γ_+ is $(u_-, L, c(L))$-cal.

Analog γ_- is $(u_+, L, c(L))$-cal.

Define $\gamma^x(t) := \begin{cases} \gamma_+^x(t) & t \geq 0 \\ \gamma_-^x(t) & t \leq 0 \end{cases}$

γ^x is $(u_\pm, L, c(L))$-cal.

This means that $u_\pm$ are diff. at $x \in \mathcal{I}_{(u_-,u_+)}$ and $du_-(x) = du_+(x) = \frac{\partial L}{\partial v}(x,v)$ and we define

$$\tilde{\mathcal{I}}_{(u_-,u_+)} = \left\{ (x,v) \mid x \in \mathcal{I}_{(u_-,u_+)},\ d_x u_\pm = \frac{\partial L}{\partial v}(x,v) \right\}$$

Recall:

time free min for $c(L)$

Mañé set: $\tilde{\mathcal{N}} = \{(x,v) \mid \exists\, \gamma$ semistatic, $t \in \mathbb{R}$ s.t. $\gamma(t) = x, \dot\gamma(t) = v\}$

$A_{L+c(L)}(\gamma|_{[a,b]}) = \Phi(\gamma(a),\gamma(b)) \quad \forall a<b$

$=: \inf\limits_{T>0} \Phi(\gamma(a),\gamma(b);T)$

$=: \inf\limits_{T>0} \inf\limits_{\sigma:[0,T]\to M} A_{L+c(L)}(\sigma)$

Aubry set: $\tilde{\mathcal{A}} = \{(x,v) \mid \exists\, \gamma$ static, $t \in \mathbb{R}$ s.t. $\gamma(t) = x, \dot\gamma(t) = v\}$

semistatic & $\Phi(\gamma(a),\gamma(b)) = -\Phi(\gamma(b),\gamma(a))$

Thm: The Mañé set is given by
$$\tilde{\mathcal{N}} = \bigcup_{\substack{(u_-,u_+)\\ \text{conj.}}} \tilde{\mathcal{I}}_{(u_-,u_+)}$$

proof: „$\supseteq$" $(x,v) \in \tilde{\mathcal{I}}_{(u_-,u_+)}$ for some conj. pair (u_-,u_+)

$\overset{\text{thm 5.2.2*}}{\Longrightarrow} \exists\, \gamma^x: \mathbb{R} \to M$ that is $(u_\pm, L, c(L))$-cal and $\gamma(0) = x, \dot\gamma(0) = v$

$\overset{\text{prop 5.1.12}}{\Longrightarrow} \gamma^x$ global Tonelli min. of $L + c(L)$

$\Rightarrow \gamma^x$ is semistatic

Gabriele will show that!

We need three lemmas to prove the other inclusion.

Lemma 1: $\forall x \in M$, $u^x : M \to \mathbb{R}$; $u^x(y) := -\Phi(x,y)$ is dom. by $L + c(L)$

proof $u^x(\gamma(b)) - u^x(\gamma(a)) = \Phi(\gamma(a), x) - \Phi(\gamma(b), x)$

$$\leq \Phi(\gamma(a), \gamma(b))$$

$$\leq A_{L+c(L)}(\gamma|_{[a,b]})$$

Lemma 2: Let $\gamma : \mathbb{R} \to M$ semistatic, if $y \in \omega(\gamma(t), \dot\gamma(t))$ for some $t \in \mathbb{R}$, then γ is $(u^y, L, c(L))$-cal.

proof: Take $s \leq t \leq t_n$ s.t. $\gamma(t_n) \to y$

$\Rightarrow \Phi(\gamma(s), \gamma(t)) + \Phi(\gamma(t), \gamma(t_n)) \overset{\gamma \text{ semist.}}{=} \Phi(\gamma(s), \gamma(t_n))$

For $n \to \infty$: $\Phi(\gamma(s), \gamma(t)) = u^y(\gamma(t)) - u^y(\gamma(s))$

$\Rightarrow \gamma$ is $(u^y, L, c(L))$-cal.

Lemma 3: If $\gamma : \mathbb{R} \to M$ is $(u, L, c(L))$-cal. then

$$u_-(\gamma(s)) = u(\gamma(s)) = u_+(\gamma(s)) \quad \forall s \in \mathbb{R}$$

proof: As $u \prec L + c(L) \Rightarrow u(y) - u(x) \leq \Phi(x, y, t)$

$$\Rightarrow u(y) \leq T_t^- u(y) + c(L) \cdot t$$

Conversely, $u(\gamma(s)) - u(\gamma(s-t)) = A_{L+c(L)}(\gamma|_{[s-t,s]})$

$$\Rightarrow u(\gamma(s)) \geq T_t^- u(\gamma(s)) + c(L) t$$

In total we see $u(\gamma(s)) = T_t^- u(\gamma(s)) + c(L) t$

$$= \lim_{t \to \infty} T_t^- u(\gamma(s)) + c(L) t$$

$$= u_-(\gamma(s))$$

(The case u_+ works similar)

proof of „$\subseteq$":

$(x,v) \in \tilde{\mathcal{N}} \Rightarrow \exists\, \gamma^x : \mathbb{R} \to M$ semistatic

$\overset{\text{Lem 1.2}}{\Rightarrow}$ γ^x is $(u^y, L+c(L))$-cal. for any $y \in \omega(x,v)$

$\overset{\text{KAM, Lem3}}{\Rightarrow}$ $u_-^y(\gamma^x(s)) = u^y(\gamma^x(s)) = u_+^y(\gamma^x(s))$

$\Rightarrow$ $x \in \mathcal{I}(u_+, u_-)$

<u>Thm</u> The Aubry set is given by

$$\tilde{\mathcal{A}} = \bigcap_{\substack{(u_-,u_+) \\ \text{conj.}}} \tilde{\mathcal{I}}_{(u_-,u_+)}$$

proof: „$\supseteq$" $(x,v) \in \tilde{\mathcal{I}}_{(u_-,u_+)}$ $\forall\, (u_-,u_+)$ conj.

$\Rightarrow$ $(x,v) \in \tilde{\mathcal{I}}_{(u_-^y, u_+^y)}$ $\forall y \in M$. , i.e. $\exists\, \gamma^x : \mathbb{R} \to M$ semist & u^y-cal.

$$\Rightarrow \quad -\Phi(\gamma^x(b), \gamma^x(a)) \overset{\text{semistat}}{=} \Phi(\gamma^x(a), y) - \Phi(\gamma^x(b), y)$$

$$= u^y(\gamma^x(b)) - u^y(\gamma^x(a))$$

$$\overset{u^y\text{-cal.}}{=} A_{L+c(L)}(\gamma|_{[a,b]})$$

$$\overset{\gamma \text{ semist.}}{=} \Phi(\gamma^x(a), \gamma^x(b))$$

$\Rightarrow$ γ^x stat. $\Rightarrow$ $x \in \mathcal{A}$