To conclude today

At the first part of the talk we characterized

$$\mathcal{L}^{-1}(T) = \bigcup_{\mu \in \mathcal{M}(L)} \{\text{supp}\,\mu \mid \text{minimizes } A_{L_\eta}^{N}\}$$

We expect something similar in the general case in particular that this M^c is a graph in TM.
(Levin already proved last time and we will give more detail next monday)
I conclude enunciating the Mather graph Theorem.

Theorem The set $\tilde{M}_c$ is compact and invariant under the E-L flow.

$\pi|_{\tilde{M}_c}$ is an injective map from $\tilde{M}_c$ to M and its inverse $\pi^{-1}: M_c \longrightarrow \tilde{M}_c$ is Lipschitz.

β-Mather's function:

In the KAM-torus case we have seen that instead of minimizing over $\mathcal{M}(L)$ and the modified Lagrangian we can also add some more "constraints" i.e. fixing the rotation vector an obtain the same results.
We want to generalize this approach for general Tonelli system.

Thanks to the above "discussions" it is well defined the following map:

$$\begin{aligned} H^1(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ [\eta] &\longrightarrow \int_{TM} \tilde{\eta}_c \, d\mu \end{aligned}$$

where $\mu \in \mathcal{M}(L)$ and $\tilde{\eta}: TM \longrightarrow \mathbb{R}$, $(x,v) \rightarrow \eta_x(v)$.

By duality there exists $\rho(\mu) \in H^1(M,\mathbb{R})^* \simeq H_1(M,\mathbb{R})$ such that

$$\int_{TM} \tilde{\eta}\, d\mu = \langle [c], \rho(\mu) \rangle$$

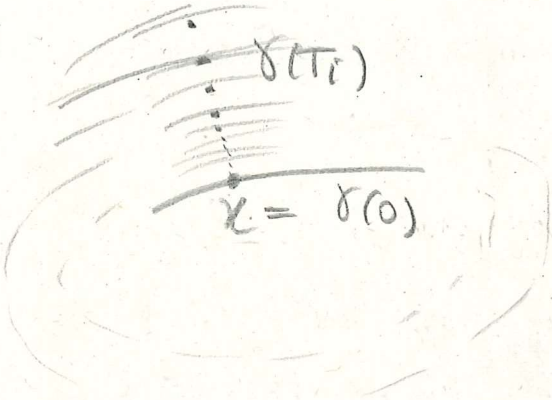
($\langle\, ,\rangle$ is the natural pairing between H^* and H_*)

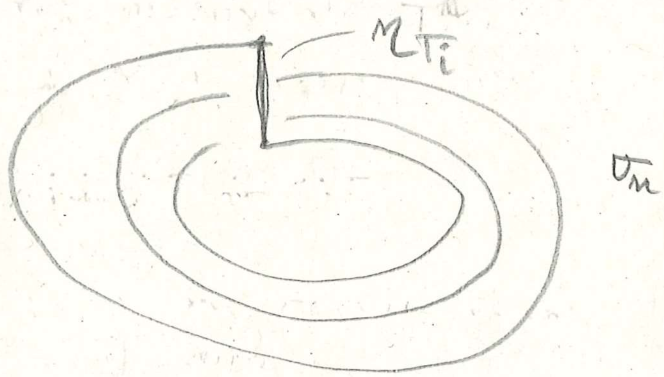
Def $\rho(\mu)$ is the rotation vector for $\mu \in \mathcal{M}(L)$.

"Geometrical interpretation"

Let $(x,v) \in TM$ and consider $\gamma(t) = \pi(\phi_L^t(x,v))$. Suppose that μ is ergodic then $\gamma(t)$ will return infinitely many times close to $\gamma(0) = x$.

We can consider a sequence of times $T_n \longrightarrow +\infty$ with $d(\gamma(T_n), \gamma(0)) \longrightarrow 0$ for $n \longrightarrow +\infty$.





σ_n is the closed loop obtained connecting $\gamma(T_i)$ and $\gamma(0)$ with the unique geodesic η_{T_i}

Now take the homology class of σ_n. It holds

$$\lim_{n\to+\infty} \frac{[\sigma_n]}{T_n} = \rho(\mu)$$

and it does not depend from the sequence $\{T_n\}$.

So in some sense the rotation vector measure the average of winds of a generic orbit on TM.

If μ is not ergodic this average has to be thought as the average on each ergodic component
($\mu = \sum_i \mu_i$ for some form)

Facts and Remarks

(1) For a given Tonelli Lagrangian L on M for every "rotation vector" $p \in H_1(M, \mathbb{R})$ there exist $\mu \in \mathbb{M}(L)$ such that $\rho(\mu) = p$. That is:

$$\mathbb{M}(L) \longrightarrow H_1(M,\mathbb{R})$$
$$\mu \longrightarrow \rho(\mu) : \{[c] \longrightarrow \int_{TM} \hat{c} \, d\mu\}$$

is continuous, and surjective. Moreover $\forall h \in H_1(M, \mathbb{R})$

$p^{-1}(h)$ is compact on $\mathbb{M}(L)$ (this is due to the continuity of p and the superlinearity of L).

(2) Since A_L is lower-semicontinuous $\forall h \in H_1(M,\mathbb{R})$ $\exists \mu \in \mathbb{M}(L)$ such that $p(\mu) = h$ and

$$A_L(\mu) = \int_{TM} L(x,v)\, d\mu = \inf_{\tilde{\mu} \in p^{-1}(h)} \int_{TM} L(x,v)\, d\tilde{\mu}$$

We can define the following function

$$\beta : H_1(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$h \longrightarrow \min_{\substack{\mu \in \mathbb{M}(L) \\ p(\mu) = h}} \int_{TM} L(x,v)\, d\mu.$$

β is called the β-Mather's function. (β is <u>Convex</u>.)

Definition A measure $\mu \in \mathbb{M}(L)$ such that $A_L(\mu) = \beta(p(\mu))$ is called action minimizing measure with homology class $p(\mu)$.

Denoting by $\mathbb{M}^h(L) = \mathbb{M}^h = \{\mu \in \mathbb{M}(L) \mid \beta(h) = A_L(\mu) \text{ and } p(\mu) = h\}$ the Mather set associated to h is

$$\tilde{\mathcal{M}}^h = \bigcup_{\mu \in \mathbb{M}^h(L)} \{\operatorname{supp} \mu\}$$

By $\mathcal{M}^h = \pi(\tilde{\mathcal{M}}^h)$ we denote the projected Mather set of homology class h.

Remark $\tilde{\mathcal{M}}^h$ is a non empty closed invariant set and is a Lipschitz graph over $\mathcal{M}^h$ i.e. $\pi^{-1}|_{\mathcal{M}^h} : \mathcal{M}^h \longrightarrow \tilde{\mathcal{M}}^h$ the inverse of π is Lipschitz.

Connection between α and β Mather's Functions

We have studied two formulations for the minimality of an invariant probability measure.

- <u>First</u> We fixed a cohomology class $[c] \in H^1(M,\mathbb{R})$ and we considered $\mu \in \mathbb{M}(L)$ such that μ minimized $A_{L_{\eta_c}}$ where L_{η_c} is the modified Lagrangian

$L_{\eta_c} = L - \tilde{\eta}_c$. So $A_{L_{\eta_c}}(\mu) = -\alpha(c)$.

· <u>Second</u> We fixed a rotation vector $h \in H_1(M, \mathbb{R})$ and we focused on minimizing prob. measure in the class of $\rho^{-1}(h)$ i.e all inv. prob. measure with hom. class h. So a such μ satisfies

$\beta(h) = A_L(\mu)$.

The goal now is to show that both the approachs lead to the same set of minimizing inv. prob measure i.e

$$\bigcup_{h \in H_1(M,\mathbb{R})} \mathbb{M}^h(L) = \bigcup_{c \in H^1(M,\mathbb{R})} \mathbb{M}_c(L).$$

In particular we will see that one method is the "dual" of the other.

Recall that

$$\alpha : H^1(M,\mathbb{R}) \longrightarrow \mathbb{R}$$

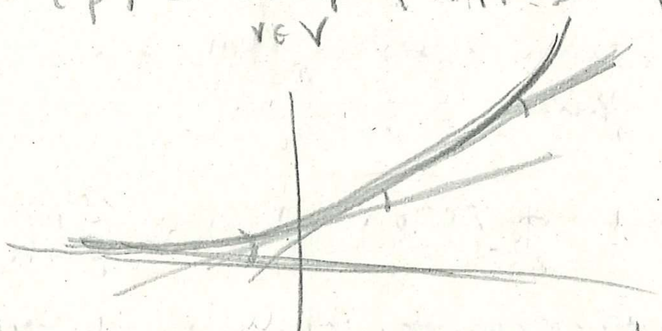
$$\text{and } \beta : \underset{\substack{\text{is} \\ H^1(M,\mathbb{R})^*}}{H_1(M,\mathbb{R})} \longrightarrow \mathbb{R}$$

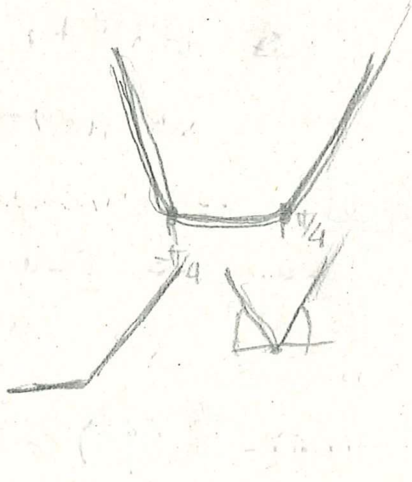
are convex function. (Simply by linearity of the integral)

In general when we deal with convex function is natural to think about the Fenchel dual. So given $f : V \longrightarrow \mathbb{R}$ convex we can consider

$f^* : V^* \longrightarrow \mathbb{R}$ defined as

$$f^*(p) = \sup_{v \in V} \left(\langle p, v \rangle - f(v) \right).$$





is like to give new coordinates by the "differential"

Proposition $\alpha^* = \beta$ and $\beta^* = \alpha$.

proof Let $c \in H^1(M,\mathbb{R})$,

$$\beta^*(c) = \max_{h \in H_1(M,\mathbb{R})} \{ \langle c,h\rangle - \beta(h) \} =$$

$$= - \min_{h \in H_1(M,\mathbb{R})} \{ \beta(h) - \langle c,h \rangle \} =$$

$$= - \min_{H_1(M,\mathbb{R})} \{ \min_{\substack{\mu \in \mathfrak{M}(L) \\ \rho(\mu)=h}} A_L(\mu) - \langle c,h\rangle \}$$

$$= - \min_{h \in H_1(M,\mathbb{R})} \{ \min_{\substack{\mu \in \mathfrak{M}(L) \\ \rho(\mu)=h}} A_L(\mu) - \langle c,\rho(\mu)\rangle \} =$$

$$= - \min A_{L_{\eta_c}}$$

$$= \alpha(c).$$

Since α and β are defined in the entire respective spaces and $\alpha = \beta^*$ we get $\alpha^* = \beta^{**} = \beta$. ∎

Moreover since α and β are finite everywhere α and β are superlinear.

Remark Easly we deduce the Fenchel inequalities

$\forall c \in H^1(M,\mathbb{R})$ and $h \in H_1(M,\mathbb{R})$

$$\langle c, h\rangle \leq \alpha(c) + \beta(h)$$

Recall In the integrable case α and β are smooth convex functions and they coincide respectively with the effective Hamiltonian and the effective Lagrangian (in effective coordinates)

$$H(x,p) = \mathcal{H}(p) = \alpha(c) \quad \text{and} \quad L(x,h) = \mathcal{L}(h) = \beta(h)$$

We gave a characterization for KAM-tori through the support of minimizing invariant prob. measure. That is

$$\tilde{\mathcal{M}}_c = \alpha^{-1}(\mathcal{T}_c) = \tilde{\mathcal{T}}^h = \tilde{\mathcal{M}}^h$$

Now $h \in H_1(\mathbb{T}^d, \mathbb{R})$ and $c \in H^1(\mathbb{T}^d, \mathbb{R})$ are such that

$$h = \nabla\alpha(c) \quad \text{and} \quad c = \nabla\beta(h)$$

In general α and β are just convex also continuous, so to generalize the remark above we have to introduce the notion of sub-differential.

Given $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ convex a subdifferential of f at the point $x_0 \in \mathbb{R}^n$ is a linear form $c \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$ such that

$$f(x) - f(x_0) \geq \langle c, x - x_0 \rangle \quad \forall x \in \mathbb{R}^n.$$

We denote by $\partial f(x_0) = \{ c \in (\mathbb{R}^n)^* \mid c \text{ is a sub-dif. for } f \text{ at } x_0 \}$

(1) If f is convex $\forall x_0 \in \mathbb{R}^n$ we have $\partial f(x_0) \neq \emptyset$

(2) If f is differentiable then at $x_0 \in \mathbb{R}^n$ then

$$\partial f(x_0) = \{ df_{x_0} \}.$$

Proposition The Fenchel inequality is an equality if and only if $\langle c, h \rangle = \alpha(c) + \beta(h)$ iff $c \in \partial\beta(h)$ and $h \in \partial\alpha(c)$. (This generalize the integrable case).

(2) In particular $c \in \partial\beta(h)$ iff $h \in \partial\alpha(c)$.

proof

(1) $\langle c, h \rangle = \alpha(c) + \beta(h) \iff c \in \partial\beta(h)$

$\Rightarrow$: we have $\langle c, h \rangle = \alpha(c) + \beta(h)$ then

From Fenchel inequality we get

$$\beta(h') \geq \langle c, h' \rangle - \alpha(c)$$

So

$$\beta(h') - \beta(h) \geq \langle c, h' \rangle - \alpha(c) - \beta(h) =$$

$$\geq \langle c, h' \rangle - (\alpha(c) + \beta(h)) =$$

$$\geq \langle c, h' \rangle - \langle c, h \rangle = \langle c, h' - h \rangle$$

$\Leftarrow$: From F I again $\alpha(c)+\beta(h') \geq \langle c,h'\rangle$ $\forall h' \in H_1(M,\mathbb{R})$. We need to show the reverse inequality. $(\beta(h')-\beta(h) \geq \langle c, h-h'\rangle)$

$$\alpha(c)+\beta(h) \leq \alpha(c)+\beta(h')-\langle c,h'-h\rangle$$

$$\leq \alpha(c)+\beta(h')-\langle c,h'\rangle+\langle c,h\rangle$$

$$\leq (\alpha(c)+\beta(h')-\langle c,h'\rangle)+\langle c,h\rangle$$

$$\leq \min_{h'\in H_1(M,\mathbb{R})} (\alpha(c)+\beta(h')-\langle c,h'\rangle)+\langle c,h\rangle$$

By Fenchel Duality the minimum over $H_1(M,\mathbb{R})$ is zero so

$$\leq \langle c,h\rangle$$

(2) With the same scheme we can prove that

$$\alpha(c)+\beta(h)=\langle c,h\rangle \text{ iff } h\in\partial\alpha(c)$$

then $c\in\partial\beta(h)$ iff $h\in\partial\alpha(c)$ follows. ∎

We can now show the duality of the two methods.

Proposition Let $\mu\in\mathbb{M}(L)$ an invariant prob. measure. $A_L(\mu)=\beta(\rho(\mu))$ if and only if $\exists c\in H^1(M,\mathbb{R})$ such that $A_{L_{\eta_c}}(\mu)=-\alpha(c)$ (i.e. μ minimizes $A_{L_{\eta_c}}$). (Moreover $c\in\partial\beta(\rho(\mu))$ or $\langle c,\rho(\mu)\rangle=\alpha(c)+\beta(\rho(\mu))$)

proof Let $\mu\in\mathbb{M}(L)$ such that $A_L(\mu)=\beta(\rho(\mu))$ and
($\Rightarrow$) let $c\in\partial\beta(\rho(\mu))\neq\emptyset$ (because β convex).

$$\alpha(c)=\langle c,\rho(\mu)\rangle-\beta(\rho(\mu))$$

$$=\langle c,\rho(\mu)\rangle-A_L(\mu)$$

$$=-A_{L_{\eta_c}}(\mu).$$

($\Leftarrow$) Now $A_{L_{\eta_c}}(\mu)=\min_{\tilde\mu\in\mathbb{M}(L)} A_{L_{\eta_c}}(\tilde\mu)=-\alpha(c)$. In this case we're given a cohomology class $[c]\in H^1(M,\mathbb{R})$.

So $\alpha(c) = -A_{L-\eta_c}(\mu) = +\langle c, \rho(\mu)\rangle - A_L(\mu)$

by Fenchel inequality $\langle c, \rho(\mu)\rangle \leq \alpha(c) + \beta(\rho(\mu))$

Using the definition of β : $\beta(\rho(\mu)) \leq A_L(\mu)$

$$\langle c, \rho(\mu)\rangle = \alpha(c) + A_L(\mu) \geq \alpha(c) + \beta(\rho(\mu))$$

~~Then~~ $\langle c, \rho(\mu)\rangle = \alpha(c) + \beta(\rho(\mu))$ ~~equal to say~~

$$\beta(\rho(\mu)) = A_L(\mu).$$

■

Corollary (1) Both minimizing procedures lead to the same invariant-measure set:

$$\bigcup_{c \in H^1(M,\mathbb{R})} \mathbb{M}_c(L) = \bigcup_{h \in H_1(M,\mathbb{R})} \mathbb{M}^h(L)$$

(2) $c \in \partial\beta(h) \Longleftrightarrow h \in \partial\alpha(c) \Longleftrightarrow \widetilde{\mathcal{M}}^h \subseteq \widetilde{\mathcal{M}}_c$

and

$$\widetilde{\mathcal{M}}_c = \bigcup_{h \in \partial\alpha(c)} \widetilde{\mathcal{M}}^h .$$

Point (2) explains the difference with the integrable case. In fact we have to put together all $\widetilde{\mathcal{M}}^h$ for each h belonging to the sub-differential of α in c.

Remark α is convex and superlinear so it has a minimum c_0, so $\alpha(c_0) = \min_{H^1(M,\mathbb{R})} \alpha$ and this is exactly the Mañé's strict critical value. (We will see that later)

Moreover $0 \in \partial\alpha(c_0)$ and $\beta(0) = -\alpha(c_0)$. Therefore the inv. min. measure with zero homology belong to the least possible energy level containing $\widetilde{\mathcal{M}}_{c_0}$.

$$\widetilde{\mathcal{M}}^0 \subseteq \mathcal{M}_{c_0} \quad \left(\begin{array}{c}\text{the inclusion is strict}\\ \text{if } \sharp\,\partial\alpha(c_0) > 1\end{array}\right).$$

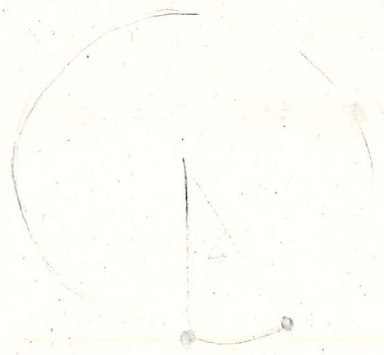
EXAMPLE : THE SIMPLE PENDULUM

$M = S^1$

$TM = S^1 \times \mathbb{R}$

$T^*M = S^1 \times \mathbb{R}$

$H^1(S^1,\mathbb{R}) \simeq H_1(S^1,\mathbb{R}) \simeq \mathbb{R}$



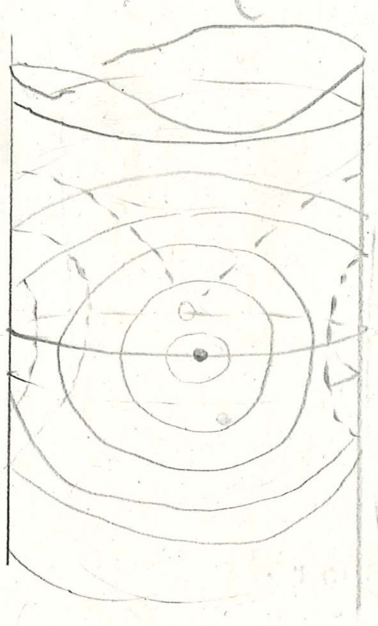
$$L : S^1 \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(\theta, v) \longrightarrow \tfrac{1}{2}v^2 + (1 - \cos(2\pi x))$$

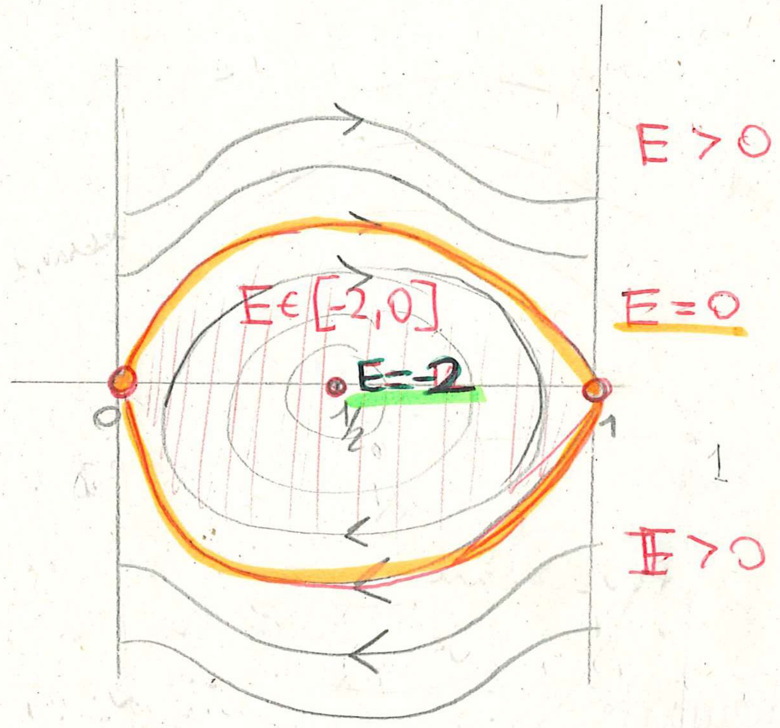
The Euler-Lagrange equation is

$$\ddot{x} = 2\pi \sin(2\pi x)$$

That is in TS^1

$$\begin{cases} \dot{x} = v \\ \dot{v} = 2\pi \sin(2\pi x) \end{cases}$$





(1) $(0,0)$ and $(\tfrac{1}{2},0)$ are fixed points of the system the first stable the second unstable.

Therefore the Dirac measures concentrated on those two points are invariant prob-measure.

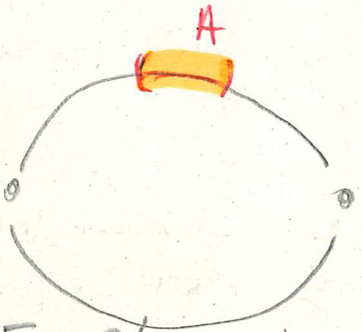
$$\int_{TM} \tilde{\eta}\, d\delta_{(1/2,0)} = \int_{TM} \tilde{\eta}\, d\delta_{(0,0)} = \tilde{\eta}_{1/2}(0) = \tilde{\eta}_0(0) = 0$$

$$\Longrightarrow \quad p(\delta_{(0,0)}) = p(\delta_{(1/2,0)}) = 0$$

$\delta(0,0)$ belongs to the energy level $E = 0$ and $\delta(1/2, 0)$ to the energy level $E = -2$.

In these two level there are not other invariant probability measures.

$E = 0$



If μ is an other invariant prob. measure

We pick $A \subset \{E=0\}$ such that $\mu(A) > 0$

$\exists \{t_i > 0\}_{i \in \mathbb{N}}$ such that $\Phi_L^{t_j}(A) \cap \Phi_L^{t_i}(A) = \emptyset \quad \forall i \neq j$

So $\mu\left(\bigcup_i \Phi^{t_i} A\right) = \sum_i \mu(A) = +\infty$. $\lightning$

In the level $E = -2$ is trivial.

(2) If $E > 0$ then the energy level $\{E(x,v) = E\}$ consists in two periodic non-contractible orbit given by

$$P_E^{\pm} = \left\{ (x,v) \in S^1 \times \mathbb{R} \;\middle|\; v = \pm\sqrt{2(1+E) - 2\cos(2\pi x)} \right\}$$

The prob. measure uniformly distributed along $P_E^{\pm}$ are invariant. Given the period of $P_E^{\pm}$

$$\underset{E>0}{T(E)} = \int_0^1 \frac{dx}{\sqrt{2(1+E) - \cos 2\pi x}}$$

we observe that

$$\rho(\mu_E^{\pm}) = \frac{\pm 1}{T(E)}$$

(Remind $\rho(\mu) = \lim_{m\to+\infty} \frac{[\sigma_m]}{T_m}$ but in this case $[\sigma_m] = \pm 1$ and $T_m = T(E)$.)

→ If the energy increases the speed increases and the period is smaller, vice versa if $T(E) \longrightarrow +\infty$ ~~then~~ $E \longrightarrow 0$ and we deduce

$$\rho(\mu_E^{\pm}) \longrightarrow 0 \quad \text{for } E \longrightarrow 0.$$

(3) $-2 < E < 0$ then the energy consists of one contractible orbit

$$P_E = \{(x,v) \in [x_E, 1-x_E] \times \mathbb{R} \mid v^2 = 2(1+E) - 2\cos(2\pi x)\}$$

$$x_E = \frac{1}{2\pi} \arccos(1+E)$$

again the prob. measure uniformly distributed are invariant and since P_E is contractible

$$\rho(\mu_E) = 0$$

Every other invariant prob. measure is obtained as convex combination of the above measures.

Remark (1) $-2 < E < 0$ the support of μ_E is not a graph over S^1 so μ_E can not minimize the action. This also implies that $\alpha(c) \geq 0$ since $M_c \subset \{E = \alpha(c)\}$.

Remark (2) $\alpha(c) = \alpha(-c)$ and the minimizing problem is ~~symmetric~~ respect the reflection

$$T: S^1 \times \mathbb{R} \longrightarrow S^1 \times \mathbb{R}$$
$$(x, v) \longrightarrow (x, -v)$$

This comes from the fact that $L(x,v) = L(x,-v)$

Then $T^* \mathbb{M}(L) = \mathbb{M}(L)$ and

$$\int (L - c\cdot v)\, d\mu = \int (L + c\cdot v)\, dT^*\mu$$

Then

$$\alpha(c) = -\min_{\mathbb{M}(L)} \int (L - c\cdot v)\, d\mu = -\min_{\mathbb{M}(L)} \int (L + cv)\, d\Pi^*_{\#}\mu = \alpha(-c)$$

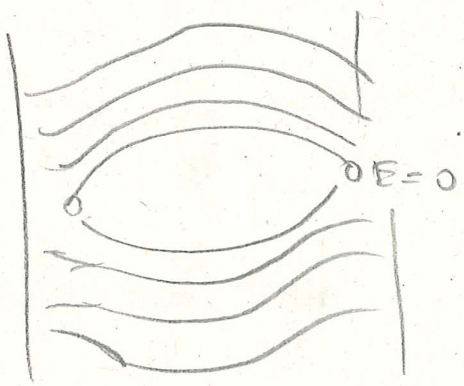
Then since α is convex $\min_{\mathbb{R}} \alpha = \alpha(0)$

- Let start by zero cohomology class. So we minimise the action respect to a non-modified Lagrangian. Observe that $L(x,v) \geq 0$ and so $A_L(\mu) \geq 0$.

In particular $A_L(\delta_{(0,0)}) = 0$ so $\delta_{(0,0)}$ minimises. Since at the zero energy level it is the only inv. prob. measure we conclude

$$\widetilde{\mathcal{M}}_{0} = \{(0,0)\}$$

- Since α is superlinear and convex for every level energy $\{E = k\}$ $k > 0$ $\exists\, c_E \mid \emptyset \neq \widetilde{\mathcal{M}}_{c_E} \subset \{E = k\}$.

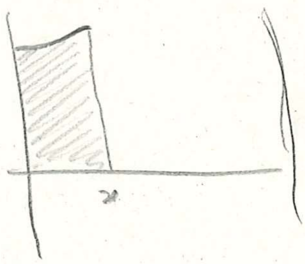


Let $E > 0$, $\mathcal{D}^+_E$ is the graph of the closed one-form

$$\eta^+_E = \sqrt{2(1+E) - \cos 2\pi x}\; dx$$

with cohomology class

$$c^+(E) = [\eta^+_E] = \int_0^1 \sqrt{2[(1+E) - \cos 2\pi x]}\; dx$$



Now $c^+(E) \nearrow$ when E increases and

$$c^+(E) \longrightarrow \int_0^1 \sqrt{2(1 - \cos 2\pi x)}\; dx = \frac{4}{\pi} \text{ when } E \to 0.$$

So we define:

$$c^+ : (0, +\infty) \longrightarrow (\tfrac{4}{\pi}, +\infty)$$
$$E \longrightarrow c^+(E)$$

Claim: μ_E^{+} (the uniform distributed inv prob measure on P_E^+) minimizes the $L c^+(E)$-action.

$$\int_{TM} L_{\eta_E^+}(x,v)\, d\mu_E^+ = \int_{TM} \left(L(x,v) - \eta_E^+(x)\cdot v\right) d\mu_E^+$$

$$\overset{\text{Fenchel equality}}{=} -\int_{TM} H(x, \eta_E(x))\, d\mu_E^+ = -E \quad (\star)$$

If $\tilde{\mu}$ is an other inv-prob-measure by Fenchel inequality (in this case)

$$\int L_{\eta_E^+}(x,v)\, d\tilde{\mu} \geq -E$$

Since the support projects on S^1 as consequence of the graph Mather's theorem

$$\tilde{\mathcal{M}}_{c^+(E)} = P_E^+ = \left\{ (x,v) \in S^1 \times \mathbb{R} \mid v = \sqrt{2(1+E-\cos 2\pi x)} \right\}$$

At the same time the rotation vector $\rho(\mu_E^+) = \frac{1}{T(E)}$

then

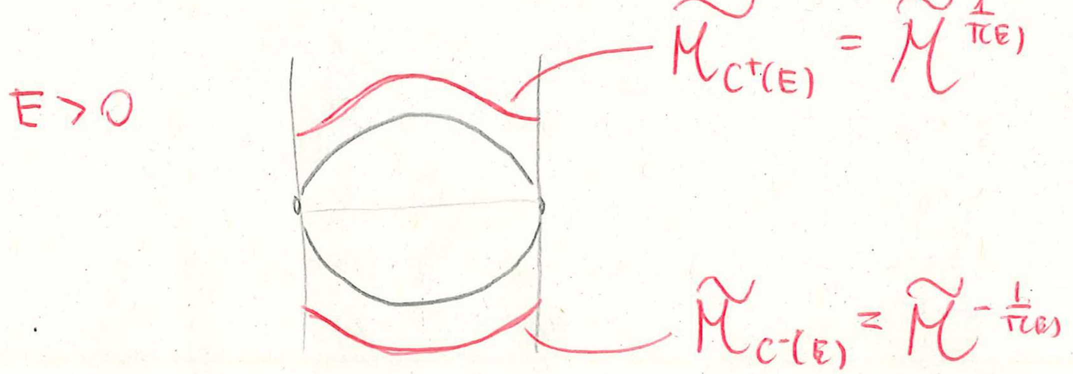
$$\tilde{\mathcal{M}}^{\frac{1}{T(E)}} = \tilde{\mathcal{M}}_{c^+(E)} = P_E^+$$

The same process could be apply to the negative real line. We get

$$c^- : (-0, +\infty) \longrightarrow (-\infty, -\tfrac{4}{\pi})$$
$$E \longrightarrow c^-(E) = -c^+(E)$$

Then

$$\tilde{\mathcal{M}}_{c^-(E)} = \tilde{\mathcal{M}}^{-\frac{1}{T(E)}} = P_E^- = \left\{ (x,v) \mid v = -\sqrt{2(1+E-\cos 2\pi x)} \right\}$$



Since

$$p(\mu_E^\pm) = \pm\frac{1}{T(E)} \longrightarrow \pm\infty \qquad \text{for } E \longrightarrow +\infty$$

$$p(\mu_E^\pm) = \pm\frac{1}{T(E)} \longrightarrow 0 \qquad \text{for } E \longrightarrow 0^+$$

this conclude the Analysis from the homological point of view.

We need to study what happens for cohomology classes in $[-\frac{4}{\pi}, \frac{4}{\pi}]$.

$$\alpha(c) = -\min_{\mu \in M(L)} A_{L\eta_c}(\mu) \quad \sim$$

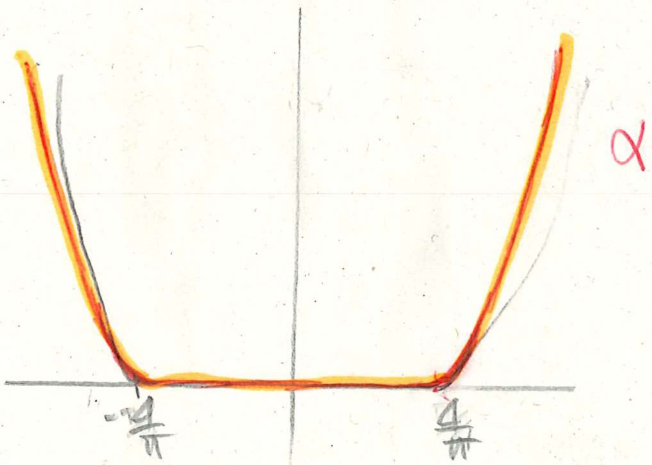
$$\alpha(c^\pm(E)) = E \quad \text{since } \alpha \text{ is continuous}$$

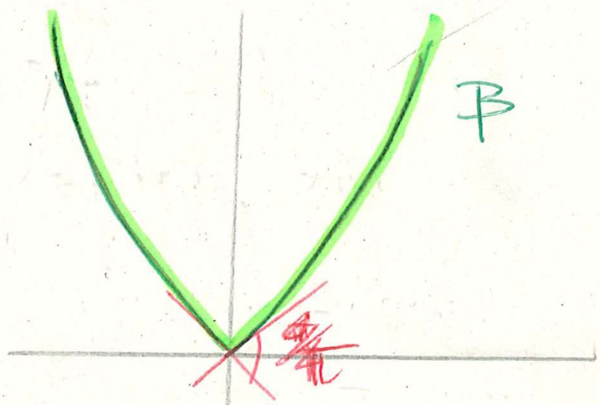
$$\lim_{E\to 0} \alpha(c^\pm(E)) = \alpha\left(\pm\frac{4}{\pi}\right) = 0$$

But α is convex and it has a minimum in 0.

So $\alpha|_{[-\frac{4}{\pi}, \frac{4}{\pi}]} = 0$. Thanks to the above argumentation

$$\tilde{M}_c = \{\operatorname{supp} \delta_{(0,0)}\} = \{(0,0)\} \quad \forall\, c \in [-\tfrac{\pi}{4}, \tfrac{\pi}{4}]$$





To conclude

→

We defined

$$C^+ : (0,+\infty) \longrightarrow (\tfrac{4}{\pi}, +\infty)$$
$$E \longrightarrow C^+(E)$$

This is ↗ then invertible. We denote by E the inverse. So E(c) is the level of the energy of the periodic orbit with cohomology class c.

$$\tilde{\mathcal{M}}_c = \begin{cases} (0,0) & c \in [-\tfrac{4}{\pi}, \tfrac{4}{\pi}] \\ P^+_{E(c)} & c > \tfrac{4}{\pi} \\ P^-_{E(-c)} & c < -\tfrac{4}{\pi} \end{cases}$$

Then

$$T : (0,+\infty) \longrightarrow (0,+\infty)$$
$$E \longrightarrow T(E)$$

is ↘ and so invertible. We denote by $\tilde{E}$ the inverse wich pick a period T and gives the energy level of the periodic orbit with period T.

$$\tilde{E} : (0,+\infty) \longrightarrow (0,+\infty)$$
$$T \longrightarrow \tilde{E}(T)$$

$$\tilde{M}^h = \begin{cases} (0,0) & h = 0 \\ P^+_{\tilde{E}(1/h)} & h > 0 \\ P^-_{\tilde{E}(-1/h)} & h < 0 \end{cases}$$