ACTION - MINIMIZING INVARIANT MEASURE AND MATHER'S SET.

Today we start to develop the Mather's theory for Tonelli system. Throught invariant measure We define an important invariant subset called the Mather's set and next monday we will show how Mather's theory is connected with the Aubry and the Mañé Point of view. In particular we will see at the end that the Mather set is included in the Aubry one.

In this first part we approach the study of invariant probability measure on a KAM-torus. For the moment our setting is the following.

$M = \mathbb{T}^d$ (d-dimensional torus) $T\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$ the tangent bundle and $T^*\mathbb{T}^d = \mathbb{T}^d \times (\mathbb{R}^d)^* \simeq \mathbb{T}^d \times \mathbb{R}^d$ the cotangent bundle. With abuse of notation we denote with $\pi : T\mathbb{T}^d \longrightarrow \mathbb{T}^d$ and $\pi : T^*\mathbb{T}^d \longrightarrow \mathbb{T}^d$ the respective projections.

Now let $L : \mathbb{T}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}$ a Tonelli Lagrangian and $H : \mathbb{T}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}$ the associated Tonelli Hamiltonian.

<u>Definition</u>: A KAM-torus with rotation vector $\rho \in \mathbb{R}^d$ is a subset $\mathcal{T} \subseteq T^*\mathbb{T}^d$ which satisfies:

(1) $\mathcal{T}$ is a C^1-Lagrangian graph i.e

$$\mathcal{T} = \{ (x, d_x u + c) \mid x \in \mathbb{T}^d \} \text{ and}$$

$u : \mathbb{T}^d \longrightarrow \mathbb{R}$ is a (at least C^2) function on the torus.

(2) $\mathcal{T}$ is Φ_H-invariant i.e $\Phi_H^t(\mathcal{T}) \subseteq \mathcal{T} \ \forall t$

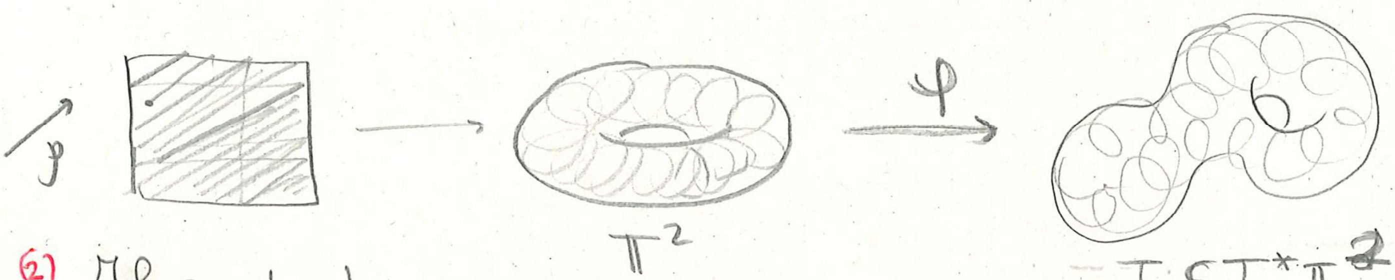
(3) $\Phi_H \big|_{\mathcal{T}}$ is conjugated to a rotational motion on $\mathbb{T}^d$ i.e $\exists \varphi : \mathbb{T}^d \longrightarrow \mathcal{T}$ diffeomorphism such that

$$\Phi_H^t \big|_{\mathcal{T}} = \varphi \circ R_\rho^t \circ \varphi^{-1} \text{ where}$$

$$R_\rho^t : \mathbb{T}^d \longrightarrow \mathbb{T}^d$$
$$x \longmapsto x + t\rho \pmod{\mathbb{Z}^d}.$$

Proprieties of $\mathcal{T}$

(1) If p is rationally independent i.e. $p\cdot k \neq 0$ $\forall k \in \mathbb{Z}^d \setminus \{0\}$ then each orbit on $\mathcal{T}$ is uniformly dense



$\mathcal{T} \subset T^*\mathbb{T}^2$

(2) If p admits some resonances that is $\exists k \in \mathbb{Z}^d \setminus \{0\} \mid p\cdot k$ then the set of resonances form a module over $\mathbb{Z}$ and $\mathcal{T}$ is foliated by a family of $(d-r)$-Tori (r is the rank of the modulus)

(3) Since $\mathcal{T}$ is ~~an~~ Φ_H-invariant proper subset of $T^*\mathbb{T}^d$ then is contained in a level of the energy E_c. Together with the fact that $\mathcal{T}$ is a Lagrangian graph we have that $u: \mathbb{T}^d \longrightarrow \mathbb{R}$ is a classical solution of the Hamilton-Jacobi equation

$$H(x, d_x u + c) = E_c$$

(This is a non-linear first order PDE and in the next talks we will see that it does not always admit a classical solution).

Invariant probability measures supported on $\mathcal{T}$.

Let μ^* be an ergodic Φ_H-invariant probability measure supported on $\mathcal{T}$ ($\operatorname{supp} \mu^* \subset \mathcal{T}$). With $\mu = \mathcal{L}^*\mu^*$ we denote the ergodic ϕ_L-invariant probability measure supported on $\mathcal{L}^{-1}(\mathcal{T})$ obtained by pulling back μ^* through Legendre's transform.

The goal is to understand the propriety of these kind of measure and give a characterisation of $\mathcal{T}$ thanks to these objects.

$\longrightarrow$

1 ρ is the μ-average of v on the generators of $H^1(\mathbb{T}^d)$ i.e along the rotational directions.

$dx_i \in H^1(\mathbb{T}^d)$, $i = 1 \dots d$, X_L on $T\mathbb{T}^d$ is on the form

$$X_L = \sum_{i=1}^{d} v_i \frac{\partial}{\partial x_i} + \left(\frac{\partial^2 L}{\partial v^2}\right)^{-1}\left(\frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial x \partial v}\cdot(v)\right)_i \frac{\partial}{\partial v_i}$$

$d\pi(X_L) = \sum_i v_i \frac{\partial}{\partial x_i}$ and $dx_i(d\pi(X_L)) = v_i$

We want to prove:

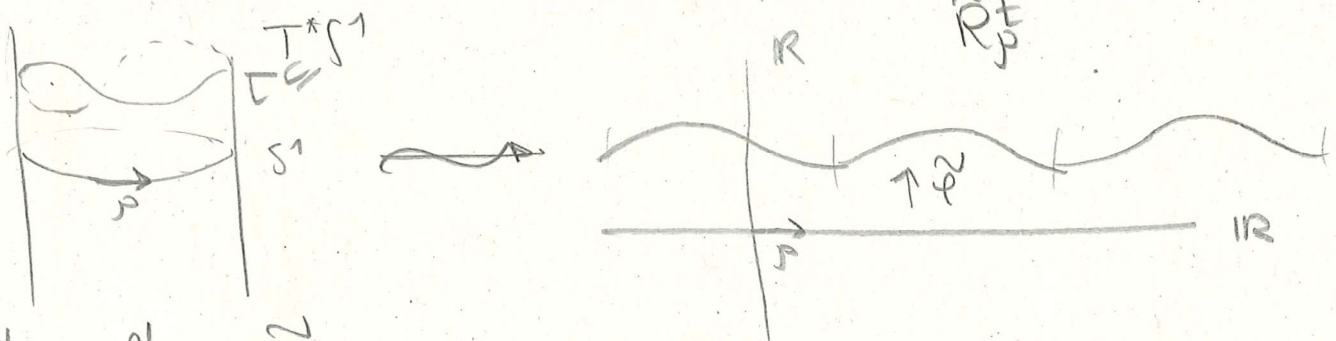
$$\int_{T\mathbb{T}^d} \begin{pmatrix} dx_1(d\pi(X_L)) \\ dx_n(d\pi(X_L)) \end{pmatrix} d\mu = \int_{T\mathbb{T}^d} \underline{v} \, d\mu$$

Let consider the universal cover $\tilde{\mathbb{T}}^d = \mathbb{R}^d$. Denote by

- $\tilde{\mathcal{T}} = \{(x, c + d\tilde{u}(x)) \mid x \in \mathbb{R}^d\}$ and $\tilde{u}$ is the $\mathbb{Z}^d$-periodic extension of u.
- $\tilde{\Phi}_H^t$ the lift of the Hamiltonian flow on $\mathbb{R}^d \times \mathbb{R}^d$
- $\tilde{\varphi}$ the lift of φ, in particular

$\tilde{\varphi}: \mathbb{R}^d \longrightarrow \tilde{\mathcal{T}}$ and again with $\tilde{R}_\rho^t(q) = q + t\rho$.

$$\begin{array}{ccc} \tilde{\mathcal{T}} & \xrightarrow{\tilde{\Phi}_H} & \tilde{\mathcal{T}} \\ \uparrow \tilde{\varphi} & & \uparrow \tilde{\varphi} \\ \mathbb{R}^d & \xrightarrow[\tilde{R}_\rho^t]{} & \mathbb{R}^d \end{array}$$



$\tilde{\varphi}: \mathbb{R}^d \longrightarrow \tilde{\mathcal{T}}$ is on the form

$$\tilde{\varphi}(q) = \left(\tilde{\varphi}_q(q),\ c + d\tilde{u}\big|_{(\tilde{\varphi}_q(q))}\right)$$

and $\tilde{\varphi}_q$ is such that $\tilde{\varphi}_q(q+k) = \tilde{\varphi}_q(q) + k \quad \forall k \in \mathbb{Z}^d$

Pick a point $(x_0, c+du(x_0)) \in \operatorname{Supp} \mu^*$ and we consider its projected orbit $\pi(\phi_H^t(x_0, c+du(x_0))) = \gamma_{x_0}(t)$

Since μ is ergodic

$$\int_{\mathbb{T}^d} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} d\mu = \lim_{m\to+\infty} \frac{1}{m} \int_0^m \dot{\tilde{\gamma}}_{x_0}(m)\, dt =$$

$$= \lim_{m\to+\infty} \frac{1}{m} \left(\tilde{\gamma}_{x_0}(m) - \tilde{\gamma}_{x_0}(0) \right) =$$

$$= \lim_{m\to+\infty} \frac{\tilde{\gamma}_{x_0}(m)}{m} =$$

$$= \lim_{m\to+\infty} \tilde{\varphi}_q \left(\tilde{\varphi}_q^{-1} (\tilde{\gamma}_{x_0}(0)) + mp \right) =$$

$$= \lim_{m\to+\infty} \frac{\tilde{\varphi}_q \left(\tilde{\varphi}_q^{-1} (\tilde{\gamma}_{x_0}(0)) + \{mp\} \right) + [mp]}{m} = p$$

$\{\ \}$ the rational part and $[\ ]$ the integer part.

② (Already seen in Levin's talk) If $f \in C^1(\mathbb{T}^d)$ then

$$\int_{\mathbb{T}^d} df_x(v)\, d\mu = 0$$

③ We called E_c the energy level of $\mathcal{T}$. Now we want to see that in somehow this quantity is related with the average of L respect μ.

In fact

$$\int_{\mathbb{T}^d} L(x,v)\, d\mu = \int_{\mathbb{T}^d} [L(x,v) - (c+du_x)(v)]\, d\mu + \int_{\mathbb{T}^d} (c+du_x)(v)\, d\mu =$$

$$= -E_c + c\int v\, d\mu = -E_c + c\cdot p.$$

thanks to Fenchel equality and the ϕ_L-invariance of μ

~~Proposition~~

Denote by $L_c = L - c\cdot v$. This new Lagrangian is still Tonelli and has the same flow of L.

Let $\mathbb{M}(L) = \{\mu \mid \mu \text{ is } \phi_L\text{-inv. prob. measure on } \mathbb{T}^d\}$

and $\mathbb{M}^\rho(L) = \{\mu \in \mathbb{M}(L) \mid \int_{\mathbb{T}^d} v \, d\mu = \rho$ i.e. μ has ρ as rotation vector $\}$

Remark In general since L and L_c have the same flow $\mathbb{M}(L) = \mathbb{M}(L_c)$

The ~~L~~ measure action is defined as

$$A_L(\tilde{\mu}) = \int_{\mathbb{T}^d} L(x,v)\, d\tilde{\mu} \quad \text{and } \mu \in \mathbb{M}(L)$$

Proposition Let μ a ϕ_L-invariant prob. measure supported on $\mathcal{L}^*(c)$ then

(1) $\forall \tilde{\mu} \in \mathbb{M}(L)$

$$\int_{\mathbb{T}^d} L_c(x,v)\, d\tilde{\mu} \geq \int_{\mathbb{T}^d} L_c(x,v)\, d\mu \quad (\star)$$

(2) $\forall \tilde{\mu} \in \mathbb{M}^\rho(L)$

$$\int_{\mathbb{T}^d} L(x,v)\, d\tilde{\mu} \geq \int_{\mathbb{T}^d} L(x,v)\, d\mu \quad (\star)(\star)$$

proof Fenchel-inequality: $(p,v)_x \leq H(x,p) + L(x,v) \quad \forall (x,v) \in \mathbb{T}^d$

(1)
$$\int_{\mathbb{T}^d} L_c(x,v)\, d\tilde{\mu} = \int_{\mathbb{T}^d} [L(x,v) - c\cdot v]\, d\tilde{\mu} = \int_{\mathbb{T}^d} [L(x,v) - (c+du)(v)]\, d\tilde{\mu}$$

$$\geq \int_{\mathbb{T}^d} -H(x, c+du)\, d\tilde{\mu} = -E_c = \int_{\mathbb{T}^d} L_c(x,v)\, d\mu$$

(2)
$$\int_{\mathbb{T}^d} L(x,v)\, d\tilde{\mu} = \int_{\mathbb{T}^d} [L(x,v) - (c+du)(v)]\, d\tilde{\mu} + \int_{\mathbb{T}^d} (c+du)(v)\, d\tilde{\mu} =$$

$$= \int_{\mathbb{T}^d} L_c\, d\tilde{\mu} + c\cdot\rho = -E_c + c\cdot\rho = \int_{\mathbb{T}^d} L(x,v)\, d\mu.$$

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Corollary

$$E_c = -\min\left\{ \int_{\mathbb{T}^d} L_c(x,v)\, d\tilde{\mu} \;\middle|\; \tilde{\mu} \in \mathbb{M}(L) \right\}$$

$$-E_c + cp = \min\left\{ \int_{\mathbb{T}^d} L(x,v)\, d\tilde{\mu} \;\middle|\; \tilde{\mu} \in \mathbb{M}^p(L) \right\}$$

A measure μ which satisfies inequality (★) is called c-action minimizing measure (or Mather's measure with cohomology class c).

A measure μ satisfying (★)(★) is called action-minimizing measure with rotational vector p.

Remarks and Facts

(1) Now $H^1(\mathbb{T}^d, \mathbb{R}) = \text{span}\{dx_1 \cdots dx_n\}$ and we look $c = \sum c_i dx_i$ as a cohomology class $[c]$.

$$\alpha : H^1(\mathbb{T}^d, \mathbb{R}) \longrightarrow \mathbb{R}$$
$$[c] \longrightarrow -\min\{ A_{L_c}(\mu) \mid \mu \in \mathbb{M}(L) \}$$

is called the α-Mather function.

(2) If $\text{supp}\,\tilde{\mu}$ is not contained on T the inequalities (★) and (★)(★) are strict.

(3) We get

$$\mathcal{L}^{-1}(T) = \bigcup \left\{ \text{supp}\,\mu \;\middle|\; \mu \text{ is a c-action minimizing measure} \right\} \quad (\bullet)$$

$$= \bigcup \left\{ \text{supp}\,\mu \;\middle|\; \mu \text{ is an action minimizing measure with rotational vector } p \right\} \quad (\bullet)(\bullet)$$

$(\bullet)$ is called the Mather's set of cohomology c and denoted by $\tilde{\mathcal{M}}_c$

$(\bullet)(\bullet)$ is called the Mather's set of homology p and denoted by $\tilde{\mathcal{M}}^p$.

ACTION-MINIMIZING MEASURE AND MATHER'S SET (general case)

Now we want to generalize what we have just seen for KAM-torus.

From now M is closed orientable manifold, without

$\pi_i: TM \longrightarrow M$ and $\pi: T^*M \longrightarrow M$ the tangent and cotangent bundle with the respective projections.

$L: TM \longrightarrow \mathbb{R}$ a Tonelli Lagrangian with

$H: T^*M \longrightarrow \mathbb{R}$ the associated Hamiltonian

Setting We denote again by

$$\mathbb{M}(L) = \{\mu \mid \mu \text{ is } \Phi_L\text{-invariant prob. measure on } TM \text{ and } \int L(x,v)\, d\mu < +\infty\}$$

Facts

① $\mathbb{M}(L) \neq \emptyset$ Thanks to the Tonelli's conditions, the level set of the energy $E(x,v) = \frac{\partial L}{\partial v}(x,v)(v) - L(x,v)$ are compact and invariant subsets of TM. A flow on a compact metric space always admits an invariant probability measure.

Then each non empty energy-level sets $\mathcal{E}(k) = \{E(x,v) = k\}$ contains at least one Φ_L invariant prob. measure.

② **Topology on $\mathbb{M}(L)$** We will assume that $\mathbb{M}(L)$ is endowed with the "Vague Topology" that is the weak* topology induced by $C^0_\ell(TM) = \{f: TM \longrightarrow \mathbb{R}$ continuous such that $\sup_{TM} \frac{|f(x,v)|}{1+\|v\|} < +\infty\}$ i.e. continuous functions with at most linear growth on the fibers of TM.

So $\mathbb{M}(L) \subset C^0_\ell(TM)^*$ and on the right hand of the inclusion the topological space is metrizable.

Moreover $\mathbb{M}(L)$ in this topology is a compact and convex subset of $C^0_\ell(TM)^*$.

③ **Action Measure A_L** For $\mu \in \mathbb{M}(L)$ we define is action as:

$$A_L(\mu) = \int L(x,v)\, d\mu .$$

In particular $A_L: \mathbb{M}(L) \longrightarrow \mathbb{R}$ is a lower semicontinuous functional.

In fact define $A_{L,k}(\mu) = \int_{TM} \min\{L,k\}\, d\mu$

$$|A_{L,K}(\mu) - A_{L,K}(\eta)| \leq K\, d(\mu, \eta)$$

then $A_{L,K}$ is Lipschitz and in particular continuous and $A_L(\mu) = \lim_{K \to +\infty} A_{L,K}(\mu)$ so upper limit of continuous function.

Since A_L is ~~la~~-S-C on a compact and convex domain there exists $\mu \in \mathbb{M}(L)$ such that μ minimizes A_L.

$$\exists \mu \in \mathbb{M}(L) \;\; | \;\; A_L(\mu) = \inf_{\tilde{\mu} \in \mathbb{M}(L)} A_L(\tilde{\mu})$$

A such μ is called an action-minimizing measure (of cohomology class $[0]$).

In order to define invariant subset through the support of those measure we need to vemote our Lagrangian in a manner such that the dynamic is the same but in some sense we can focus on different energy levels. As for the case of KAM-tori we deform L with a closed 1-form.

Given $\eta \in \Omega^1(M)$, $d\eta = 0$ we can look η as a function

$$\tilde{\eta} : TM \longrightarrow \mathbb{R}$$
$$(x, v) \longrightarrow \langle \eta_x, v \rangle$$

The new Tonelli Lagrangian is defined as

$$L_\eta(x, v) = L(x, v) - \tilde{\eta}(x, v).$$

The associated Hamiltonian is given by

$$L^*_\eta(x, p) = \sup_{v \in T_x M} \{ \langle p, v \rangle - L(x, v) - \langle \eta_x, v \rangle \} =$$

$$= \sup_{x \in T_x M} \{ \langle p - \eta_x, v \rangle - L(x, v) \} = H(x, \eta + p)$$

<u>Facts</u> We have already seen during this seminar that

1. L and L_η has the same E-L flow
2. If η is exact then $\int_{TM} \tilde{\eta}\, d\mu = 0 \;\; \forall \mu \in \mathbb{M}(L)$.

This implies that deforming L with an exact form the action A_L does not change then we just need to look at the cohomology class of $H^1(M,\mathbb{R})$.

In fact $\forall \mu \in \mathbb{M}(L)$, $\forall \eta \in \Omega^1(M)$ and $f \in C^\infty(M)$

$$\int_{TM} [L(x,v) - (\eta_x + df_x)(x,v)]\, d\mu = \int_{TM} [L(x,v) - \tilde{\eta}(x,v)]\, d\mu$$

Definition Denoting by η_c a closed 1-form in the cohomology class $[c] \in H^1(M,\mathbb{R})$, we call $\mu \in \mathbb{M}(L)$ which minimizes $A_{L\eta_c}$ a c-action-minimizing measure or Mather's measure with cohomology class $[c]$.

Remark the cohomology class for a measure depends only on the Lagrangian, changing L with $L-\eta$ we just "translate" all cohomology class by $[\eta] \in H^1(M,\mathbb{R})$.

We can define now the Mather's α-function

$$\alpha : H^1(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$[c] \longrightarrow -\min_{\mu \in \mathbb{M}(L)} A_{L\eta_c}(\mu)$$

α is well defined and convex on $\mathbb{M}(L)$ (just by linearity of the integral)

(We will see next time the connection of α with the Mañé critical value of L_{η_c}).

Mather's set

$$\mathbb{M}_c = \mathbb{M}_c(L) = \{ \mu \in \mathbb{M}(L) \mid A_{L\eta_c}(\mu) = -\alpha(c) \}$$

and $\tilde{\mathcal{M}}_c = \bigcup_{\mu \in \mathbb{M}_c} \{\mathrm{supp}\,\mu\} \subset TM$ is Mather set of cohomology class c.

with $\mathcal{M}_c = \pi(\tilde{\mathcal{M}}_c) \subset M$ the projected Mather set.

Remark $\tilde{\mathcal{M}}_c$ is non empty invariant and closed on TM. (Historically $\tilde{\mathcal{M}}_c = \overline{\bigcup_{\mu \in \mathbb{M}_c} \{\mathrm{supp}\,\mu\}}$ but since $\mathbb{M}_c$ is separable with a density argument we can prove that it is closed).

To conclude today

At the first part of the talk we characterized

$$\mathcal{L}^{-1}(\mathcal{T}) = \bigcup_{\mu \in \mathcal{M}(L)} \{ \text{supp}\,\mu \mid \text{minimizers } A_{L_{\eta}}^{N} \}$$

We expect something similar in the general case in particular that this M^c is a graph in TM.
(Levin already proved last time and we will give more detail next monday)
I conclude enunciating the Mather graph Theorem.

Theorem The set $\tilde{M}_c$ is compact and invariant under the E-L flow.

$\pi|_{\tilde{M}_c}$ is an injective map from $\tilde{M}_c$ to M and its inverse $\pi^{-1} : M_c \longrightarrow \tilde{M}_c$ is Lipschitz.

B-Mather's function:

In the KAM-torus case we have seen that instead of minimizing over $\mathcal{M}(L)$ and the modified Lagrangian we can also add some more "constraints" i.e fixing the rotation vector we obtain the same results.
We want to generalize this approach for general Tonelli systems.

Thanks to the above "discussions" it is well defined the following map:

$$\begin{aligned} H^1(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ [\eta] &\longrightarrow \int_{TM} \tilde{\eta}_c \, d\mu \end{aligned}$$

where $\mu \in \mathcal{M}(L)$ and $\tilde{\eta} : TM \longrightarrow \mathbb{R}$, $(x,v) \rightarrow \eta_x(v)$.

By duality there exists $\rho(\mu) \in H^1(M, \mathbb{R})^* \simeq H_1(M, \mathbb{R})$ such that

$$\int_{TM} \tilde{\eta} \, d\mu = \langle [c], \rho(\mu) \rangle$$

($\langle , \rangle$ is the natural pairing between H^* and H_*)

Def $\rho(\mu)$ is the rotation vector for $\mu \in \mathcal{M}(L)$.