Algebraic and Geometric Cutting and Pasting of Manifolds

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Abstract

This report is structured in two main parts: the first part is devoted to the study of the geometric effects of cutting and pasting of manifolds, while the second gives an algebraic approach to this relation. In Chapter 1, I present a survey of the main ideas in [Kre73] and [Neu75]. In this chapter, I include examples and state some conclusions which are derived from [Kre73], but not explicitly stated in this reference. In chapter 2, I present my current work on an L-theoretic interpretation of algebraic cutting and pasting.

Contents

List of Figures

Introduction

The signature of a closed oriented n-dimensional manifold M^n is denoted by $\sigma(M) \in \mathbb{Z}$, and is defined to be zero if the dimension of M is not divisible by 4. If $n = 4k$ then $\sigma(M)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form $(H^{2k}(M, \mathbb{R}), \lambda)$, where

$$\lambda : (H^{2k}(M, \mathbb{R}) \times (H^{2k}(M, \mathbb{R}) \longrightarrow \mathbb{R}; (u, v) \mapsto \langle u \cup v, [M] \rangle .$$

The additivity of the signature was proved by Novikov:

$$\sigma(M_1 \cup_h M_2) = \sigma(M_1) + \sigma(M_2)$$

for any diffeomorphism $h : \partial M_1 \to \partial M_2$.

Furthermore Jänich proved in [Jae68] that the signature is the only invariant with this additive property.

The idea of cutting and pasting grew out of a series of papers by Jänich ([Jae66], [Jae68], [Jae69]) which studied the Novikov additivity of the signature and the additivity properties of the Euler characteristic. The results by Jänich where reviewed and extended in [Kre73] by Kreck, Karras, Neumann and Ossa.

In [Kre73] the theory of cut and paste invariants or briefly SK (Schneiden und Kleben) invariants is discussed, and the SK groups are defined. The reading of this book was my starting point this year, and at the end of the first term I wrote an account on "SK and SKK groups"[1], and gave a survey talk about this for the "Young Women in Topology" conference taking place in the Hausdorff Institute in Bonn in December 2011. Precise statements related to the definition of the cut and paste invariants and the SK groups illustrated with examples can be found in the first chapter of this report. A cut and paste invariant is a function on closed smooth manifolds M which is unchanged if one cuts M along a codimension 1 submanifold into two pieces and glues them back using a different diffeomorphism. The SK groups are then the Grothendieck-type groups of all closed manifolds of a fixed dimension modulo the cut and paste relation.

In the first chapter, I explain the relation between the cut and paste groups and surgery theory.

The process of a k-surgery on an n-dimensional manifold M consists of removing a framed k-embedding $f : S^k \times D^{n-k} \hookrightarrow M^n$ and replacing it by $D^{k+1} \times S^{n-k-1}$,

[1] The SKK-groups are a weaker version of the SK groups. The main idea here is that difference of the invariants only depends on the gluing diffeomorphism. The SKK groups are defined in [Kre73] and can be identified with Reinhart's vector field bordism groups ([Rei63]).

with effect the n-dimensional manifold

$$M' = \overline{M^n - f(S^k \times D^{n-k})} \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1}.$$

In general, the choice of different embeddings affects the result of surgery. In this report we derive from statements in [Kre73] that in SK, the result of surgery does not depend on the embedding.

Two different cut and paste groups are defined in [Kre73], the SK groups and the bordism $\overline{SK}$ groups. The relations between these two groups and the cobordism groups $\Omega_n(X)$ is reviewed in this report.

In [Neu75] Neumann gives the computation of the $SK_2(X)$ group, and in this same paper he relates the SK-groups to the non-multiplicativity of the signature. If $F^m \longrightarrow E^{4k} \longrightarrow M^n$ is a fibration of closed oriented manifolds, it is known that under certain conditions $\sigma(E) = \sigma(F) \times \sigma(M)$. But this equality does not always hold. The reduced SK-groups were shown by Neumann to be obstruction groups for the multiplicativity of the signature.

In chapter 2 of this report I present my current work on providing an L-theoretic interpretation of the work of Karras, Kreck, Neumann and Ossa on the connection between the open book decompositions, the SK and $\overline{SK}$ groups and the non-multiplicativity of the signature in fibre bundles. The L-theoretic interpretation is given for the $\overline{SK}$ groups by Ranicki in Remark 30.30 of [Ran98]. In this report, I give the L-theoretic definition of SK groups, which I denote as SKL groups. Even though this idea follows very closely to the geometric formulation of the SK groups, to my knowledge, the SKL groups have not been previously defined in the Literature.

Chapter 1

Geometric Cutting and Pasting

1.1 $SK_n(X)$ and the cutting and pasting semi-group

The set of equivalence classes of oriented manifolds in a space X modulo the relation created by cutting and pasting gives rise to the definition of SK-groups. This cut and paste relation can be described as follows:

Definition 1.1.1. *Cut and paste operations on a manifold M are realized as follows: Cut a closed n-dimensional smooth manifold M along a codimension 1 manifold F which has trivial normal bundle. After performing this cut we obtain a manifold with two boundary components, each of them a copy of F. Pasting back these boundary components by a diffeomorphism $h : F \to F$, results in a new manifold $M(F, h)$.*

Example 1.1.2. *Start with $M = S^1$ and cut along the codimension 1 manifold $F = S^0$. Paste the boundaries using a diffeomorphism $h \neq Id$ as follows:*

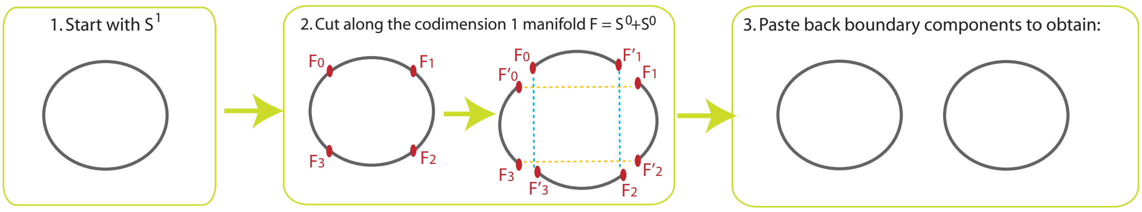


Figure 1.1: $[S^1] = 2[S^2]$ under cutting and pasting

In this case the map $h : F \longrightarrow F'$ is given by mapping $F_0 \mapsto F_3'$; $F_1 \mapsto F_0'$; $F_2 \mapsto F_1'$; $F_3 \mapsto F_2'$.

The manifold we obtain is $M(S^0 \sqcup S^0, h) = S^1 \sqcup S^1$. So $S^1 \sqcup S^1$ can be obtained from a single copy of S^1 by a cutting and pasting operation.

Example 1.1.3. *In this example we will see that orientation need not be preserved in the process of cutting and pasting. Starting with a torus $M = T^2$, cut along a codimension 1 manifold $F = S^1$. Paste back the two copies of S^1 which are the boundaries of the cylinder using the orientation - reversing automorphism $h : S^1 \longrightarrow S^1; z \mapsto z^{-1}$ to obtain $M(S^1, h) =$ Klein bottle:*

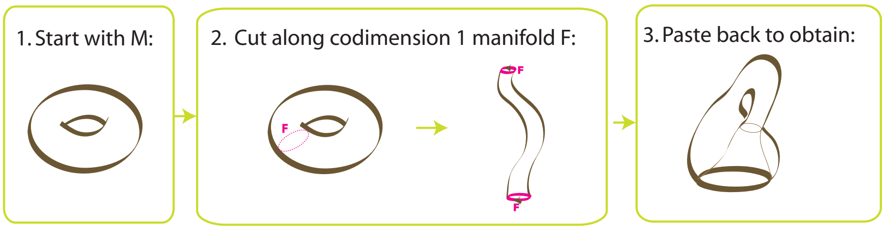


Figure 1.2: Torus = Klein bottle under cutting and pasting

Definition 1.1.4. ***(The cutting and pasting semigroup $\mathscr{M}_n(X)/\sim_{SK}$)*** *For any connected space X, $\mathscr{M}_n(X)/\sim_{SK}$ is the semigroup of equivalence classes of pairs (M, f), with M a closed n-dimensional manifold and $f : M \longrightarrow X$ a map, subject to the following equivalence relation:*
$(M, f) \sim_{SK} (M', f')$ if one can be obtained from the other by a sequence of cutting and pasting operations with a map to X.

Remark 1.1. We will later prove that the cut and paste semigroup $\mathscr{M}_n(X)/\sim_{SK}$ is an abelian group (with operation induced by disjoint union), which we denote by $SK_n(X)$. An inverse of an equivalence class of manifolds $[N^n, *]$ is an equivalence class $-[N^n, *]$ such that

$$-[N^n, *] + [N^n, *] = 0 \in \mathscr{M}_n(X)/\sim_{SK}$$

We will later prove that the inverse of an element $(N, f) \in \mathscr{M}_n(X)/\sim_{SK}$, is given by

$$-[N, f] = [-N, f] - \chi(N)[S^n, *] \in \mathscr{M}_n(X)/\sim_{SK}$$

where $-N$ is the manifold N with reversed orientation and $\chi(N)$ is the Euler characteristic of N.

Remark 1.2. From now on we will denote the cut and paste semigroup $\mathscr{M}_n(X)/\sim_{SK}$ as "$SK_n(X)$", and we will drop the quotes when we prove that $SK_n(X)$ is in fact a group.

In the first place, we will describe some manifolds which are $0 \in$ "$SK_n(X)$". In example 1.1.2 we have seen that $2[S^1]$ can be obtained from a cut and paste operation on $[S^1]$, so

$$[S^1, *] = 2[S^1, *] \in \text{"}SK_1(X)\text{"}$$

and hence, $[S^1, *] = 0 \in$ "$SK_1(X)$"

Remark 1.3. (From [Kre73]) The product of singular manifolds (M, f) induces the bilinear map

$$SK_n(X) \oplus SK_m(Y) \longrightarrow SK_{n+m}(X \times Y)$$

Example 1.1.5. *Consider the n-dimensional manifold $F^{n-1} \times S^1$. Similarly to what happens for S^1, $[F^{n-1} \times S^1] = 2[F^{n-1} \times S^1] \in$ "SK_n", so consequently, $[F^{n-1} \times S^1] = 0 \in$ "SK_n".*

Example 1.1.6. *In this example we explain why mapping tori are zero in* ”$SK_n(X)$”.

The mapping torus is a twisted double:

$$T(h) = F \times [0, 1/2] \cup_{h \cup 1} F \times [1/2, 1]$$

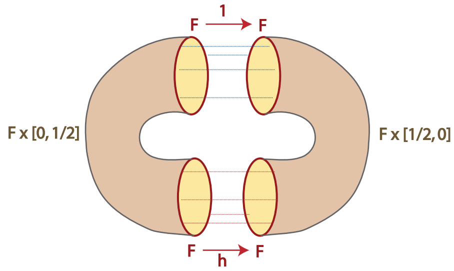


Figure 1.3: A mapping torus is a twisted double

So in ”$SK_n(X)$”, *the mapping torus is always a boundary:*

$$T(h : F \to F) = T(1 : F \to F) = F \times S^1 = 0 \in \text{”}SK_n(X)\text{”},$$

which follows from the previous example.

Before we start on a more formal approach to the definition of the inverses, we give the following example which shows how the inverse of S^2 can be found.

Example 1.1.7. *In this example, we will find a representative for the inverse of* $[S^2] \in$ ”$SK_2(X)$”.

If we take $F = S^1$ *in example 1.1.5 we see that* $[S^1 \times S^1] = 0 \in$ ”$SK_2(X)$”. *So, in particular,* $2[S^1 \times S^1] = 0 \in$ ”$SK_2(X)$”. *We will now perform a sequence of cut and paste operations on the zero class* $0 = [S^1 \times S^1 + S^1 \times S^1] \in$ ”$SK_2(X)$” *to obtain the inverse of* $[S^2]$.

1. Start with $S^1 \times S^1 \sqcup S^1 \times S^1$: *2. Cut along* $S^0 \times S^1$: *3. Paste back boundaries:*

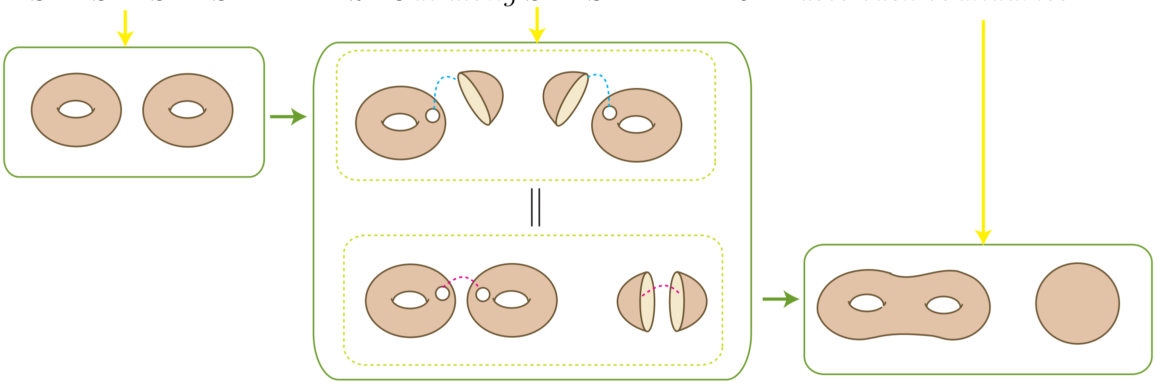


Figure 1.4: $-[S^2] = [\Sigma_2] =$ [surface of genus 2] $\in$ ”$SK_2(X)$”

So we find that the inverse of $[S^2] \in$ ”$SK_2(X)$” *is,*

$$\boxed{-[S^2] = [\Sigma_2] = [\text{surface of genus 2}] \in \text{”}SK_2(X)\text{”}}$$

1.2 The behaviour of "$SK_n(X)$" under Surgery

In this section we will develop some machinery that will allow us to finally prove that the semigroup "$SK_n(X)$" is actually an abelian group, and we will also be able to proof that in "$SK_n(X)$" the result of surgery is independent of the chosen embedding for that surgery.

Proposition 1.2.1. *The following relation was proved in [Neu71]: let A, B and C be n-dimensional manifolds with boundary such that $\partial A = \partial B = \partial C$, and consider diffeomorphisms of the boundaries $f : \partial A \longrightarrow \partial B, \quad g : \partial B \longrightarrow \partial C$* and *$h : \partial A \longrightarrow \partial C$, then*

$$\boxed{A \cup_f B + B \cup_g C = C \cup_h A + B \cup B \in \text{"}SK_n(X)\text{"}}$$

Proof.

$$\begin{aligned}(A \cup_f B) + (B \cup_g C) &= (A + B) \cup_{f+g} B + C \\ &= (A + B) \cup_{h+1} (B + C) \\ &= (A \cup_h C) + (B \cup_1 B) \in \text{"}SK_n(X)\text{"}\end{aligned}$$

□

Proposition 1.2.2. *(This is lemma 1.6 in [Kre73]) Let M be an n-dimensional manifold and M' be the effect of surgery on $S^k \times D^{n-k} \subset M$, $M' = \overline{(M \backslash S^k \times D^{n-k})} \cup D^{k+1} \times S^{n-k-1}$,*

$$M + S^n = M' + S^k \times S^{n-k} \in \text{"}SK_n(X)\text{"}$$

Proof. To prove that $M + S^n = M' + S^k \times S^{n-k} \in$ "$SK_n(X)$" we are going to use the identity

$$A \cup B + B \cup C = A \cup C + B \cup B \in \text{"}SK_n(X)\text{"}$$

We now will rewrite this identity using the following inputs for A, B and C:

- $A = \overline{M - (S^k \times D^{n-k})}$,
- $B = S^k \times D^{n-k}$,
- $C = D^{k+1} \times S^{n-k-1}$

$$\begin{aligned}M + S^n &= [(\overline{M - (S^k \times D^{n-k})}) \cup (S^k \times D^{n-k})] + [(S^k \times D^{n-k}) \cup (D^{k+1} \times S^{n-k-1})] \\ &= [(\overline{M - (S^k \times D^{n-k})}) \cup (D^{k+1} \times S^{n-k-1})] + [(S^k \times D^{n-k}) \cup (S^k \times D^{n-k})] \\ &= M' + S^k \times S^{n-k}\end{aligned}$$

Since $A \cup B = M$, $B \cup C = S^n$, $A \cup C = M'$ and $B \cup B = S^k \times S^{n-k}$.

□

The following Proposition (which appears as Corollary 1.7. in [Kre73]) shows that some products of two spheres are $0 \in$ "$SK_n(X)$".

Proposition 1.2.3. *(This corresponds to Corollary 1.7 of [Kre73])*
In "$SK_n(X)$"

$$[S^k \times S^{n-k}, *] = \begin{cases} 2[S^n, *], & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

Proof. This is a consequence of Proposition 1.2.2, because if we choose $M = S^n$ then the result of surgery on an embedding $S^k \times D^{n-k}$ is $M' = S^{k+1} \times S^{n-k-1}$ and then the equation of lemma 1.6 in [Kre73] becomes

$$[S^n, *] + [S^n, *] = [S^{k+1} \times S^{n-k-1}, *] + [S^k \times S^{n-k}, *] \in \text{"}SK_n(X)\text{"}$$

and taking $k = 0$ here, we obtain,

$$\begin{aligned}[S^n, *] + [S^n, *] &= [S^1 \times S^{n-1}, *] + [S^0 \times S^n, *] \\ &= [S^1 \times S^{n-1}, *] + [S^n, *] + [S^n, *] \in \text{"}SK_n(X)\text{"}\end{aligned}$$

so that $[S^1 \times S^{n-1}, *] = 0 \in$ "$SK_n(X)$" and the Corollary follows by induction. □

Proposition 1.2.4. *(This corresponds to Corollary 1.8 of [Kre73]) Let M' be the result of surgery on $S^k \times D^{n-k} \subset M$ and Y be the trace of the surgery, then*

$$[M, f] = [M', f'] - (\chi(Y) - \chi(M))[S^n] \in \text{"}SK_n(X)\text{"}.$$

Proof. Here we use again the equation in Proposition (1.2.2),

$$M + S^n = M' + S^k \times S^{n-k} \in \text{"}SK_n(X)\text{"}$$

and use the result of corollary 1.7 in [Kre73]. By this corollary, for k even, we have $M + S^n = M' + 2S^n$ i.e, $M - S^n = M'$ And for k odd, $M + S^n = M'$ So that,

$$[M, f] = [M', f'] - (-1)^{k+1}[S^n, *]$$

and

$$\begin{aligned}\chi(Y) &= \chi(M \times I) + \chi(D^{k+1} \times D^{n-k}) - \chi(S^k \times D^{n-k}) \\ &= \chi(M) + (-1)^{k+1}\end{aligned}$$

so Corollary 1.8 in [Kre73] follows. □

As promised before, we will now show that the group given by the Grothendieck construction of the cut and paste semigroup is actually equal to the semigroup itself.

Proposition 1.2.5. *The cut and paste semigroup* "$SK_n(X)$" *is an abelian group (with operation induced by disjoint union), which we denote by $SK_n(X)$.*

The inverse of an element in $[N, f] \in$ *"*$SK_n(X)$*", is given by*

$$-[N, f] = [-N, f] - \chi(N)[S^n, *] \in \text{"}SK_n(X)\text{"},$$

where $-N$ *is the manifold* N *with reversed orientation and* $\chi(N)$ *is the Euler characteristic of* N*.*

Proof. In the statement of Corollary (1.2.4) take $(M, f) = (N \sqcup -N, f \sqcup -f)$ and also $M' = \varnothing$ so that $Y = N \times I$. Then,

$$[N \sqcup -N, f \sqcup f] = -(\chi(N \times I) - \chi(M))[S^n]$$

$$[N, f] + [-N, f] = \chi(N)[S^n]$$

and hence,

$$[N, f] + [-N, f] - \chi(N)[S^n] = 0 \in SK_n(X)$$

□

Example 1.2.6. *The formula for the inverses allows us to see why odd dimensional spheres are* $0 \in SK_{2k+1}(X)$*.*

$$\begin{aligned} -[S^{2k+1}, *] &= [-S^{2k+1}, *] - \chi(S^{2k+1})[S^{2k+1}, *] \\ &= [-S^{2k+1}, *] \\ &= [S^{2k+1}, *] \end{aligned}$$

which implies that $2[S^{2k+1}, *] = 0 \in SK_{2k+1}(X)$ *and consequently*

$$[S^{2k+1}, *] = 0 \in SK_{2k+1}(X).$$

Example 1.2.7. *Note that by the definition of the cut and paste relation it holds that for n-dimensional closed manifolds* M *and* N*,*

$$M \# N + S^n = M + N \in SK_n(X).$$

In example 1.1.7 we saw that the surface of genus 2*, denoted by* Σ_2 *satisfies,*

$$[\Sigma_2] + [S^2] = S^1 \times S^1 = 0 \in SK_2(X)$$

Now, a surface of genus 3 *is*

$$\Sigma_3 = \Sigma_2 \# S^1 \times S^1 \in SK_2(X)$$

Using equation 1.2.7,

$$\Sigma_3 + S^2 = \Sigma_2 + S^1 \times S^1 = \Sigma_2 \in SK_2(X)$$

It follows by induction that

$$\Sigma_{g+1} + S^2 = \Sigma_g \in SK_2(X)$$

and consequently,

$$\Sigma_g = -(g-1)S^2 \in SK_2(X)$$

where Σ_g *is a surface of genus* g*.*

Remark 1.4. Note that the cut and paste semigroup actually equals the SK-group, so all the results mentioned up to now hold not only for "$SK_n(X)$" but also for $SK_n(X)$.

Proposition 1.2.8. *In* $SK_n(X)$ *the result of surgery does not depend on the embedding.*

Proof. Note that by Proposition (1.2.2),

$$M' = M + S^n - S^k \times S^{n-k} \in SK_n(X)$$

The LHS of this equation is M', which is the result of surgery on M. The RHS is $M + S^n - S^k \times S^{n-k}$ which does not depend on the embedding. Hence the corollary follows.

□

1.3 Cutting and pasting invariants

Definition 1.3.1. *A cut and paste invariant is a function* λ *which takes values in an abelian group* G *and satisfies the following identity,*

$$\lambda(M_1 \cup_f M_2) = \lambda(M_1 \cup_g M_2) \in G$$

where M_1 *and* M_2 *are* n*-dimensional manifolds with boundary* $\partial M_1 = \partial M_2$*, and* f *and* g *are diffeomorphisms* $f, g : \partial M_1 \to \partial M_2$*.*

A natural question to ask is: what are invariants under cut and paste relation?

Proposition 1.3.2. *The* ***Euler characteristic*** *is an* SK *invariant.*

Proof. This can be proved by considering the Poincaré formula

$$\chi(M_1 \cup M_2) = \chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2) \in \mathbb{Z},$$

since $\chi(M_1 \cup M_2)$ does not depend on the diffeomorphism used to glue M_1 and M_2. □

Proposition 1.3.3. *The* ***Signature*** *is an* SK *invariant.*

Proof. By the Novikov additivity formula $\sigma(M_1 \cup M_2) = \sigma(M_1) + \sigma(M_2)$, we see that it is independent of the diffeomorphism used to glue M_1 and M_2. □

Proposition 1.3.4. *A function* λ *sending manifolds to their bordism class,*

$$\lambda : M^n \longrightarrow [M] \in \Omega_n$$

is ***not*** *an* SK*-invariant.*

Proof. An n-dimensional manifold M fibers over S^1 if it is the mapping torus $M = T(h)$ of an automorphism $h : F \longrightarrow F$ of an $(n-1)$-dimensional manifold F as defined by

$$T(h) = F \times [0,1]/\{(x,0) \sim (h(x),1) | x \in F\}.$$

Note that the projection $F \times I \longrightarrow I$ induces the fibration

$$T(h) \longrightarrow S^1.$$

As was explained in example (1.1.6):

$$T(h) = F \times [0,1/2] \cup_{h \cup 1} F \times [1/2,1].$$

So in SK, the mapping torus is always a boundary

$$T(h : F \to F) = T(1 : F \to F) = F \times S^1 = \partial(F \times D^2) \in SK_n(X).$$

The following counterexample shows that bordism is not an SK invariant, since in Ω_n, the mapping torus $T(h : F \to F)$ is in general not a boundary:

$\Omega_5 = \mathbb{Z}_2$ (and in general Ω_{2k+1}) consists of elements of order 2, which are classified by the Stiefel-Whitney numbers. Consider the following automorphisms of $\mathbb{C}P^2$,

- The identity: $\mathbb{C}P^2 \xrightarrow{1} \mathbb{C}P^2$,
- Complex conjugation: $\mathbb{C}P^2 \xrightarrow{h} \mathbb{C}P^2$.

The **DeRham invariant** is defined by (See Lemma 4.4 in [Kre84]):

$$dR(h) = w_2(T(h)) \cdot w_3(T(h)) = \chi_{1/2}(T;\mathbb{Z}_2) - \chi_{1/2}(T;\mathbb{Q}) \in \mathbb{Z}_2$$

where h is a diffeomorphism of a 4-dimensional manifold, w_2 and w_3 are the second and third Stiefel-Whitney classes, and $\chi_{1/2}$ is the Kervaire semicharacteristic.

So the DeRham invariant $dR(1)$ of the mapping torus $T(1 : \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2)$ is 0, while $dR(h)$ of the mapping torus $T(h : \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2)$ is 1.

Hence we deduce that bordism is not an SK invariant.

Remark 1.5. Alternatively we can note from the definition of the inverses, that manifolds belonging to the same bordism class such as S^2 and a surface of genus 2, belong to different classes in $SK_*(X)$.

□

1.4 Cutting and pasting bordism groups $\overline{SK}_n(X)$

In this section we will define the cut and paste bordism groups, and we will see how these groups relate to the cobordism groups and to the SK-groups.

Definition 1.4.1. *The cut and paste bordism group $\overline{SK}_n(X)$ is $SK_n(X)$ factored by the subgroup $I_n \subseteq SK_n(X)$ generated by all elements which have a representative that is a boundary in X.*

Theorem 1.4.2. *(Theorem 1.1 in SK book) SK and $\overline{SK}$ are related by the exact sequence,*

$$0 \longrightarrow I_n \longrightarrow SK_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0$$

where I_n is the subgroup of $SK_n(X)$ of manifolds which bound in X, that is, $I_n \subset SK_n(X)$ is the subgroup of $SK_n(X)$ which is generated by $[S^n, *]$.

$$I_n = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Proof. What we first need to see in order to prove this theorem is that I_n is indeed generated by $[S^n, *]$. By definition $I_n = \text{Ker}(SK_n(X) \longrightarrow \overline{SK}_n(X))$ is generated by all classes of singular manifolds $[M, f]$ such that (M, f) bounds in X. So we first note that by Corollary 1.8 in [Kre73], if M is the boundary of a manifold Y^{n+1} then,

$$[M, f] = \chi(Y^{n+1})[S^n, *], \tag{1.6}$$

so that each of the (M^n, f) which bound, are multiples of $[S^n, *]$.

If Y^{n+1} is a closed manifold, then it has no boundary component, so that it follows from [Kre73] Corollary 1.8 (taking M_1 and M_2 in this corollary to be both $\varnothing$) that,

$$\chi(Y^{n+1})[S^n] = 0.$$

This expression allows us to compute the order of $[S^n]$ in SK_n and hence I_n.

First consider n even. In this case we note that the Euler characteristic of a compact odd dimensional manifold Y^{2k+1} is 0, hence $[S^n]$ is an element of infinite order in I_n, so that for n even, $[S^n]$ generates $I_n = \mathbb{Z}$.

Now consider n odd, $n = 2k+1$. In this case we have $\chi(Y^{2k+2})[S^{2k+1}] = 0$. If $Y^{2k+2} = S^{2k+2}$ then $\chi(Y^{2k+2}) = 2$ and we have that $[S^{2k+1}]$ has order at most 2 in SK_{2k+1}

$$2[S^{2k+1}] = 0 \Longrightarrow [S^{2k+1}] = 0.$$

On the other hand, if $\chi(Y^{2k+2})$ is odd, then

$$(2a+1)[S^{2k+1}] = 0,$$

but we already know that $[S^{2k+1}]$ has at most order 2, so this implies that $[S^{2k+1}] = 0$. Consequently, $I_n = 0$, when n is odd.

□

Definition 1.4.3. *$F_n(X) \subseteq \Omega_n(X)$ is the subgroup of the bordism classes of closed n-dimensional manifolds which fiber over S^1.*

$$F_n(X) = \{[M] \in \Omega_n(X) \mid \sigma(M) = 0\}.$$

$F_n(X)$ is also defined in Remark 30.30 of [Ran98] to be

$$F_n(X) = im(DB_{n-1}(X) \longrightarrow \Omega_n(X)),$$

where $DB_*(X)$ are the twisted double bordism groups of definition 30.5 (i) in [Ran98].

Remark 1.7. In the geometric setting, the subgroup $I_n \subseteq SK_n(X)$ is generated by $[S^n]$. All spheres are doubles, which means that they are elements in the twisted double bordism group $DB_n(X)$ represented by $(D^n, S^{n-1}, 1, f, g)$. But $[S^n] = 0 \in \Omega_n$ so that $I_n \subseteq \ker(D : DB_{n-1}(X) \longrightarrow \Omega_n(X))$.

Theorem 1.4.4. *(Theorem 1.2 of [Kre73]) Let $F_n(X) \subseteq \Omega_n(X)$ be subgroup of the bordism classes of closed n-dimensional manifolds which fiber over S^1, then the following sequence is exact,*

$$0 \longrightarrow F_n(X) \longrightarrow \Omega_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0.$$

Proof. We need to show that

$$\mathrm{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X)) = \mathrm{Im}(F_n(X) \longrightarrow \Omega_*(X)).$$

$\mathrm{Im}(F_n(X) \longrightarrow \Omega_*(X))$ will consist of those bordism classes in $\Omega_*(X)$ with representatives which fiber over S^1. So we have to show the following:

(i) First we want to show that if (M^n, f) fibers over S^1 it represents $0 \in \overline{SK}_n$, i.e. it belongs to $\mathrm{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X))$:

An n-dimensional manifold M^n fibers over S^1 if it is the mapping torus $M = T(h)$ of an automorphism $h : F \longrightarrow F$ of an $(n-1)$- dimensional manifold F as defined by $T(h) = F \times [0,1]/\{(x,0) = (h(x),1) \mid x \in F\}$. The mapping torus, as explained before, is a twisted double,

$$T(h) = F \times [0,1/2] \cup F \times [1/2,1].$$

So, in SK_* it is equivalent to the boundary

$$T(1 : F \to F) = F \times S^1 = \partial(F \times D^2).$$

By the definition of $\overline{SK}_*$, a boundary represents the zero class in $\overline{SK}_*$, so (i) holds.

(ii) We also need to prove the reverse inclusion:

$$\mathrm{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X)) \subset F_*(X).$$

This means that $\text{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X))$ has to be generated by classes in $\Omega_*(X)$ which have representatives which fiber over S^1, or what is the same, classes that have a mapping torus as representative.

So we note that $\text{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X))$ is generated by classes of the form $[M, h] - [M', h']$, where $[M', h']$ is obtained from $[M, h]$ by cutting and pasting.

What we now need to see is if such classes are mapping tori:

Consider $M = M_1 \cup_f M_2$ and $M' = M_1 \cup_g M_2$ where both M and M' have been cut along an $(n-1)$-dimensional manifold F and then pasted back by diffeomorphisms

$$f, g : \partial M_1 \longrightarrow \partial M_2.$$

We now construct a bordism Y in the following way:

Start by considering $M_1 \times I$ and $M_2 \times I$ and gluing as in the following picture. (Note that boundary components in the following figures are shown in light brown).

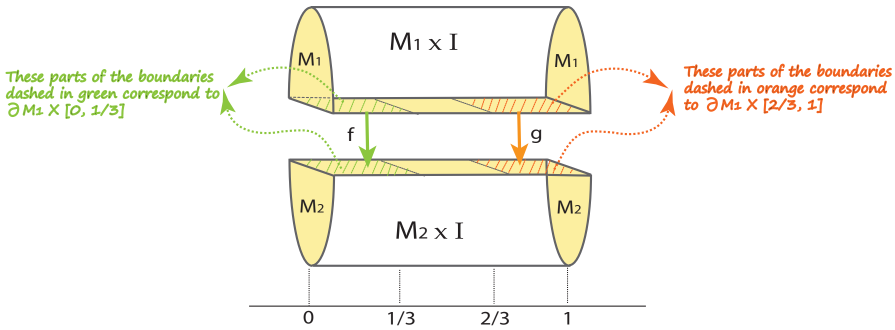


Figure 1.5: Constructing a bordism of SK-equivalent manifolds

After gluing $\partial M_1 \times [0, 1/3]$ and $\partial M_2 \times [0, 1/3]$ via f, and gluing $\partial M_1 \times [2/3, 1]$ and $\partial M_2 \times [2/3, 1]$ via g, we obtain the following bordism Y,

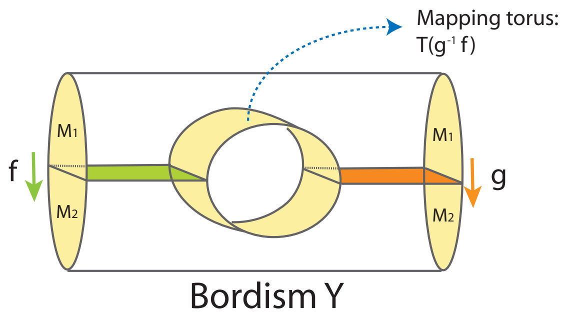


Figure 1.6: bordism = $\overline{SK}$-equivalence + fiber over S^1

If we look more closely at what happens in $[1/3, 2/3]$ we see:

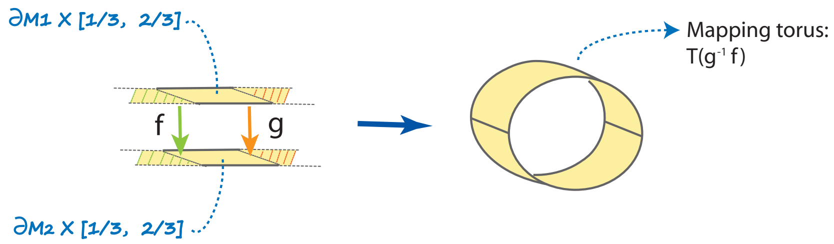


Figure 1.7: Diffeomorphism of twisted double and mapping torus

So this part of the boundary is formed by

$$\partial M_1 \times [1/3, 2/3] \cup \partial M_2 \times [1/3, 2/3],$$

by joining $(x, 1/3)$ to $(f(x), 1/3)$ and also joining $(x, 2/3)$ to $(g(x), 2/3)$, where $x \in \partial M_1$. This is diffeomorphic to $\partial M_1 \times I$ with $(x, 0)$ identified with $(g^{-1}f(x), 1)$ using the diffeomorphism defined by $(x, t) \to (x, t - 1/3)$ for $x \in \partial M_1$ and $(y, t) \to (g^{-1}(y), 4/3 - t)$ for $y \in \partial M_2$. Hence the part of the boundary that we are now describing is the mapping torus $T(g^{-1}f)$, i.e., it is a fiber bundle over the circle S^1 with fiber $\partial M_1 = \partial M_2$.

The bordism Y has boundary,

$$\partial Y = (M_1 \cup_f M_2) - (M_1 \cup_g M_2) - T(g^{-1}f).$$

From this we see that the classes of singular manifolds $[M, h] - [M', h']$ are just classes represented by mapping tori. Hence $\mathrm{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X)) \subset F_*(X)$.

Thus (i) and (ii) hold, and this proves that the sequence is exact. $\square$

1.5 Computation of $SK_n(X)$

In what follows we will assume the space X to be path connected.

In general $SK_n(X) \neq SK_n$. The map from the space X to a point, $X \longrightarrow *$ induces a map $SK_n(X) \longrightarrow SK_n(*) = SK_n$. The kernel of this map will be denoted by the reduced SK-group.

Definition 1.5.1. *(**reduced** SK**-group**) Let X be path connected. The reduced SK-group, $\widetilde{SK}_n(X)$ is the kernel of the map $SK_n(X) \longrightarrow SK_n$, induced by $X \longrightarrow *$. Since X is path connected, the map $* \longrightarrow X$ induces a splitting so that*

$$SK_n(X) = \widetilde{SK}_n(X) \oplus SK_n.$$

Remark 1.8. The computation of SK_n is known, so one of the tools to compute $SK_n(X)$ will be investigate the reduced SK-group $\widetilde{SK}_n(X)$.

Firstly we give a brief account of the computation of $\overline{SK}_n$ and SK_n.

Proposition 1.5.2. *The $\overline{SK}$ groups are given by,*

$$\overline{SK}_n \cong \begin{cases} \mathbb{Z} \quad \text{with basis} \quad [\mathbb{C}P^{n/2}] & \text{for } n \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

Proof. This result follows from the exact sequence

$$0 \longrightarrow F_n \longrightarrow \Omega_n \longrightarrow \overline{SK}_n \longrightarrow 0,$$

$F_n = \{[M] \in \Omega_n \mid \sigma(M) = 0\}$ and all the torsion parts in Ω_n are contained in F_n, so the only possible generator for $\overline{SK}_n$ is $\mathbb{C}P^{n/2}$, which has signature 1. ☐

Remark 1.9. note that the signature induces the isomorphism

$$\sigma : \overline{SK}_{4k} \cong \mathbb{Z}.$$

Since the generator of $\overline{SK}_{4k}$ is $[\mathbb{C}P^{2k}]$, and $\sigma(\mathbb{C}P^{2k}) = 1$, thus generating $\mathbb{Z}$

Example 1.5.3. *Note that $\Omega_8 = \mathbb{Z} \oplus \mathbb{Z}$ is generated by $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$. But these belong to same equivalence class in $\overline{SK}_8$ since $\mathbb{C}P^4 - \mathbb{C}P^2 \times \mathbb{C}P^2$ has zero signature.*

Proposition 1.5.4. *The SK_n groups are given by*

$$SK_n \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{with basis} \quad [S^n], [\mathbb{C}P^{n/2}] & \text{for } n \equiv 0 \pmod 4 \\ \mathbb{Z} & \text{with basis} \quad [S^n] & \text{for } n \equiv 2 \pmod 4 \\ 0 & & \text{for } n \equiv 1 \pmod 2 \end{cases}$$

Proof. The computation of the SK_n groups follows from the computations of $\overline{SK}_n$ given in Proposition (1.5.2) and of I_n given in the proof of Theorem (1.4.2) and the exact sequence

$$0 \longrightarrow I_n \longrightarrow SK_n \longrightarrow \overline{SK}_n \longrightarrow 0.$$

This exact sequence splits since the map $(\chi - \sigma)/2 : SK_n \to I_n$ gives a retraction map to the inclusion map $I_n \hookrightarrow SK_n$.

The map $(\chi - \sigma)/2 \oplus \sigma$ defines the isomorphism

$$(\chi - \sigma)/2 \oplus \sigma : SK_{4k} \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}.$$

First note that for a closed $4k$-dimensional manifold $(\chi(M) - \sigma(M))/2 \in \mathbb{Z}$. This follows from Lemma II.1 in [Ker56]. Any $4k$-dimensional manifold $M^{4k} \in SK_{4k}$ can be written as a linear combination of $4k$-dimensional spheres and complex projective planes, i.e,

$$M^{4k} = i[S^{4k}] + j[\mathbb{C}P^{2k}] - jk[S^{4k}].$$

Recall that $\chi(S^{4k}) = 2$, $\chi(\mathbb{C}P^{2k}) = 2k + 1$ and $\sigma(S^{2k}) = 0$, $\sigma(\mathbb{C}P^{2k}) = 1$. So that, $\chi(M) = 2i + (2k + 1) - 2jk$ and $\sigma(M) = j$. Consequently,

$$i = (\chi(M) - \sigma(M))/2 \quad \text{and} \quad j = \sigma(M).$$

Similarly, the map $\chi/2$ defines the isomorphism

$$\chi/2 : SK_{4k+2} \xrightarrow{\cong} \mathbb{Z}.$$

A $(4k+2)$-dimensional manifold $M \in SK_{4k+2}$ can be written as $M^{4k+2} = p[S^{4k+2}]$ and $\chi(S^{4k+2}) = 2$ so $p = \chi(M)/2$.

$\square$

Proposition 1.5.5. *The exact sequence from 1.4.2,*

$$0 \longrightarrow I_n \longrightarrow SK_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0$$

splits.

Proof. If n is **odd**, then $I_n = 0$ and the sequence becomes,

$$0 \longrightarrow SK_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0,$$

which implies that,

$$SK_n(X) \cong \overline{SK}_n(X).$$

So we see that it splits trivially.

When n is **even**, the map defined by

$$\frac{(\chi - \sigma)}{2} : SK_n \longrightarrow I_n$$

gives a retraction map to the inclusion map $I_n \hookrightarrow SK_n$. See 1.5. Hence the sequence splits. $\square$

As was mentioned before, the computation of $SK_n(X)$ is far more complicated than that of SK_n. Although some facts are known, the complete computation has not been done. The computation $SK_2(X)$ is given in [Neu75].

Proposition 1.5.6. *Here we now quote some relevant results which are proved in [Neu75]:*

(i) *If* $\pi_1(X) = 1$ *then* $\widetilde{SK}_n(X) = 0$.

(ii) *If* $\pi_1(X)$ *is finite then* $\widetilde{SK}_n(X)$ *is torsion.*

(iii) *If* $X \longrightarrow Y$ *induces isomorphisms for* $\pi_i(X) \longrightarrow \pi_i(Y)$ *for* $0 \leqslant i \leqslant n-1$, *then* $SK_q(X) \cong SK_q(Y)$ *for* $q \leqslant n$.

(iv) $\mathrm{Ker}(\overline{SK}_n(X) \longrightarrow \overline{SK}_n) = \mathrm{Ker}(SK_n(X) \longrightarrow SK_n)$.

(v) *the subgroup* $I_n(X) \subset SK_n(X)$ *is independent of* X, *so that*

$$I_n(X) \cong I_n = \mathrm{Ker}(SK_n \longrightarrow \overline{SK}_n).$$

(vi) $\widetilde{SK}_n(X) = \{[M, f] \in \overline{SK}_n(X) \mid \sigma(M) = 0\}$

Proof. For a proof of these results see [Neu75]. $\square$

Summarizing some of the results presented int [Neu75], we can draw the following braid of exact sequences:

$$\begin{tikzcd}
I_n \arrow[rr, bend left] \arrow[dr] & & SK_n \arrow[rr, bend left] \arrow[dr] & & 0 \\
& SK_n(X) \arrow[ur] \arrow[dr] & & \overline{SK}_n \arrow[ur] \arrow[dr] & \\
\widetilde{SK}_n(X) \arrow[ur] \arrow[rr, bend right] & & \overline{SK}_n(X) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

Proposition 1.5.7. $\overline{SK}_{2k+1}(X) = 0$.

Proof. Every odd dimensional manifold has open book decomposition for $n \geqslant 6$ (see [Law78]), and from the sequence

$$0 \longrightarrow F_n(X) \longrightarrow \Omega_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0$$

we know that any n dimensional manifold with an open book decomposition represents $0 \in \overline{SK}_n(X)$, hence the result follows. $\square$

Remark 1.10. Note that not all manifolds with open book decomposition are zero in $SK_n(X)$. From the computation of I_n,

$$I_n = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

we know that $[S^{2k}, f] \neq 0 \in I_{2k}$ and consequently $[S^{2k}, f] \neq 0 \in SK_{2k}$. But S^{2k} has open book decomposition as,

$$\begin{aligned} S^{2k} &= S^{2k-2} \times D^2 \cup D^{2k-1} \times S^1 \\ &= S^{2k-2} \times D^2 \cup T(h : D^{2k-1} \to D^{2k-1}). \end{aligned}$$

Proposition 1.5.8. $SK_{2k+1}(X) = 0$.

Proof. This result follows directly by combining the computation of I_n, (which is zero for n odd), the result in proposition 1.5.7, and theorem 1.4.2. $\square$

Chapter 2

Algebraic Cutting and Pasting

In this chapter we will define the algebraic analogues of the cut and paste operations given in chapter 1. In particular the cutting and pasting ε-symmetric groups $SKL^*(A,\varepsilon)$ (where A is a ring with involution), which are the algebraic analogues of the $SK_*(X)$ groups.

2.1 The algebraic cut and paste semigroup

In what follows we are going to denote the set of n-dimensional symmetric Poincaré complexes by $\mathscr{L}^n(A,\varepsilon)$,

$$\mathscr{L}^n(A,\varepsilon) = \{n\text{-dimensional symmetric Poincaré complexes over} A\}$$

Definition 2.1.1. *Let $c_1 = (f_1 : C \longrightarrow D, (\delta_D\varphi, \varphi))$ and $c_2 = (f_2 : C \longrightarrow E, (\delta_E\varphi, \varphi))$ be n-dimensional ε-symmetric pairs. Also consider the self homotopy equivalence of the $(n-1)$-dimensional ε-symmetric complex (C,φ),*

$$(h,\chi) : (C,\varphi) \longrightarrow (C,\varphi).$$

The union of the symmetric pairs c_1 and c_2 results in an n-dimensional ε-symmetric Poincaré complex $c_1 \cup_{(h,\chi)} -c_2$ which is defined as follows,

$$\begin{aligned} c_1 \cup_{(h,\chi)} -c_2 &= (D \cup_h -E, \delta_D\varphi \cup_\chi \delta_E\varphi) \\ &= (f_1 h : C \longrightarrow D, (\delta_D\varphi + f_1\chi f_1^*, \varphi)) \cup (f_2 : C \longrightarrow E, (-\delta_E\varphi, -\varphi)) \end{aligned}$$

where the chain complex $D \cup_h -E$ is the following mapping cone,

$$D \cup_h -E = \mathscr{C}(\begin{pmatrix} f_1 h \\ f_2 \end{pmatrix} : C \longrightarrow D \oplus E),$$

with differentials,

$$\begin{pmatrix} d_D & (-)^r f_1 h & 0 \\ 0 & d_C & 0 \\ 0 & (-)^r f_2 & d_E \end{pmatrix} :$$

$$(D \cup_h E)_r = D_r \oplus C_{r-1} \oplus E_r \longrightarrow (D \cup_h E)_{r-1} = D_{r-1} \oplus C_{r-2} \oplus E_{r-1}$$

and the symmetric structure is given by

$$(\delta_D\varphi\cup_\chi -\delta_E\varphi) = \begin{pmatrix} \delta_D\varphi_s + f_1\chi_s f_1^* & 0 & 0 \\ (-)^{n-r}\varphi_s h^* f_1^* & (-)^{n-r+s+1}T_\varepsilon\varphi_{s-1} & 0 \\ 0 & (-)^s f_2\varphi_s & -\delta_E\varphi_s \end{pmatrix} :$$

$$(D\cup_h E)^{n-r+s} = D^{n-r+s}\oplus C^{n-r-1+s}\oplus E^{n-r+s} \longrightarrow (D\cup_h E)_r = D_r\oplus C_{r-1}\oplus E_r$$

Proposition 2.1.2. *Two n-dimensional symmetric Poincaré complexes (A,φ_A) and (B,φ_B) in $\mathscr{L}^n(A,\varepsilon)$ are SKL equivalent if there exist n-dimensional ε-symmetric Poincaré pairs $c_1 = (f_1 : C \longrightarrow D, (\delta_D\varphi,\varphi))$ and $c_2 = (f_2 : C \longrightarrow E, (\delta_E\varphi,\varphi))$ such that*

$$(A,\varphi_A) = c_1 \cup_{(h,\chi)} c_2 \qquad \text{and} \qquad (B,\varphi_B) = c_1 \cup_{(g,\rho)} c_2$$

Proof. Here we show that Proposition 2.1.2 establishes an equivalence relation on $\mathscr{L}^n(A,\varepsilon)$.

(i) Reflexive: Let $(A,\varphi_A) = c_1\cup_{(h,\chi)} c_2$, as $[c_1\cup_{(h,\chi)} c_2 \sim_{SKL} c_1\cup_{(h,\chi)} c_2]$ then (A,φ_A) is SKL equivalent to itself.

(ii) Symmetric: If $(A,\varphi_A)\sim_{SKL}(B,\varphi_B)$ then by definition, $(A,\varphi_A) = c_1\cup_{(h,\chi)} c_2$ and $(B,\varphi_B) = c_1\cup_{(g,\rho)} c_2$. And $c_1\cup_{(g,\rho)} c_2$ is equivalent to $c_1\cup_{(h,\chi)} c_2$ so this implies that $(B,\varphi_B)\sim_{SKL}(A,\varphi_A)$.

(iii) Transitive: Let c_1 and c_2 be Poincaré pairs such that

$$(A,\varphi_A) = c_1 \cup_{(h,\chi)} c_2 \qquad \text{and} \qquad (B,\varphi_B) = c_1 \cup_{(g,\rho)} c_2$$

and suppose that there exists an ε-symmetric Poincaré complex (P,φ_P) which is SKL equivalent to (B,φ_B). Then (P,φ_P) can be written as

$$(P,\varphi_P) = c_1\cup_{(j,\alpha)} c_2$$

But $c_1\cup_{(j,\alpha)} c_2 \sim_{SKL} c_1\cup_{(h,\chi)} c_2$ so that (P,φ_P) is also SKL equivalent to (A,φ_A).

□

Definition 2.1.3. ***(Algebraic cutting and pasting semigroup)*** *"$SKL^n(A,\varepsilon)$" is the semigroup of equivalence classes subject to the equivalence relation described in Proposition 2.1.2.*

Example 2.1.4. *From Proposition 2.1.2 we note that twisted doubles and untwisted doubles are equivalent in "$SKL^n(A,\varepsilon)$".*
That is, if $c_1 = (f : C \longrightarrow D, (\delta\varphi,\varphi))$ is a symmetric pair and there are self homotopy equivalences, $(h,\chi) : (C,\varphi) \longrightarrow (C,\varphi)$ and $(1,0) : (C,\varphi) \longrightarrow (C,\varphi)$, then,

$$c_1\cup_{(h,\chi)} c_1 = c_1\cup_{(1,0)} c_1 \in \text{"}SKL^n(A,\varepsilon)\text{"}$$

Here the algebraic twisted and untwisted doubles are defined as in 30.8 in [Ran98]:

A ***twisted double*** $c_1 \cup_{(h,\chi)} c_1$ *of* $c_1 = (f : C \longrightarrow D, (\delta\varphi, \varphi))$ *is,*

$$\begin{aligned} c_1 \cup_{(h,\chi)} -c_1 &= (D \cup_h D, \delta\varphi \cup_\chi \delta\varphi) \\ &= (fh : C \longrightarrow D, (\delta\varphi + f\chi f^*)) \cup (f : C \longrightarrow D, (-\delta\varphi, -\varphi)) \end{aligned}$$

where the chain complex $D \cup_h D$ *is*

$$D \cup_h D = \mathscr{C}(\begin{pmatrix} fh \\ f \end{pmatrix} : C \longrightarrow D \oplus D),$$

with differentials,

$$\begin{pmatrix} d_D & (-)^r fh & 0 \\ 0 & d_C & 0 \\ 0 & (-)^r f & d_D \end{pmatrix} :$$

$$(D \cup_h D)_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow (D \cup_h D)_{r-1} = D_{r-1} \oplus C_{r-2} \oplus D_{r-1}$$

and the symmetric structure is given by

$$(\delta\varphi \cup_\chi -\delta\varphi) = \begin{pmatrix} \delta\varphi_s + f\chi_s f^* & 0 & 0 \\ (-)^{n-r}\varphi_s h^* f^* & (-)^{n-r+s+1} T_\varepsilon \varphi_{s-1} & 0 \\ 0 & (-)^s f\varphi_s & -\delta\varphi_s \end{pmatrix} :$$

$$(D \cup_h D)^{n-r+s} = D^{n-r+s} \oplus C^{n-r-1+s} \oplus D^{n-r+s} \longrightarrow (D \cup_h D)_r = D_r \oplus C_{r-1} \oplus D_r$$

The ***untwisted double*** $c_1 \cup_{(1,0)} c_1$ *of* $c_1 = (f : C \longrightarrow D, (\delta\varphi, \varphi))$ *is,*

$$\begin{aligned} c_1 \cup_{(1,0)} -c_1 &= (D \cup_{1:C \longrightarrow C} D, \delta\varphi \cup_0 \delta\varphi) \\ &= (f : C \longrightarrow D, (\delta\varphi, \varphi)) \cup (f : C \longrightarrow D, (-\delta\varphi, -\varphi)) \end{aligned}$$

where the chain complex $D \cup_{1:C \longrightarrow C} D$ *is*

$$D \cup_1 D = \mathscr{C}(\begin{pmatrix} f \\ f \end{pmatrix} : C \longrightarrow D \oplus D),$$

with differentials,

$$\begin{pmatrix} d_D & (-)^r f & 0 \\ 0 & d_C & 0 \\ 0 & (-)^r f & d_D \end{pmatrix} :$$

$$(D \cup_h D)_r = D_r \oplus C_{r-1} \oplus D_r \longrightarrow (D \cup_h D)_{r-1} = D_{r-1} \oplus C_{r-2} \oplus D_{r-1}$$

and the symmetric structure is given by

$$(\delta\varphi \cup_0 -\delta\varphi) = \begin{pmatrix} \delta\varphi_s & 0 & 0 \\ (-)^{n-r}\varphi_s f^* & (-)^{n-r+s+1} T_\varepsilon \varphi_{s-1} & 0 \\ 0 & (-)^s f\varphi_s & -\delta\varphi_s \end{pmatrix} :$$

$$(D \cup_h D)^{n-r+s} = D^{n-r+s} \oplus C^{n-r-1+s} \oplus D^{n-r+s} \longrightarrow (D \cup_h D)_r = D_r \oplus C_{r-1} \oplus D_r$$

2.2 The behaviour of algebraic surgery in the cut and paste category.

In this section we will present some results which are algebraic analogues of those given in Section 1.2. Like in that section, these results will allow us to give a general formula for inverses in "$SKL^n(A,\varepsilon)$", and we will also see that in "$SKL^n(A,\varepsilon)$" the result of algebraic surgery is independent of the embedding.

Proposition 2.2.1. *Consider the following symmetric pairs:*

$$c_1 = (f_1 : C \longrightarrow D, (\delta_D\varphi, \varphi))$$
$$c_2 = (f_2 : C \longrightarrow E, (\delta_E\varphi, \varphi))$$
$$c_3 = (f_3 : C \longrightarrow F, (\delta_F\varphi, \varphi))$$

then,

$$\boxed{(c_1 \cup_{(h,\chi)} c_2) \oplus (c_2 \cup_{(g,\rho)} c_3) = (c_1 \cup_{(j,\alpha)} c_3) \oplus (c_2 \cup_{(1,0)} c_2) \in \text{"}SKL^n(A,\varepsilon)\text{"}}$$

Proof.

$$\begin{aligned}
(c_1 \cup_{(h,\chi)} c_1) \oplus (c_2 \cup_{(g,\rho)} c_3) &= [(D \cup_h E), \delta_D\varphi \cup_\chi \delta_E\varphi] \oplus [(E \cup_g F), \delta_E\varphi \cup_\rho \delta_F\varphi] \\
&= [(D \cup_h E) \oplus (E \cup_g F), (\delta_D\varphi \cup_\chi \delta_E\varphi) \oplus (\delta_E\varphi \cup_\rho \delta_F\varphi)] \\
&= [(D \oplus E) \cup_{h+g} (E \oplus F), (\delta_D\varphi \oplus \delta_E\varphi) \cup_{\chi+\rho} (\delta_E\varphi \oplus \delta_F\varphi)] \\
&= [(D \oplus E) \cup_{j+1} (E \oplus F), (\delta_D\varphi \oplus \delta_E\varphi) \cup_\alpha (\delta_E\varphi \oplus \delta_F\varphi)) \\
&= [(D \cup_j F) \oplus (E \cup_1 E), (\delta_D\varphi \cup_\alpha \delta_F\varphi) \oplus (\delta_E\varphi \cup_0 \delta_E\varphi)] \\
&= (D \cup_j F, \delta_D\varphi \cup_\alpha \delta_F\varphi) \oplus (E \cup_1 E, \delta_E\varphi \cup_0 \delta_E\varphi) \\
&= (c_1 \cup_{(j,\alpha)} c_3) \oplus (c_2 \cup_{(1,0)} c_2) \in \text{"}SKL^n(A,\varepsilon)\text{"}
\end{aligned}$$

□

Proposition 2.2.2. *In this proposition we give an algebraic analog of the result in proposition 1.2.2*

$$(C(M), \varphi_M) \oplus (C(S^n), \varphi_{S^n}) = (C(M'), \varphi_{M'}) \oplus (C(S^k \times S^{n-k}), \varphi_{S^k \times S^{n-k}}) \in \text{"}SKL^n(A,\varepsilon)\text{"}$$

Proof. Here we first note that geometrically the input of surgery, M is the union of the framed embedding $U = S^k \times D^{n-k}$ and the complement of this embedding in

M, which we call $M_0 = \overline{M\backslash U}$. This complement M_0 is not modified in the surgery process, but the framed embedding U is substituted by $U' = D^{k+1} \times S^{n-k-1}$. Note that $\partial U = \partial U'$. So the result of surgery is $M' = M_0 \cup U'$. See the Appendix (3) for a more detailed explanation. The geometric situation is therefore as follows:

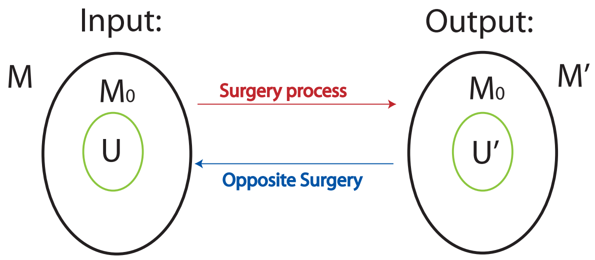


Figure 2.1: Input and Output of geometric surgery

In algebraic surgery an analogous situation can be described. Here we define the following corresponding symmetric pairs:

$$\begin{aligned} c_1 &= (f_1 : C(S^k \times S^{n-k-1}) \longrightarrow C(M_0), (\delta_{C(M_0)}\varphi, \varphi)) \\ c_2 &= (f_2 : C(S^k \times S^{n-k-1}) \longrightarrow C(U), (\delta_{C(U)}\varphi, \varphi)) \\ c_3 &= (f_2 : C(S^k \times S^{n-k-1}) \longrightarrow C(U'), (\delta_{C(U')}\varphi, \varphi)) \end{aligned}$$

The strategy now will be to use the identity in Proposition 2.2.1 gluing together the pairs c_1, c_2 and c_3 that have just been defined.

Gluing together c_1 and c_2:
We are now going to glue c_1 and c_2 along $C(S^k \times S^{n-k-1})$, using a self homotopy equivalence of $(C(S^k \times S^{n-k-1}), \varphi_{C(M)})$,

$$(h, \chi) : (C(S^k \times S^{n-k-1}), \varphi) \longrightarrow (C(S^k \times S^{n-k-1}), \varphi)$$

We get:

$$(C(M), \varphi_{C(M)}) = c_1 \cup_{(h,\chi)} -c_2 = (C(M_0) \cup_h C(U), \delta\varphi_{C(M_0)} \cup_\chi \delta\varphi_{C(U)})$$

where the chain complex $C(M_0) \cup_h C(U)$ is

$$C(M_0) \cup_h C(U) = \mathscr{C}(\begin{pmatrix} f_1 h \\ f_2 \end{pmatrix} : C(S^k \times C^{n-k-1}) \longrightarrow C(M_0) \oplus C(U)),$$

This double mapping cone has differentials,

$$\begin{pmatrix} d_{C(M_0)} & (-)^r f_1 h & 0 \\ 0 & d_{C(S^k \times C^{n-k-1})} & 0 \\ 0 & (-)^r f & d_{C(U)} \end{pmatrix} :$$

$(C(M_0) \cup_h C(U))_r = C(M_0)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U)_r \longrightarrow (C(M_0) \cup_h C(U))_{r-1} = C(M_0)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U)_{r-1}$

Gluing together c_1 and c_3:
In a similar way, we are now going to glue

$$\begin{aligned} c_1 &= (f_1 : C(S^k \times S^{n-k-1}) \longrightarrow C(M_0), (\delta_{C(M_0)}\varphi, \varphi)) \\ c_3 &= (f_2 : C(S^k \times S^{n-k-1}) \longrightarrow C(U'), (\delta_{C(U')}\varphi, \varphi)) \end{aligned}$$

along the chain complex $C(S^k \times S^{n-k-1})$, using another self homotopy equivalence of $(C(S^k \times S^{n-k-1}), \varphi)$,

$$(j, \alpha) : (C(S^k \times S^{n-k-1}), \varphi) \longrightarrow (C(S^k \times S^{n-k-1}), \varphi)$$

We get:

$$(C(M'), \varphi_{C(M')}) = c_1 \cup_{(h,\chi)} -c_3 = (C(M_0) \cup_h C(U'), \delta\varphi_{C(M_0)} \cup_\chi \delta\varphi_{C(U')})$$

where the chain complex $C(M_0) \cup_h C(U')$ is

$$C(M_0) \cup_h C(U') = \mathscr{C}(\begin{pmatrix} f_1 j \\ f_3 \end{pmatrix} : C(S^k \times C^{n-k-1}) \longrightarrow C(M_0) \oplus C(U')),$$

The differentials for this double mapping cone are,

$$\begin{pmatrix} d_{C(M_0)} & (-)^r f_1 j & 0 \\ 0 & d_{C(S^k \times C^{n-k-1})} & 0 \\ 0 & (-)^r f_3 & d_{C(U')} \end{pmatrix} :$$

$(C(M_0) \cup_h C(U'))_r = C(M_0)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U')_r \longrightarrow (C(M_0) \cup_h C(U'))_{r-1} = C(M_0)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U')_{r-1}$

Remark 2.1. Note that this differential is essentially the same as the more usual form given for the differential of the result of algebraic surgery, as given in section 3 of [Ran01]. For a detailed explanation of this and a detailed explanation of the algebraic interpretation of M_0 see the Appendix 3

Gluing together c_2 and c_3:
We will now glue together the pairs c_2 and c_3

$$\begin{aligned} c_1 &= (f_1 : C(S^k \times S^{n-k-1}) \longrightarrow C(U), (\delta_{C(U)}\varphi, \varphi)) \\ c_3 &= (f_2 : C(S^k \times S^{n-k-1}) \longrightarrow C(U'), (\delta_{C(U')}\varphi, \varphi)) \end{aligned}$$

using the self homotopy equivalence of $(C(S^k \times S^{n-k-1}), \varphi)$,

$$(g, \rho) : (C(S^k \times S^{n-k-1}), \varphi) \longrightarrow (C(S^k \times S^{n-k-1}), \varphi)$$

We get:

$$(C(S^n), \varphi_{C(S^n)}) = c_2 \cup_{(g,\rho)} -c_3 = (C(U) \cup_h C(U'), \delta_{C(U)}\varphi \cup_\chi \delta_{C(U')}\varphi)$$

where the chain complex $C(U) \cup_h C(U')$ is

$$C(U) \cup_h C(U') = \mathscr{C}(\begin{pmatrix} f_2 g \\ f_3 \end{pmatrix} : C(S^k \times C^{n-k-1}) \longrightarrow C(U) \oplus C(U')),$$

The differentials for this double mapping cone are,

$$\begin{pmatrix} d_{C(U)} & (-)^r f_1 j & 0 \\ 0 & d_{C(S^k \times C^{n-k-1})} & 0 \\ 0 & (-)^r f_3 & d_{C(U')} \end{pmatrix} :$$

$(C(M_0) \cup_h C(U'))_r = C(M_0)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U')_r \longrightarrow (C(M_0) \cup_g C(U'))_{r-1} = C(M_0)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U')_{r-1}$

Gluing together c_2 and c_2: We will now write down the algebraic untwisted double of c_2 :

$$(C(S^k \times S^{n-k}), \varphi_{C(S^k \times S^{n-k})}) = c_2 \cup_{(1,0)} -c_2 = (C(U) \cup_1 C(U), \delta_{C(U)}\varphi \cup_0 \delta_{C(U)}\varphi)$$

where the chain complex $C(U) \cup_1 C(U)$ is

$$C(U) \cup_1 C(U) = \mathscr{C}(\begin{pmatrix} f_2 \\ f_2 \end{pmatrix} : C(S^k \times C^{n-k-1}) \longrightarrow C(U) \oplus C(U)),$$

The differentials for this double mapping cone are,

$$\begin{pmatrix} d_{C(U)} & (-)^r f_2 & 0 \\ 0 & d_{C(S^k \times C^{n-k-1})} & 0 \\ 0 & (-)^r f_2 & d_{C(U)} \end{pmatrix} :$$

$(C(U) \cup_1 C(U))_r = C(U)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U)_r \longrightarrow (C(U) \cup_1 C(U))_{r-1} = C(U)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U)_{r-1}$

We now apply the identity in Proposition 2.2.1

For the LHS we have,

$$(C(M), \varphi_M) \oplus (C(S^n), \varphi_{S^n}) = (c_1 \cup -c_2) \oplus (c_2 \cup -c_3)$$

which has differentials

$$\begin{pmatrix} d_{C(M_0)} & (-)^r f_1 h & 0 & 0 & 0 & 0 \\ 0 & d_{C(S^k \times S^{n-k-1})} & 0 & 0 & 0 & 0 \\ 0 & (-)^r f_2 & d_{C(U)} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{C(U)} & (-)^r f_2 g & 0 \\ 0 & 0 & 0 & 0 & d_{C(S^k \times S^{n-k-1})} & 0 \\ 0 & 0 & 0 & 0 & (-)^r f_3 & d_{C(U')} \end{pmatrix} :$$

$$[C(M_0)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U)_r] \oplus [C(U)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U')_r] \longrightarrow$$
$$[C(M_0)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U)_{r-1}] \oplus [C(U)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U')_{r-1}]$$

Similarly for the RHS we have,

$$(C(M'), \varphi_{M'}) \oplus (C(S^n), \varphi_{S^n}) = (c_1 \cup -c_2) \oplus (c_2 \cup -c_3)$$

which has differentials

$$\begin{pmatrix} d_{C(M_0)} & (-)^r f_1 g & 0 & 0 & 0 & 0 \\ 0 & d_{C(S^k \times S^{n-k-1})} & 0 & 0 & 0 & 0 \\ 0 & (-)^r f_3 & d_{C(U')} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{C(U)} & (-)^r f_2 & 0 \\ 0 & 0 & 0 & 0 & d_{C(S^k \times S^{n-k-1})} & 0 \\ 0 & 0 & 0 & 0 & (-)^r f_2 & d_{C(U)} \end{pmatrix} :$$

$$[C(M_0)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U')_r] \oplus [C(U)_r \oplus C(S^k \times S^{n-k-1})_{r-1} \oplus C(U')_r] \longrightarrow$$
$$[C(M_0)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U)_{r-1}] \oplus [C(U)_{r-1} \oplus C(S^k \times S^{n-k-1})_{r-2} \oplus C(U)_{r-1}]$$

So the LHS and the RHS are chain equivalent in "$SKL^n(A, \varepsilon)$", and hence the result follows.

$\square$

Proposition 2.2.3. *In "$SKL^n(A, \varepsilon)$",*

$$(C(S^k \times S^{n-k}), \varphi_{(S^k \times S^{n-k})}) \simeq \begin{cases} (C(S^n), \varphi_{S^n}) \oplus (C(S^n), \varphi_{S^n}) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Proof. In this proof we use Proposition 2.2.2 with $(C, \varphi) = (C(S^n), \varphi_{S^n})$. $(C(S^n), \varphi_{S^n})$ is chain equivalent to $[(\mathbb{Z} \oplus S^n\mathbb{Z}), \varphi]$, and consequently, $(C', \varphi') \simeq [(\mathbb{Z} \oplus S^{k+1}\mathbb{Z}) \oplus (S^{n-k-1}\mathbb{Z} \oplus S^n\mathbb{Z}), \varphi]$. Substituting this in Proposition 2.2.2 and taking $k = 0$, we find that $[(\mathbb{Z} \oplus S^1\mathbb{Z} \oplus S^{n-1}\mathbb{Z} \oplus S^n\mathbb{Z}), \varphi] \simeq 0 \in$ "$SKL^n(A, \varepsilon)$" and the result follows by induction. $\square$

Proposition 2.2.4. *Let (C, φ) be a symmetric Poincaré complex and (C', φ') the result after algebraic surgery on (C, φ) with data $(f : C \longrightarrow D, (\delta\varphi, \varphi))$. The trace of such an algebraic surgery is the $(n+1)$-dimensional symmetric Poincaré cobordism between (C, φ) and (C', φ') is $((f f') : C \oplus C' \longrightarrow D', (0, \varphi \oplus \varphi'))$. An algebraic description of Corollary 1.8 in [Kre73] is as follows:*

$$(C, \varphi) = (C', \varphi') - (\chi(D') - \chi(C))[(C(S^n), \varphi_{S^n})]$$

.

Proof. In general an algebraic surgery can be broken down into a sequence of elementary surgeries, subject to a K-theoretic restriction [2]. In this case we are

[2]See Proposition 4.7 (iii) in [Ran80]

dealing with free modules, so this restriction does not exist and the algebraic surgery is a composition of elementary surgeries. So we first consider D' to be the trace of an elementary surgery of type $(k, n-k-1)$ on a $(n+1)$-dimensional ε-symmetric pair $(f : C \longrightarrow D, (\delta\varphi, \varphi))$ over A. Then from Proposition 2.2.2 and Proposition 2.2.3, we can write,

$$\begin{aligned} (C, \varphi) =& (C', \varphi') \oplus [C(S^n), \varphi] && \text{for } k \text{ even} \\ (C, \varphi) \oplus & [C(S^n), \varphi] = (C', \varphi') && \text{for } k \text{ odd} \end{aligned}$$

So like in the geometric case, we need to see that $\chi(D') - \chi(C) = (-1)^{k+1}$.

In the case of an algebraic elementary surgery, $D_* = S^{n-k}A$, which is concentrated in dimension $(n-k)$, and $D^{n-*+1} = S^{k+1}A$. So that,

$$\chi(D^{n-*+1}) = \sum_{r=0}^{\infty} (-)^r rank_A(D^{n-*+1}) = (-1)^{k+1}$$

Since $D'_r = C_r \oplus D^{n-r+1}$ then

$$\chi(D') - \chi(C) = (-1)^{k+1}$$

Hence the result follows.

For a wider explanation on the effect of algebraic surgery on a symmetric Poincaré complex see the Appendix 3.

□

By algebraic analogy with the expression for the inverses in $SK_n(X)$ (given in Proposition 1.2.5), the inverses in $SKL^n(A, \varepsilon)$ are as follows,

Definition 2.2.5. *The inverse in $SKL^n(A, \varepsilon)$ of an n-dimensional symmetric Poincaré complex (C, φ) is*

$$(C, -\varphi) - \chi(C)[(C(S^n), \varphi_{S^n}]$$

where the Euler characteristic of the chain complex is given by

$$\chi(C) = \sum_{r=0}^{\infty} (-)^r rank_A(C_r) \in \mathbb{Z}$$

The algebraic semigroup "$SKL^n(A, \varepsilon)$" contains inverses and is an abelian group. Nevertheless this group is not 4-periodic since the double skew suspension maps,

$$\begin{aligned} "SKL^n(A, \varepsilon)" &\xrightarrow{\overline{S}^2} "SKL^{n+4}(A, \varepsilon)" \\ (C, \varphi) &\longmapsto (\overline{S}^2 C, \overline{S}^2 \varphi) \end{aligned}$$

and $\overline{S}^2 C(S^n) \neq C(S^{n+4})$. To avoid this we are make appropriate identifications in the definition of the $SKL^n(A, \varepsilon)$ group.

Definition 2.2.6. *The algebraic semigroup "$SKL^n(A,\varepsilon)$" contains inverses and is an abelian group, by identifying $C(S^n)$ and $C(S^{n+4})$ we obtain the 4-periodic $SKL^n(A,\varepsilon)$ group,*

$$SKL^n(A,\varepsilon) = \text{"}SKL^n(A,\varepsilon)\text{"}/C(S^n) \sim C(S^{n+4}).$$

2.3 Relation between algebraic cutting and pasting and ε-symmetric L-theory

In this section we are going to define exact sequences analogous to those presented in Section 1.4. To this purpose, we are first going to define the cut and paste ε-symmetric algebraic bordism groups $\overline{SK}L^n(A,\varepsilon)$. These groups are defined in Remark 30.30 in [Ran98]. (Note that the notation used in [Ran98] differs slightly from the notation we use in this report, so that the groups which we denote by $\overline{SK}L^n$ are called SKL^n there).

Remark 2.2. The twisted double L-groups $DBL^*(A,\varepsilon)$ which feature in the next definitions, are discussed in section 30D of [Ran98]. $DBL^n(A,\varepsilon)$ is the cobordism group of $(n+1)$-dimensional ε-symmetric Poincaré complexes over A with twisted double structure. These groups are the algebraic analogues of the twisted double bordism groups $DB_*(X)$.

Definition 2.3.1. *(from Remark 30.30 in [Ran98]) The ε-symmetric cut and paste $\overline{SK}L$ groups are the quotients of the ε-symmetric L-groups*

$$\overline{SK}L^n(A,\varepsilon) = L^n(A,\varepsilon)/\sim$$

by the equivalence relation generated by

$$C \cup_f -D \sim C \cup_g -D$$

for n-dimensional ε-symmetric Poincaré pairs $(C,\partial C)$, $(D,\partial D)$ and homotopy equivalences $f,g : \partial C \longrightarrow \partial D$.

Definition 2.3.2. *The groups $FL^n(A,\varepsilon) \subseteq L^n(A,\varepsilon)$ are the algebraic analogues of the groups $F_n(X) \subseteq \Omega(X)$, which are defined in Theorem (1.3c) in [Kre73].*

$$FL^n(A,\varepsilon) = \mathrm{im}(D : DBL^{n-1}(A,\varepsilon) \longrightarrow L^n(A,\varepsilon))$$

Proposition 2.3.3. *(i) The groups from definitions 2.3.2 and 2.3.1 fit into the following short exact sequence,*

$$0 \longrightarrow FL^n(A,\varepsilon) \longrightarrow L^n(A,\varepsilon) \longrightarrow \overline{SK}L^n(A,\varepsilon) \longrightarrow 0$$

(ii) (From Remark 30.30 in [Ran98]) The ε-symmetric $\overline{SK}L^n(A,\varepsilon)$ groups are

the images of the ε-symmetric L-groups in the asymmetric L-groups

$$\begin{aligned}\overline{SK}L^n(A,\varepsilon) &= \operatorname{coker}(D : DBL^{n-1}(A,\varepsilon) \longrightarrow L^n(A,\varepsilon)) \\ &= \operatorname{im}(L^n(A,\varepsilon) \longrightarrow LAsy^n(A))\end{aligned}$$

with $\overline{SK}L^{2*+1}(A,\varepsilon) = 0$.

Proof. (i) Similarly to the situation in the geometric case, $FL^n(A,\varepsilon)$ is a subgroup of $L^n(A,\varepsilon)$ that can be identified as the group of algebraic mapping tori. The proof of exactness of the sequence is in two stages, just as the proof of Theorem 1.4.4. Consider the ε-symmetric Poincaré pair $(C,\partial C)$, and let $T_A(h,\chi)$ be the A-coefficient mapping torus of $(h,\chi) : (\partial C,\varphi) \longrightarrow (\partial C,\varphi)$, as it is defined in (24.3) [Ran98]. First we want to show that every algebraic mapping torus is a zero in $\overline{SK}L^n(A,\varepsilon)$. By the definition of $\overline{SK}L^n(A,\varepsilon)$, we know that for any self homotopy equivalence $(h,\chi) : (\partial C,\varphi) \longrightarrow (\partial C,\varphi)$, $T_A(h,\chi)$ will be equivalent to the A-coefficient algebraic mapping torus of $1 : (\partial C,\varphi) \longrightarrow (\partial C,\varphi)$, which we denote by $T_A(\partial C,\varphi)$, and

$$T_A(\partial C,\varphi) = (\partial C,\varphi) \otimes \sigma^*(S^1;\mathbb{Z}) = (\partial C \oplus \partial C_{*-1},\theta)$$

is a null-cobordism with
$\theta_s = \begin{pmatrix} 0 & (-)^s\varphi_s \\ (-)^{n-r-2}\varphi_s & (-)^{n-1-r+s}T_\varepsilon\varphi_{s-1} \end{pmatrix} : \partial C^{n-1-r+s} \oplus \partial C^{n-2-r+s} \longrightarrow \partial C_r \oplus \partial C_{r-1}$
Hence $FL^n(A,\varepsilon) \subseteq \operatorname{Ker}(L^n(A,\varepsilon) \longrightarrow \overline{SK}L^n(A,\varepsilon))$.

Now we need to show the reverse inclusion, $\operatorname{Ker}(L^n(A,\varepsilon) \longrightarrow \overline{SK}L^n(A,\varepsilon)) \subseteq FL^n(A,\varepsilon)$. This kernel is generated by classes of the form $(E,\theta) - (E',\theta')$, where (E',θ') is obtained from (E,θ) by cutting and pasting. That is, if $E = C \cup_f -D$ then $E' = C \cup_g -D$. A cobordism between them give a pair of pants:

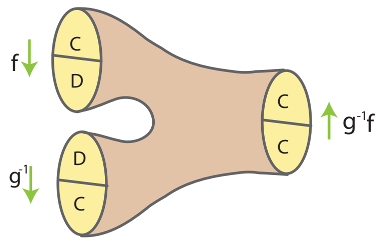


Figure 2.2: Cobordism between cut and paste equivalent Poincaré complexes and algebraic twisted doubles

For the definition of a twisted double cobordism between a twisted double $(C \cup_h C, \delta\varphi \cup_\chi \delta\varphi)$ and the A-coefficient algebraic mapping torus $T_A(h,\chi)$ see the proof of Proposition 30.20 (ii) in [Ran98].

Since the algebraic twisted double is cobordant to the A-coefficient mapping torus, then the two cut and paste equivalent ε-symmetric Poincaré complexes $(E,\theta) = (C \cup_f -D,\theta)$ and $(E',\theta') = (C \cup_g -D,\theta')$ are cobordant to the A-coefficient algebraic twisted double $T_A(h,\chi)$.

(ii) This follows directly from the proof part (i) of this Proposition and Proposition 30.11 in [Ran98].

□

Definition 2.3.4. *The algebraic analog of $I_n \subseteq SK_n(X)$ is $IL^n \subseteq SKL^n(A,\varepsilon)$, which is the subgroup of cut and paste classes of n-dimensional ε-symmetric Poincaré complexes such that,*

$$IL^n = \{[(C,\varphi)] \in SKL^n(A,\varepsilon) \mid \sigma((C,\varphi)) = 0\}$$

Proposition 2.3.5. (i) *The algebraic cutting and pasting group $SKL^n(A,\varepsilon)$ fits into the short exact sequence,*

$$0 \longrightarrow IL^n \longrightarrow SKL^n(A,\varepsilon) \longrightarrow \operatorname{im}(L^n(A,\varepsilon) \to LAsy^n(A,\varepsilon)) \longrightarrow 0$$

(ii) *This short exact sequence splits, so that,*

$$SKL^n(A,\varepsilon) \cong IL^n \oplus \operatorname{im}(L^n(A,\varepsilon) \to LAsy^n(A,\varepsilon))$$

Proof. The proofs of both (i) and (ii) are similar to the geometric case.

□

2.4 Computations of cut and paste L-theoretic groups

Proposition 2.4.1. *The algebraic cut and paste bordism groups $\overline{SK}L^*(\mathbb{Z})$ are given by the following computation,*

$$\overline{SK}L^n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 0 \pmod 4 \\ 0 & \text{othervise} \end{cases}$$

Proof. The $\overline{SK}L^*(\mathbb{Z})$ groups fit into the short exact sequence,

$$0 \longrightarrow FL^n \longrightarrow L^n(\mathbb{Z}) \longrightarrow \overline{SK}L^n(\mathbb{Z}) \longrightarrow 0$$

The computation of the symmetric $L^n(\mathbb{Z})$ groups is

$$L^n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 0 \pmod 4 \\ \mathbb{Z}_2 & \text{for } n \equiv 1 \pmod 4 \\ 0 & \text{for } n \equiv 2 \pmod 4 \\ 0 & \text{for } n \equiv 3 \pmod 4 \end{cases}$$

In the proof of Proposition 4.3.1 in [Ran81] it is explained that for $n \equiv 0 \pmod 4$, the generator of the symmetric L-group $L^0(\mathbb{Z}) = \mathbb{Z}$ is represented by the non-singular symmetric form over $\mathbb{Z}$, $(\mathbb{Z}, 1 \in Q^{+1}(\mathbb{Z}))$ of signature $1 \in \mathbb{Z}$. And for $n \equiv 1 \pmod 4$, the generator of $L^1(\mathbb{Z}) = \mathbb{Z}_2$ is represented by the non-singular symmetric formation over $\mathbb{Z}$ of deRham invariant $1 \in \mathbb{Z}_2$.

$$\left(\mathbb{Z}\oplus\mathbb{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in Q^+(\mathbb{Z}\oplus\mathbb{Z}); \left(\text{im}\begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\right), \left(\text{im}\begin{pmatrix} 1 \\ -2 \end{pmatrix} : \mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\right)\right)$$

For the computation of $\overline{SKL}^*(\mathbb{Z})$ we observe that:

For $n \equiv 0 \pmod 4$, the signature map $L^0(\mathbb{Z}) \xrightarrow{\sigma} \overline{SKL}^0(\mathbb{Z})$ sends the generator of $L^0(\mathbb{Z})$ to 1, which generates $\mathbb{Z}$. Hence $\overline{SKL}^0(\mathbb{Z}) = \mathbb{Z}$.

For $n \equiv 1 \pmod 4$, note that from proposition 2.3.3 we have the exact sequence

$$0 \longrightarrow FL^1(\mathbb{Z}) \xrightarrow{\cong} L^1(\mathbb{Z}) \longrightarrow (\text{im}(L^1(\mathbb{Z}) \to LAsy^1(Z))) \longrightarrow 0,$$

But in general, $LAsy^n(A)$ is two periodic and $LAsy^{(2*+1)} = 0$ so the map

$$DBL^{2i}(A,\varepsilon) \longrightarrow L^{2i+1}(A,\varepsilon)$$

is surjective so in particular

$$FL^1(\mathbb{Z}) = im(D : DBL^0(\mathbb{Z}) \longrightarrow L^1(\mathbb{Z})) \cong L^1(\mathbb{Z})$$

and hence in the sequence,

$$0 \longrightarrow FL^1(\mathbb{Z}) \xrightarrow{\cong} L^1(\mathbb{Z}) \longrightarrow \overline{SKL}^1(\mathbb{Z}) \longrightarrow 0,$$

$\overline{SKL}^1(\mathbb{Z})$ is zero.

For $n \equiv 2, 3 \pmod 4$, we observe that the symmetric L-groups over $\mathbb{Z}$ are 0. From this it follows directly that both $\overline{SKL}^{4k+2}(\mathbb{Z})$ and $\overline{SKL}^{4k+3}(\mathbb{Z})$ are zero. $\square$

Proposition 2.4.2. *The algebraic cutting and pasting groups of $\mathbb{Z}$ are given by,*

$$SKL^n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{for } n \equiv 0 \pmod 4 \\ 0 & \text{for } n \equiv 1 \pmod 4 \\ \mathbb{Z} & \text{for } n \equiv 2 \pmod 4 \\ 0 & \text{for } n \equiv 3 \pmod 4 \end{cases}$$

Proof. For $n \equiv 0 \pmod 4$:

Consider the sequence,

$$0 \longrightarrow IL^0 \longrightarrow SKL^0(\mathbb{Z}) \longrightarrow \overline{SKL}^0(\mathbb{Z}) \longrightarrow 0$$

IL^0 is generated by $[C(S^0)]$ which is the non-singular symmetric form

$$(\mathbb{Z}\oplus\mathbb{Z}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \in SKL^0(\mathbb{Z})$$

From proposition 2.4.1, we know that $\overline{SKL}^0(\mathbb{Z}) = \mathbb{Z}$ and is generated by the non-singular symmetric form $(\mathbb{Z}, 1 \in Q^+(\mathbb{Z}))$. From Proposition 2.3.5 we know that the sequence splits, so that there is an isomorphism

$$(\chi - \sigma)/2 \oplus \sigma : SKL^0(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

For $n \equiv 1, 3 \pmod 4$, both IL^n and $\overline{SK}L^n(\mathbb{Z})$ are zero so $SKL^n(\mathbb{Z})$ is also zero in this case.

For $n \equiv 2 \pmod 4$, $\overline{SK}L^n(\mathbb{Z}) = 0$ which implies that there is an isomorphism

$$\chi/2 : SKL^n(\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

So this gives us the full computation of the 4-periodic $SKL^n(\mathbb{Z})$ group.

$\square$

Chapter 3

Further ideas

At this point, it would be interesting to investigate the following ideas:

(i) As mentioned before, the reduced SK groups are obstruction groups for the multiplicativity of the signature. At this point it would be interesting to investigate if an algebraic analog of this statement is possible.

(ii) We have given the computation of the $SKL^n(A, \varepsilon)$ groups when $A = \mathbb{Z}$. It would be interesting to have a computation of these groups for other rings A. In particular, how can $SKL^n(\mathbb{Z}[\pi_1(X)])$ be computed?

(iii) It would be interesting to define the algebraic analog of the SKK groups.

(iv) The SK and the SKK groups are now being studied by M. Kreck, jointly with P. Teichner. They are relating the idea of these groups to TQFT, but to my knowledge their recent results have not yet been published, so I am looking forward to attend a talk by M. Kreck in the topology seminar in Edinburgh at the beginning of October.

Appendix A

Surgery "dictionary"

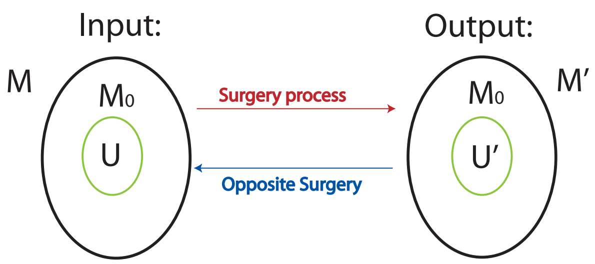
A.1 Geometric surgery

- **Input:**
  - **Manifold M:** An n-dimensional manifold M.
  - **Framed embedding:** $U = S^k \times D^{n-k} \subset M$
  - **Complement of embedding:** $M_0 = \overline{(M \backslash S^k \times D^{n-k})}$.
- **Output:**
  - **Effect of surgery:** The effect of surgery on $S^k \times D^{n-k} \subset M$ is the n-dimensional manifold given by,

$$M' = M_0 \cup D^{k+1} \times S^{n-k-1}$$

  - **Dual framed embedding:** $U' = D^{k+1} \times S^{n-k-1} \subset M'$
  - **Complement of the dual embedding:** $M_0' = \overline{(M' \backslash D^k \times S^{n-k-1})}$. Note that the complement in M' of the dual framed embedding is also M_0, i.e,

$$M_0' = M_0 = \overline{M \backslash S^k \times D^{n-k}}$$



- **Trace of surgery:** The trace of the surgery is the cobordism $(W; M, M')$ given by attaching a $(k+1)$ handle at $S^k \times D^{n-k} \subset M$, so that,

$$W = M \times I \cup D^{k+1} \times D^{n-k}$$

  Also note the following homotopy equivalences,

$$W \simeq M \cup_x D^{k+1} \simeq M' \cup_{x'} D^{n-k}$$

  where $x : S^k \longrightarrow M$ is the inclusion $S^k \times \{0\} \subset S^k \times D^{n-k} \subset M$, and similarly $x' : S^{n-k-1} \longrightarrow M'$ is the inclusion $S^{n-k-1} \times \{0\} \subset S^k \times D^{n-k} \subset M$

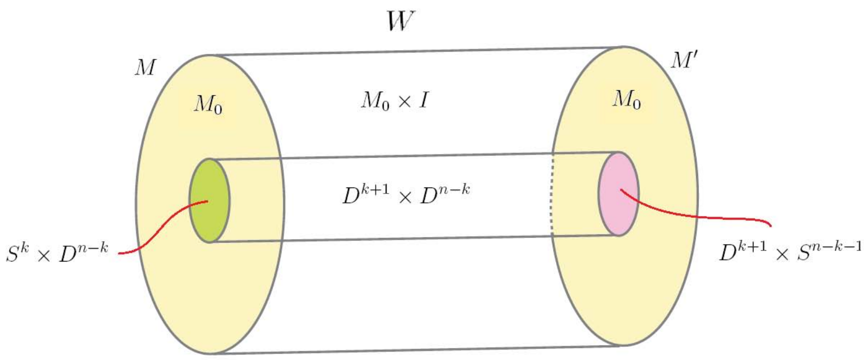


Figure A.1: Cobordism of Surgeries

- **Homology effect:** the homology effect of surgery is to kill $x \in H_k(M)$ so that $H_k(W) = H_k(M)/\langle x \rangle$ with $\langle x \rangle \subseteq H_k(M)$ is the subgroup generated by x.

Braids of exact sequences relating these chain complexes:

- **Braid 1**

$$\begin{tikzcd}
H_r(M_0) \arrow[rr, bend left] \arrow[dr] & & H_r(M) \arrow[rr, bend left] \arrow[dr] & & H_r(M, W') \arrow[dr] & \\
& H_{r+1}(W, M \cup M') \arrow[ur] \arrow[dr] & & H_rC(M, M_0) \arrow[ur] \arrow[dr] & & \dots \\
H_{r+1}(W, M') \arrow[ur] \arrow[rr, bend right] & & \mathbf{H} \arrow[ur] \arrow[rr, bend right] & & H_r(M_0) \arrow[ur] &
\end{tikzcd}$$

  Where $\mathbf{H} = H_{r+1}(D^{k+1} \times D^{n-k}, \partial(D^{k+1} \times D^{n-k}))$

- **Braid 2**

$$
\begin{tikzcd}
H_{r+1}(W, M) \arrow[dr] \arrow[rr, bend left, red] & & H_r(M) \arrow[dr, red] \arrow[rr, bend left, blue] & & H_r(M, W') \arrow[dr, blue] & \\
& H_{r+1}(W, M \cup M') \arrow[ur, blue] \arrow[dr] & & H_r(W) \arrow[ur, green] \arrow[dr, red] & & \dots \\
H_{r+1}(W, M') \arrow[ur, blue] \arrow[rr, bend right, green] & & H_r(M') \arrow[ur, green] \arrow[rr, bend right] & & H_r(W, M) \arrow[ur] &
\end{tikzcd}
$$

- **Braid 3**

$$
\begin{tikzcd}
\dots \arrow[dr] \arrow[rr, bend left, red] & & H_r(M_0) \arrow[dr, red] \arrow[rr, bend left, blue] & & H_r(M, U) \arrow[dr, blue] & \\
& H_r(\partial U) \arrow[ur, blue] \arrow[dr] & & H_r(M) \arrow[ur, green] \arrow[dr, red] & & H_r(\partial U)_{*-1} \\
\dots \arrow[ur, blue] \arrow[rr, bend right, green] & & H_r(U) \arrow[ur, green] \arrow[rr, bend right] & & H_r(M, M_0) \arrow[ur] &
\end{tikzcd}
$$

where $U = S^k \times D^{n-k}$

A.2 Algebraic surgery

Note: here "=" stands for chain equivalent

- **Input:**
  - **Chain complex:** $(C(M), \varphi)$
  - **Surgery data:** The data for algebraic surgery on an n-dimensional symmetric Poincaré complex (C, φ) is an $(n+1)$- dimensional symmetric pair $(f : C \longrightarrow D, (\delta\varphi, \varphi))$
  - **Chain complex of the embedding:** $C(S^k \times D^{n-k})$ is chain equivalent to $\mathbb{Z} \oplus S^k\mathbb{Z}$, and $\dot{C}(S^k \times D^{n-k}) = C(W, M)_{*+1}$
  - **Chain complex D:** this is the relative chain complex

$$C(W, M') = D = S^{n-k}\mathbb{Z}$$

  - **Dual chain complex D^{n-*}:** this is the dual chain complex

$$C(W, M)_{*+1} = D^{n-*} = S^k\mathbb{Z} \quad \text{and} \quad C(W, M) = D^{n-*+1} = S^{k+1}\mathbb{Z}$$

  - **Chain complex of the complement:** $C(M_0)$ fits into the long exact sequence,

$$\longrightarrow C(M_0) \longrightarrow C(M) \longrightarrow C(M, M_0) \longrightarrow$$

and
$$\begin{aligned} C(M, M_0) &= C(S^k \times D^{n-k}, S^k \times S^{n-k-1}) = C(S^k) \otimes C(D^{n-k}, S^{n-k-1}) \\ &= (\mathbb{Z} \oplus S^k\mathbb{Z}) \otimes (S^{n-k}\mathbb{Z}) = S^{n-k}\mathbb{Z} \oplus S^n\mathbb{Z} \end{aligned}$$

Hence $S^{n-k-1}\mathbb{Z} \oplus S^{n-1}\mathbb{Z} \longrightarrow C(M_0) \longrightarrow C(M) \longrightarrow C(M, M_0) = S^{n-k}\mathbb{Z} \oplus S^n\mathbb{Z}$ so that

$$C(M_0) = C(M) \oplus S^{n-k-1}\mathbb{Z} \oplus S^{n-1}\mathbb{Z}$$

- **Output:**
  - **Effect of surgery on the chain complex (C, φ):** the effect of surgery is $(C(M'), \varphi')$, where
$$\begin{pmatrix} d_C & 0 & (-1)^{n+1}\varphi_0 f^* \\ (-1)^r f & d_D & (-1)^r\delta\varphi_0 \\ 0 & 0 & d_D^* \end{pmatrix} :$$
$$C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \longrightarrow C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n-r+2}$$
  - **Chain complex of the dual embedding:** $C(D^{k+1} \times S^{n-k-1})$ is chain equivalent to $\mathbb{Z} \oplus S^{n-k-1}\mathbb{Z}$ and $\dot{C}(D^{k+1} \times S^{n-k-1}) = C(W, M')_{*+1}$

– **Mapping cone $\mathscr{C}(f)$:** the algebraic mapping cone of the chain map $f : C \longrightarrow D$ is the chain complex with

$$d_{\mathscr{C}(f)} = \begin{pmatrix} d_D & (-1)^r f \\ 0 & d_C \end{pmatrix} : \mathscr{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow \mathscr{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2}$$

The mapping cone $\mathscr{C}(f : C \longrightarrow D)$ is chain equivalent to the dimension shifted relative chain complex $C(W, M \cup M')_{*+1}$. Note the following pushout diagram:

$$\begin{tikzcd}
\left(D_{*+1} \oplus D^{n-*}, \begin{pmatrix} d_D & 0 \\ 0 & d_D^* \end{pmatrix}\right) \arrow[r, "(0\ 1)"] \arrow[d, "\begin{pmatrix} 1 & \delta\varphi_0 \\ 0 & \varphi_0 f^* \end{pmatrix}"'] & (D^{n-*}, d_D^*) \arrow[d, "\varphi_0 f^*"'] \\
\left(\mathscr{C}(f)_{*+1}, \begin{pmatrix} d_D & (-)^{r+1} f \\ 0 & d_C \end{pmatrix}\right) \arrow[r, "(0\ 1)"] & (C(M), d_C)
\end{tikzcd}$$

Braids of exact sequences relating these chain complexes:

- **Braid 1**

$$\begin{tikzcd}
C(M_0) \arrow[rr, bend left] \arrow[dr] & & C(M) \arrow[rr, bend left] \arrow[dr] & & C(M, W') \arrow[dr] & \\
& C(W, M \cup M')_{*+1} \arrow[ur] \arrow[dr] & & C(M, M_0) \arrow[ur] \arrow[dr] & & \ldots \\
C(W, M')_{*+1} \arrow[ur] \arrow[rr, bend right] & & S^n\mathbb{Z} \arrow[ur] \arrow[rr, bend right] & & C(M_0) \arrow[ur] &
\end{tikzcd}$$

- **Braid 2**

$$\begin{tikzcd}
C(W, M)_{*+1} \arrow[rr, bend left] \arrow[dr] & & C(M) \arrow[rr, bend left] \arrow[dr] & & C(M, W') \arrow[dr] & \\
& C(W, M \cup M')_{*+1} \arrow[ur] \arrow[dr] & & C(W) \arrow[ur] \arrow[dr] & & \ldots \\
C(W, M')_{*+1} \arrow[ur] \arrow[rr, bend right] & & C(M') \arrow[ur] \arrow[rr, bend right] & & C(W, M) \arrow[ur] &
\end{tikzcd}$$

- **Braid 3**

$$\begin{tikzcd}
\ldots \arrow[rr, bend left] \arrow[dr] & & C(M_0) \arrow[rr, bend left] \arrow[dr] & & C(M, U) \arrow[dr] & \\
& C(\partial U) \arrow[ur] \arrow[dr] & & C(M) \arrow[ur] \arrow[dr] & & C(\partial U)_{*-1} \\
\ldots \arrow[ur] \arrow[rr, bend right] & & C(U) \arrow[ur] \arrow[rr, bend right] & & C(M, M_0) \arrow[ur] &
\end{tikzcd}$$

where $U = S^k \times D^{n-k}$

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