Multiplicativity of the signature of fibre bundles, SK groups and $TQFT$

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Abstract

Let $F \to E \to B$ be a fibre bundle. Here we seek the necessary conditions to obtain multiplicativity of the signature of fibre bundles, i.e. when does it hold that $\sigma(E) = \sigma(B)\sigma(F)$. We will also present the relationship with the cutting and pasting SK-groups and the asymmetric L-theory group.

The signature

The signature of a closed oriented n-dimensional manifold M^n is denoted by $\sigma(M) \in \mathbb{Z}$, and is defined to be zero if the dimension of M is not divisible by 4. If $n = 4k$ then $\sigma(M)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the non-singular symmetric intersection form $(H^{2k}(M, \mathbb{R}), \lambda)$, where

$$\lambda : H^{2k}(M, \mathbb{R}) \times H^{2k}(M, \mathbb{R}) \longrightarrow \mathbb{R}; (u, v) \mapsto \langle u \cup v, [M] \rangle .$$

The additivity of the signature was proved by Novikov:

$$\sigma(M_1 \cup_h M_2) = \sigma(M_1) + \sigma(M_2)$$

for any orientation preserving diffeomorphism $h : \partial M_1 \to \partial M_2$.

Cobordism groups

- Two n-dimensional manifolds M and M' are cobordant if there exists a manifold W^{n+1} such that $\partial W = M \sqcup M'$. Cobordism groups of manifolds are denoted by Ω_n.

Various algebraic L-theoretic groups can be defined in a similar way :

- An algebraic symmetric Poincaré complex (C, ϕ) over a ring with involution A is an A-module chain complex C with symmetric Poincaré duality ϕ The **symmetric** L-groups are cobordism groups of algebraic symmetric Poincaré complexes,

  $$(C, \phi) \in L^n(A).$$

  The symmetric signature of a geometric Poincaré complex X, $\sigma^*(X)$ is the cobordism class of the A-module chain complex (C, ϕ) over $A = \mathbb{Z}[\pi_1(X)]$, with $C = C(\tilde{X})$. ($\tilde{X}$ = Universal cover of X)
- The **quadratic** L-groups are cobordism groups of algebraic quadratic Poincaré complexes, $(C, \psi) \in L_n(A)$. These are the Wall surgery obstruction groups.
- The **visible** L-groups $VL^n(B)$ of a simplicial space B are the cobordism groups of globally Poincaré n-dimensional cycles over B. The following forgetful maps

  $$L_n(\mathbb{Z}[\pi_1(B)]) \longrightarrow VL^n(B); (C, \psi) \longmapsto (C, (1 + T)\psi)$$
  $$VL^n(B) \longrightarrow L_n(\mathbb{Z}[\pi_1(B)]); (C, \psi) \longmapsto (C(\tilde{B}), \phi^B)$$

  are isomorphisms modulo 8 torsion.
- An **asymmetric** Poincaré complex (C, λ) is a chain complex with Poincaré duality. The asymmetric $LAsy$-groups are cobordism groups of asymmetric Poincaré complexes,

  $$(C, \lambda) \in LAsy^n(A)$$

  The asymmetric signature of a geometric Poincaré complex X, $\sigma Asy^*(X)$ is the cobordism class of the $\mathbb{Z}[\pi_1(X)]$-module chain complex (C, λ).

Transfer maps in L-theory

For a fibration $F^m \to E \to B^n$ there exist transfer maps:

$$p^! : L_n(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{n+m}(\mathbb{Z}[\pi_1(E)])$$
$$p^! : VL^n(B) \longrightarrow VL^{n+m}(E)$$
$$p^! : LAsy^n(\mathbb{Z}[\pi_1(B)]) \longrightarrow LAsy^{n+m}(\mathbb{Z}[\pi_1(E)])$$

The transfer maps allow us to express the total space of the fiber bundle algebraically in a convenient way in terms of the fiber and base space, since $p^! = (C(\tilde{F}), \alpha, U)^!$ with

- $C(\tilde{F})$ is a $\mathbb{Z}[\pi_1(E)]$-module chain complex and $\tilde{F}$ is the pullback from the universal cover $\tilde{E}$ of E,
- $\alpha : C(\tilde{F}) \longrightarrow C(\tilde{F})^{n-*}$,
- $U : \mathbb{Z}[\pi_1(B)] \longrightarrow H_0(\mathrm{Hom}(C(\tilde{F}), C(\tilde{F})))$. U is determined by the fiber transport and encodes the information about the action of $\pi_1(B)$ on the fiber F.

There is no transfer map for the symmetric L-groups, since not every element in $L^n(\mathbb{Z}[\pi_1(B)])$ can be realized geometrically as the symmetric signature of a Poincaré complex. For a detailed description of transfer maps in L-theory see [11].

References

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The non-multiplicativity of the signature

For an untwisted product of spaces X and Y the signature is multiplicative: $\sigma(X \times Y) = \sigma(X)\sigma(Y)$. In a fiber bundle $F \to E \to B$, $\pi_1(B)$ acts on $H^*(F)$ by fibre transport. Thus the signature of the total space may not be the product of the signatures of the base space and the fibre, so that in general $\sigma(E) \neq \sigma(B)\sigma(F)$. The following questions arise in this context:

- Under what conditions does a fibre bundle have multiplicative signature? (Chern, Hirzebruch, Serre (1957) [4])
- Can we find examples of non-multiplicative fibre bundles? (Atiyah (1969) [2], Kodaira (1969) [9])
- On what does the value of $\sigma(E)$ depend? (Meyer (1973) [12])
- How can we express the difference $\sigma(E) - \sigma(B)\sigma(F)$?

Sufficient conditions for multiplicativity

Chern, Hirzebruch and Serre (1957)

Theorem (See [4]) Let $F \to E \to B$ be a fiber bundle then *if the fundamental group* $\pi_1(B)$ *acts trivially on* $H^*(F, \mathbb{Q})$ then

$$\sigma(E) = \sigma(B)\sigma(F).$$

Examples: Atiyah (1969) and Kodaira (1967)

Atiyah [2] and Kodaira [9] constructed non-multiplicative examples with $\pi_1(B)$ acting non-trivially on $H^*(F, \mathbb{Q})$: With B and F compact oriented surfaces of genus 129 and 6 respectively, $\sigma(E) = 2^8 \neq \sigma(B)\sigma(F)$.

Multiplicativity mod 4

Hambleton, Korzeniewski, Ranicki (2007)

Theorem (See [5]) Let $F \to E \to B$ be a fibre bundle of closed, connected, compatibly oriented manifolds. Then

$$\sigma(E) \equiv \sigma(B)\sigma(F) \pmod 4$$

Multiplicativity mod 8

A. Korzeniewski (geometric theorem, 2005)

Theorem ([10]) Let $F^{4m} \longrightarrow E^{4n+4m} \longrightarrow B^{4n}$ be a Poincaré fibration such that the action of $\pi_1(B)$ on $(H_{2m}(F; \mathbb{Z})/torsion) \otimes \mathbb{Z}_2$ is trivial then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod 8$$

$\Longleftrightarrow$

A. Korzeniewski (algebraic theorem, 2005)

Theorem ([10]) Let $(C(\tilde{B}), \phi^B)$ be a $4n$ dimensional visible symmetric complex over $\mathbb{Z}[\pi_1(B)]$ and let (A, α, U) be a $\mathbb{Z}_2$-trivial $(\mathbb{Z}, 2m)$-symmetric representation. Then

$$\sigma((A, \alpha, U) \otimes (C, \phi)) \equiv \sigma(C, \phi)\sigma(A, \alpha) \pmod 8$$

One of the key tools in the proof given in [10] is firstly to give an analogous algebraic version of the theorem in terms of visible Poincaré complexes (stated above) and then use a theorem of Morita which states that the signature modulo 8 is the Arf-Brown invariant of the $\mathbb{Z}_4$-valued Pontryagin square. It is also possible to use the transfer map in asymmetric L-theory in this context (as in Ranicki [14], Chapter 30), which allows to gain a new insight of the problem by giving an asymmetric version of the theorem.

Cutting and pasting: SK-groups

Cut and paste operations on a manifold M are realized as follows: Cut a closed n-dimensional smooth manifold M along a codimension 1 manifold F which has trivial normal bundle. After performing this cut we obtain a manifold with two boundary components, each of them a copy of F. Pasting back these boundary components by a diffeomorphism $h : F \to F$, results in a new manifold $M(F, h)$.

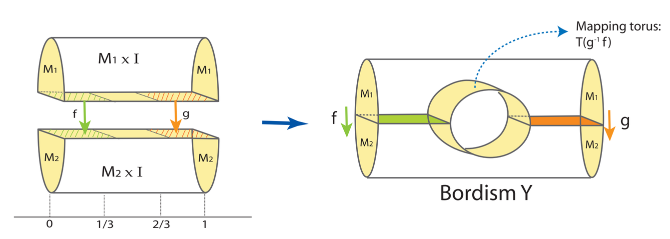
The set of equivalence classes of oriented manifolds in a space X modulo the relation created by cutting and pasting gives rise to the definition of SK-groups (See [8]).

The signature is a cut and paste invariant. Jänich (1968)

Let $A = M_1 \cup_h M_2$ and $B = M_1 \cup_g M_2$ be two closed n-dimensional manifolds, and $h, g : \partial M_1 \to \partial M_2$ be orientation preserving diffeomorphisms. By the Novikov additivity of the signature: $\sigma(A) = \sigma(M_1 \cup_h M_2) = \sigma(M_1) + \sigma(M_2) = \sigma(M_1 \cup_g M_2) = \sigma(B)$. (See [7])

$\widetilde{SK}(X) \cong$ im$(\sigma Asy : \Omega_n(X) \longrightarrow LAsy^n(\mathbb{Z}[\pi_1(X)])$ Ranicki (1998)

Let $F_n(X) \subseteq \Omega_n(X)$ be the subgroup of the bordism classes of closed n-dimensional manifolds which fibre over S^1, then the cut and paste bordism groups are defined **geometrically** as $\widetilde{SK}_n(X) \cong \Omega_n(X) \backslash F_n(X)$.



Neumann (1975)

Theorem (Neumann) If $F^m \to E^{4k} \to B^n$ is a fibration with $\sigma(B) = 0$ and $\sigma(E) \neq 0$, so that $\sigma(E) \neq \sigma(F)\sigma(B)$, then $[B, f : B \to BG]$ generates a free $SK_*(BG)$-module.

The asymmetric signature of a mapping torus is zero: $\sigma Asy(T(h)) = 0 \in LAsy^*(\mathbb{Z}[\pi_1(T(h))]$ See [14, Remark 30.30]. Hence,

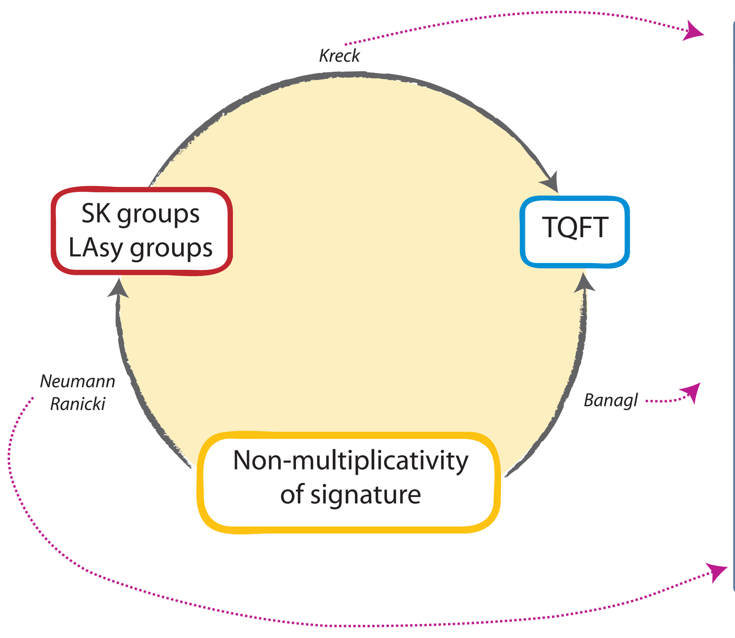
$$\widetilde{SK}(X) \cong \mathrm{Im}(\sigma Asy : \Omega_n(X) \longrightarrow LAsy^n(\mathbb{Z}[\pi_1(X)])$$ (See [14, Proposition 30.6])

Neumann's theorem can be proved **algebraically** by using the transfer map in asymmetric L-theory:

$$\begin{array}{ccccccc} LAsy^n(\mathbb{Z}[\pi_1(B)]) & \xrightarrow{p^!} & LAsy^{4k}(\mathbb{Z}[\pi_1(E)]) & \longrightarrow & LAsy^{4k}(\mathbb{Z}) & \longrightarrow & L^{4k}(\mathbb{Z}) = \mathbb{Z} \\ \sigma Asy(B) & \longmapsto & \sigma Asy(E) & \longmapsto & & & \sigma(E). \end{array}$$

Note that if $\sigma(E) \neq 0$ then $\sigma Asy(B)$ has infinite order in $LAsy^n(\mathbb{Z}[\pi_1(B)])$. Then $0 \neq \sigma Asy(B) \in \mathrm{Im}(\sigma Asy : \Omega_n(B) \longrightarrow LAsy^n(\mathbb{Z}[\pi_1(B)])$.

Relating ideas



From the topological point of view, a field theory gives a way to compute an invariant by cutting a manifold into simple pieces and make the function effectively computable. The next question to ask is: Which invariants of closed manifolds are partition functions of the field theory? No complete answer is known to this question. Nevertheless Kreck has proved that SK-invariants are partition functions of the field theory. (Work in progress. See [1])

Very recently Markus Banagl has described in [3] how the non-multiplicativity of the signature of fibre bundles gives rise to certain TQFTs.

Neumann [13, theorem 3.1] (stated above) and Ranicki [14] (algebraic interpretation)

What next?

The difference $\sigma(E) - \sigma(F)\sigma(B)$ is related to the Browder-Livesay invariant for double covers, which has been previously studied by Hambleton and Milgram [6]. At the moment I am identifying why the assumption of the trivial $\mathbb{Z}_2$ action of $\pi_1(B)$ is necessary and how to formulate a general result on the multiplicativity of the signature if this assumption is eliminated.