A note on SKK groups

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This is a brief account on the definition of the SKK groups and SKK invariants given in [Kre73]

1 SKK groups and SKK -invariants

1.1 SKK groups

Definition 1.1. *Let $\mathcal{M}_n$ be the semigroup of diffeomorphism classes of closed oriented n-dimensional manifolds. Factoring this semigroup by the following relation*

$$(M_1 \cup_f M_2) + (M_3 \cup_g M_4) = (M_1 \cup_g M_2) + (M_3 \cup_f M_4)$$

where $\partial M_1 = \partial M_2 = \partial M_3$ and f and g are diffeomorphisms of these boundaries, we obtain the semigroup $\mathcal{M}_n / \sim_{SKK}$.

The SKK_n group is the Grothendieck group of the semigroup $\mathcal{M}_n / \sim_{SKK}$.

The SKK groups are identified in Theorem 4.4 of [Kre73] with Reinhart's vector field cobordism groups ([Rei63]).

1.2 SKK invariants

Definition 1.2. *An invariant λ is called SK-controlled (SKK) if*

$$\lambda(M_1 \cup_f M_2) - \lambda(M_1 \cup_g M_2) := \lambda(f, g) \tag{1}$$

depends only on the diffeomorphism $f, g : \partial M_1 \to \partial M_2$ and not on the choice of M_1 and M_2.

Remark 2. We note that all SK -invariants are also SKK invariants. When the correction term $\lambda(f, g) = 0$ then Equation (1) becomes,

$$\lambda(M_1 \cup_f M_2) = \lambda(M_1 \cup_g M_2)$$

which is the requirement for λ to be an SK invariant.

Proposition 1.3. ***Euler characteristic*** *is an SKK invariant.*

Proof. This follows from the fact that the Euler characteristic is an SK invariant. (See [Kre73]). □

Remark 3. In order to describe the Euler characteristic as a bordism invariant, Reinhart introduced the concept of vector field cobordism in [Rei63]. By doing this he implicitly describes the SKK groups, since two manifolds are vector field cobordant if and only if they are equivalent in SKK_n.

Proposition 1.4. *The* ***Signature*** *is an SKK invariant.*

Proof. The signature is an SK invariant, and hence also an SKK invariant. □

Nevertheless, it is important to note that **some SKK invariants are not SK invariants**, as the following propositions show.

Proposition 1.5. ***Bordism*** *is an SKK invariant.*

Proof. We will prove later on (with Theorem 1.7) that there exists a surjective homomorphism from SKK_* to Ω_* which sends oriented manifolds to their cobordism class. Note that two manifolds which are cobordant (i.e. are in the same cobordism class) differ in SKK by a multiple of a sphere. □

Proposition 1.6. *The* ***Kervaire semi-characteristic*** *is an SKK invariant.*

Proof. First recall that the Kervaire semi-characteristic is defined as

$$\chi_{1/2}(M^{4k+1}) = \sum_{i=0}^{2k} b_i(M) \pmod 2,$$

where $b_i(M)$ is the ith betti number. See [Ker56].

We consider the closed oriented manifolds

$$M_1 \cup_f -M_2 \quad \text{and} \quad M_1 \cup_g -M_2,$$

which are obtained from each other by cutting and pasting, and f and g are diffeomorphisms of the boundary $f, g : \partial M_1 \to \partial M_2$.

By definition an SKK invariant depends only on these diffeomorphisms f and g. So this means that if the Kervaire semi-characteristic $\chi_{1/2}$ is an SKK invariant, then we will be able to express the following "correction term",

$$\chi_{1/2}(f,g) := \chi_{1/2}(M_1 \cup_f -M_2) - \chi_{1/2}(M_1 \cup_g -M_2)$$

by an expression involving only f and g, and not M_1 or M_2.

In [Ker56] it is shown that for an even dimensional manifold Y with boundary,

$$\chi_{1/2}(\partial Y) = \chi(Y) - \sigma(Y) \qquad (\text{mod}2)$$

where $\chi(Y)$ is the Euler characteristic and $\sigma(Y)$ is the signature of Y.

In this case we consider ∂Y to have dimension $4k+1$, so $\sigma(Y^{4k+1}) = 0$, and then

$$\chi_{1/2}(\partial Y) = \chi(Y) \pmod 2$$

Now we consider Y to be a bordism which is constructed as in Lemma 1.9 of [Kre73], and has boundary,

$$\partial Y = (M_1 \cup_f -M_2) - (M_1 \cup_g -M_2) - (T : \partial M_1 \xrightarrow{g^{-1}f} \partial M_1)$$

so the expression $\chi_{1/2}(\partial Y) = \chi(Y) \pmod 2$ becomes

$$\chi_{1/2}(M_1 \cup_f -M_2) - \chi_{1/2}(M_1 \cup_g -M_2) - \chi_{1/2}(T : \partial M_1 \xrightarrow{g^{-1}f} \partial M_1) = \chi(Y) \pmod 2$$

From this we deduce that the correction term $\chi_{1/2}(f,g)$ is defined as

$$\chi_{1/2}(f,g) := \chi_{1/2}(T : \partial M_1 \xrightarrow{g^{-1}f} \partial M_1) - \chi(Y) \pmod 2$$

But we still need to write $\chi_{1/2}(f,g)$ as an expression involving only f and g, so we will now use the computation of $\chi(Y)$:

$$\chi(Y) = \chi(M_1) + \chi(M_2) - 2\chi(\partial M_1).$$

When considered modulo 2, the term $2\chi(\partial M)$ disappears, so that

$$\chi_{1/2}(f,g) := \chi_{1/2}(T : \partial M_1 \xrightarrow{g^{-1}f} \partial M_1) - [\chi(M_1) + \chi(M_2)] \pmod 2$$

We also have that

$$\begin{aligned} \chi(M_1 \cup M_2) &= \chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2) \\ &= \chi(M_1) + \chi(M_2) - \chi(\partial M_1) \end{aligned}$$

$(M_1 \cup M_2)$ is a closed orientable manifold so that $\chi(M_1 \cup M_2)$ is even. Hence $\chi(M_1 \cup M_2) = 0 \pmod 2$. That is,

$$\chi(M_1) + \chi(M_2) = \chi(\partial M_1) \pmod 2$$

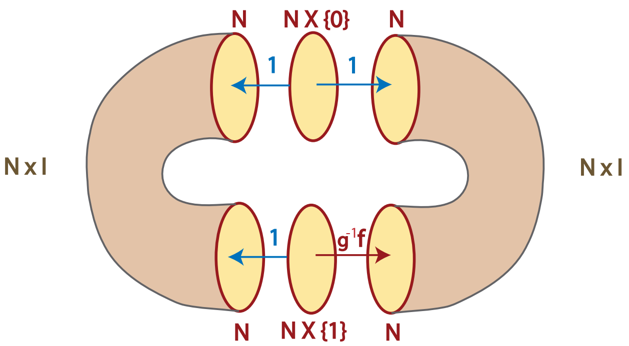
This implies that,

$$\chi_{1/2}(f,g) := \chi_{1/2}(T : \partial M_1 \xrightarrow{g^{-1}f} \partial M_1) - \chi(\partial M_1) \pmod 2$$

For simplicity we will write $\partial M_1 = N$ and we will also write the mapping torus $T : \partial M_1 \xrightarrow{g^{-1}f} \partial M_1$ as $N_{g^{-1}f}$.

The mapping torus is defined as

$$T : \partial M_1 \xrightarrow{g^{-1}f} \partial M_1 = N_{g^{-1}f} = (N \times I) \cup_{g^{-1}f \times 1} (N \times I)$$



This gives rise the following Mayer-Vietoris sequence,

$$\ldots \longrightarrow H_*(N \times \{0,1\}) \longrightarrow H_*(N \times I) \oplus H_*(N \times I) \longrightarrow H_*(T(g^{-1}f)) \longrightarrow \ldots$$

We note that

$$H_*(N \times 0,1) \cong H_*(N) \oplus H_*(N)$$

and

$$H_*(N \times I) \oplus H_*(N \times I) \cong H_*(N) \oplus H_*(N)$$

so the following maps in the Mayer-Vietoris sequence,

$$H_*(N \times \{0,1\}) \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & g^{-1}f \end{pmatrix}} H_*(N \times I) \oplus H_*(N \times I)$$

correspond to

$$\begin{array}{ccc} H_*(N) \oplus H_*(N) & \longrightarrow & H_*(N) \oplus H_*(N) \\ \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \uparrow & & \uparrow \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \\ H_*(N) & \xrightarrow{1-g^{-1}f} & H_*(N) \end{array}$$

This means that we can consider the exact sequence:

$$\ldots \longrightarrow H_*(N) \xrightarrow{1-g^{-1}f} H_*(N) \longrightarrow H_*(T(g^{-1}f)) \longrightarrow \ldots$$

In this sequence all dimensions except the middle dimension pair off by Poincaré duality, so the "*correction term*" $\chi_{1/2}(f,g)$ will be given by the rank of the map in the middle dimension,

$$H_{2k}(N) \xrightarrow{1-g^{-1}f} H_{2k}(N)$$

That is, $\chi_{1/2}(f,g) = \text{rank}(1 - g^{-1}f) \pmod 2$

We have now achieved an expression for this correction term depending only on the diffeomorphisms f and g, so we deduce that the Kervaire semi-characteristic is an SKK invariant.

□

Theorem 1.7. *Theorem 4.2 in the SK-book*

Let $I'_n \subset SKK_n$ be the cyclic subgroup generated by $[S^n]$. Then,

$$I'_n = \begin{cases} \mathbb{Z} & n \equiv 0 \pmod 2 \\ \mathbb{Z}_2 & n \equiv 1 \pmod 4 \\ 0 & n \equiv 3 \pmod 4 \end{cases}$$

Furthermore there exists an split exact sequence,

$$0 \longrightarrow I'_n \longrightarrow SKK_n \longrightarrow \Omega_n \longrightarrow 0$$

Proof. To achieve the computation of I'_n given above, we need to consider the order of $[S^n, *]$ in SKK_n. So we do this for the different possible values of n. Recall that by Lemma 4.3 in [Kre73] we have that

$$\chi(M^{n+1})[S^n] = 0 \text{ where } M^{n+1} \text{ is a closed manifold.}$$

- $n \equiv 0 \pmod 2$, i.e, $n = 2k$:

The argument in this case is the same as the one given in the proof of Theorem 1.1 in [Kre73].

- $n \equiv 3 \pmod 4$, i.e, $n = 4k + 3$:

Since $\chi(M^{n+1})[S^n] = 0$, then in this case we have $\chi(M^{4k+4})[S^{4k+3}] = 0$. Firstly we can deduce from this that $[S^{4k+3}]$ has at most order 2, since $\chi(S^{2m}) = 2$, then,

$$\chi(S^{2m})[S^{4k+3}] = 0 \quad \Longrightarrow \quad 2[S^{4k+3}] = 0$$

so in general, if the M^{n+1} has even Euler Characteristic, then $[S^{4k+3}] = 0$.

If we consider M^{n+1} to have odd Euler characteristic, that is $\chi(M^{n+1}) = 2a + 1$ then we will have,

$$\begin{aligned} 0 &= \chi(M^{4k+4})[S^{4k+3}] \\ &= (2a+1)[S^{4k+3}] \\ &= 2a[S^{4k+3}] + [S^{4k+3}] \\ &= [S^{4k+3}] \end{aligned}$$

Hence we deduce that $[S^{4k+3}] = 0$ and consequently $I'_{4k+3} = 0$

- $n \equiv 1 \pmod 4$, i.e, $n = 4k + 1$:

In this case, $[S^{4k+1}]$ also has order at most 2 in SKK, since $\chi(S^{4k+2}) = \chi(S^{2m}) = 0$ First we note that if M is orientable then $\chi(M^{4k+2})$ is even.

We will show by contradiction that $[S^{4k+1}] \neq 0 \in SKK_{4k+1}$. So suppose that $[S^{4k+1}] = 0$ in SKK_{4k+1}.

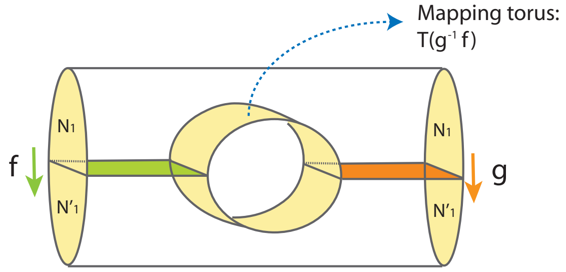
By the definition of SKK, we know that there exist orientable manifolds N_i and N'_i, where $i = 1, 2$ and $\partial N_1 = \partial N_2$ and $\partial N'_1 = \partial N'_2$ and diffeomorphisms $f, g : \partial N_i \longrightarrow \partial N'_i$ such that,

$$(N_1 \cup_f -N'_1) - (N_2 \cup_g -N'_2) = (N_2 \cup_f -N'_2) - (N_1 \cup_g -N'_1) \tag{4}$$

and if we are assuming that $[S^{4k+1}] = 0$ then we can also write Equation (4) as,

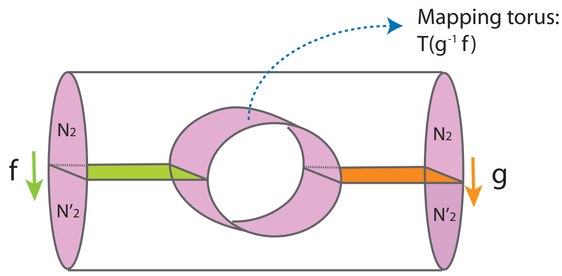
$$S^{4k+1} + (N_1 \cup_f -N'_1) - (N_2 \cup_g -N'_2) = (N_2 \cup_f -N'_2) - (N_1 \cup_g -N'_1)$$

Following the same procedure as in Theorem 1.2 in [Kre73], we can construct two bordisms Y_1 and Y_2 defined as follows: Let Y_1 be the bordism with boundary $\partial Y_1 = (N_1 \cup_f -N'_1) - (N_1 \cup_g -N'_1) - T(g^{-1}f)$,



Bordism Y_1

and Y_2 be the bordism with boundary $\partial Y_2 = (N_2 \cup_f -N'_2) - (N_2 \cup_g -N'_2) - T(g^{-1}f)$,



Bordism Y₂

Note that mapping torus $T(g^{-1}f)$ is the same mapping torus in both bordisms Y_1 and Y_2.

This means that the disjoint union of Y_1 and Y_2 has boundary,

$$\begin{aligned}\partial(Y_1 \sqcup Y_2) =& [(N_1 \cup_f -N_1') - (N_1 \cup_g -N_1') - T(g^{-1}f)] + [(N_2 \cup_f -N_2') - (N_2 \cup_g -N_2') - T(g^{-1}f)] \\ &= [(N_1 \cup_f -N_1') - (N_2 \cup_g -N_2') - T(g^{-1}f)] + [(N_2 \cup_f -N_2') - (N_1 \cup_g -N_1') - T(g^{-1}f)]\end{aligned}$$

Using the relation established in (4), we can rewrite this as follows

$$\begin{aligned}\partial(Y_1 \sqcup Y_2) =& [(N_1 \cup_f -N_1') - (N_2 \cup_g -N_2') - T(f^{-1}g)] \\ &- [S^{4k+1} + (N_1 \cup_f -N_1) - (N_2 \cup_g -N_2) - T(f^{-1}g)]\end{aligned}$$

If we now paste pairwise the boundaries in this expression, then we obtain a manifold with one boundary component, S^{4k+1}. So if we glue an $4k+2$-dimensional disc along this boundary, then the manifold $Y_1 \cup Y_2 \cup D^{4k+2}$ is a closed $4k+2$-dimensional manifold M^{4k+2}

We now compute Euler characteristic $\chi(M^{4k+2})$.

$$\chi(M^{4k+2}) = \chi(Y_1 \cup Y_2 \cup D^{4k+2}) = \chi(Y_1 \cup Y_2) + 1$$

So we need to compute,

$$\chi(Y_1 \cup_{(N_1 \cup_f -N_1') - (N_2 \cup_g -N_2') - T(f^{-1}g)} Y_2)$$

This is given by

$$\chi(Y_1 \cup Y_2) = \chi(Y_1) + \chi(Y_2) - [\chi(N_1 \cup_f -N_1') + \chi(N_2 \cup_g -N_2') + \chi(T(f^{-1}g))] \quad (5)$$

By the computation of $\chi(Y_i)$ given before, we know that

$$\chi(Y_i) = \chi(N_i) + \chi(N_i') - 2\chi(\partial N_i)$$

so substituting appropriately in Equation (**??**) we obtain

$$\begin{aligned}\chi(Y_1 \cup Y_2) =& [\chi(N_1) + \chi(N_1') - 2\chi(\partial N_1)] + [\chi(N_2) + \chi(N_2') - 2\chi(\partial N_2)] - \\ &- \chi(N_1 \cup -N_1') - \chi(N_2 \cup -N_2') - \chi T(g^{-1}f)] \\ \\ =& [\chi(N_1) + \chi(N_1') - 2\chi(\partial N_1)] + [\chi(N_2) + \chi(N_2') - 2\chi(\partial N_2)] - \\ &- [\chi(N_1) + \chi(N_1') - \chi(\partial N_1)] - [\chi(N_2) + \chi(N_2') - \chi(\partial N_2)] - 0\end{aligned}$$

so rearranging we obtain,

$$\chi(Y_1 \cup Y_2) = -\chi(\partial N_1) - \chi(\partial N_2) = -2\chi(\partial N_1) \quad (6)$$

Hence,

$$\chi(M^{4k+2}) = \chi(Y_1 \cup Y_2 \cup D^{4k+2}) = \chi(Y_1 \cup Y_2) + 1 = 1 - 2\chi(\partial N_1)$$

But $1-2\chi(\partial N_1)$ is always odd, and this is a contradiction since a $4k+2$-dimensional closed manifold always has even Euler characteristic. Hence we deduce that the assumption that $[S^{4k+1}] = 0 \in SKK_{4k+1}$ is false. Hence $[S^{4k+1}] \neq 0$ and $[S^{4k+1}]$ has order 2 in SKK_{4k+1} so from this we deduce that $I'_{4k+1} = \mathbb{Z}_2$.

Finally we note that the exact sequence

$$0 \longrightarrow I'_n \longrightarrow SKK_n \longrightarrow \Omega_n \longrightarrow 0$$

splits:

- For $n = 2k$ we have that the Euler characteristic gives a map

$$SKK_{2k} \longrightarrow \mathbb{Z} = I'_{2k}$$

- For $n = 4k+3$, the sequence splits trivially because $I'_{4k+3} = 0$
- For $n = 4k+1$, the Kervaire semi-characteristic provides a retraction map,

$$SKK_n \longrightarrow \mathbb{Z}_2 = I'_{4k+1}$$

  which provides an inverse of $I'_n \longrightarrow SKK_n$

Thus for any possible value of n there exists a retraction map, so the sequence splits.

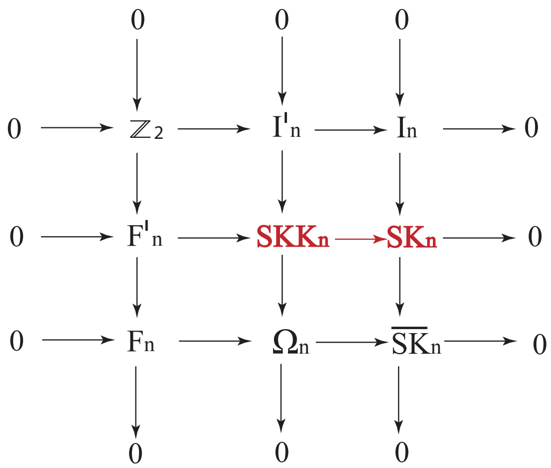
□

1.3 Relating concepts

Here we present a diagram relating the exact sequences from the theorems 1.1 and 1.2 in [Kre73], and 1.7 mentioned in this account.

Through this diagram of exact sequences, it becomes clear that the difference in the groups I_n from Theorem 1.1 in [Kre73] and I'_n from 1.7 is given by the Kervaire semi-characteristic. Similarly for F_n which is introduced in Theorem 1.2 in [Kre73] and F'_n which does not figure in this book.

Also with this diagram we establish the relation between SKK_n and SK_n which is defined as a surjective homomorphism. This is homomorphism is not discussed in [Kre73].



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