Homotopy Theory, Poincaré Duality for Singular Spaces, and String Theory

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Main Topic of Talk:

$\boxed{\textbf{Poincaré Duality for Singular Spaces}}$

Fails for ordinary homology.
Example: $X^3 = \text{Susp}(T^2),\ b_1 = 0,\ b_2 = 2.$

Solutions:

- L^2-cohomology (Cheeger)
- Intersection Homology (Goresky, MacPherson)

We will propose a new (nonisomorphic) Ansatz.

MOTIVATION.

1. General emphasis on **spatial constructions** in modern algebraic topology: Try to work on the level of spaces/spectra as long as possible, pass to homology/homotopy groups as late as possible.

2. Will address certain problems in **string theory** for which $H^\bullet_{(2)}, IH_\bullet$ are too small. ($H^\bullet_{(2)}, IH_\bullet$ miss some dual cycles.)

1. Spatial Philosophy:

X^n stratified pseudomanifold.

$$
\begin{array}{ccc}
X & \longrightarrow & \boxed{I^{\bar{p}}X} \\
\text{space} & & \text{“}\textbf{intersection} \\
 & & \textbf{space}\text{”} \\
\downarrow & & \downarrow \\
IC^{\bar{p}}_{\bullet}(X) & & C_{\bullet}(I^{\bar{p}}X) \\
\text{chain} & & \text{chain} \\
\text{complex} & & \text{complex} \\
\downarrow & & \downarrow \\
IH^{\bar{p}}_{\bullet}(X) & & H_{\bullet}(I^{\bar{p}}X) \\
 & & \text{ordinary} \\
 & & \text{homology}
\end{array}
$$

Requirements:

- $H_\bullet(I^{\bar{p}}X)$ should satisfy **Poincaré Duality**.

- $X \rightsquigarrow I^{\bar{p}}X$ should be as "**natural**" as possible. (It is not expected to be a functor wrt. *all* continuous maps.)

- X should be modified as little as possible (only near the singularities). The homotopy type away from the singularities should be completely preserved.

- If X is a finite cell complex, then $I^{\bar{p}}X$ should be a finite cell complex.

- $X \rightsquigarrow I^{\bar{p}}X$ should be homotopy-theoretically tractable, so as to facilitate computations.

Additional Key Benefit:

E spectrum $\rightsquigarrow E_\bullet(I^{\bar{p}}X)$.
P.D.?

Could look at:

$\pi^s_\bullet(I^{\bar{p}}X)$
$\Omega_\bullet(I^{\bar{p}}X)$
$K_\bullet(I^{\bar{p}}X)$
$L_\bullet(I^{\bar{p}}X)$— have P.D. rationally.
$\vdots$

2. String Theory.

worldsheet $\rightarrow$ target space $= M^4 \times X^6$.

X should be a Calabi-Yau space. But which one?

Conifold transition is a way to navigate within the moduli space of Calabi-Yau manifolds.

"It appears that all Calabi-Yau vacua may be connected by conifold transitions."
[J. Polchinski]

Def. A (topological) *conifold* is a compact pseudomanifold S with only isolated singularities.

2-Step Process:

1. *Deformation of complex structure:*

• X_ϵ CY 3-fold whose complex structure depends on a complex parameter ϵ.

• For small $\epsilon \neq 0$: X_ϵ is smooth.

• $\epsilon \to 0$: singular conifold S.

• Common Assumption: All singularities are nodes.

• Links $\cong S^2 \times S^3$.

• Topologically: S^3-shaped cycles in X_ϵ are collapsed.

2. *Small resolution:*

• $Y \to S$ replaces every node in S by a $\mathbb{C}P^1$.

• Y is a smooth Calabi-Yau manifold.

Conifold Transition:

$$X_\epsilon \rightsquigarrow S \rightsquigarrow Y.$$

Massless D-Branes.

- Z : 3-cycle in X_ϵ which collapses to a node in S.

- In type IIB string theory: exists a charged 3-brane that wraps around Z.

- Mass (3-brane) $\propto$ Vol(Z).

- $\Rightarrow$ 3-brane becomes **massless** in S.

- $\mathbb{C}P^1$: 2-cycle in Y which collapses to a node in S.

- In type IIA string theory: exists a charged 2-brane that wraps around $\mathbb{C}P^1$.

- Mass (2-brane) $\propto$ Vol($\mathbb{C}P^1$).

- $\Rightarrow$ 2-brane becomes **massless** in S.

Cohomology and Massless States

Rule: cohomology classes on X are manifested in four dimensions as massless particles.

- ω differential form on $T = M^4 \times X$.
- For such a form to be physically realistic:

$$d^* d\omega = 0 \text{ (“Maxwell equation”)},$$

$$d^* \omega = 0 \text{ (“Lorentz gauge condition”)}.$$

- So $\Delta_T \omega = 0$, $\Delta_T = dd^* + d^* d$ Hodge-de Rham Laplacian on T.
- Decomposition

$$\Delta_T = \Delta_M + \Delta_X.$$

- Wave equation

$$(\Delta_M + \Delta_X)\omega = 0.$$

- Interpretation: Δ_X is a kind of "mass" operator for four-dimensional fields, whose eigenvalues are masses as seen in four dimensions.

- (Klein-Gordon equation $(\Box_M + m^2)\omega = 0$ for a free particle.)

- For the zero modes of Δ_X (the harmonic forms on X), one sees in the four-dimensional reduction massless forms.

Physics and Topology of the Conifold Transition.

Type	d	X_ϵ	S	Y
Elem. Massless	2	p	p	$p+m$
	3	$q+2(n-m)$	$q+(n-m)$	q
	4	p	$p+m$	$p+m$
D-Branes	2		m (massless)	m (2-Branes, massive)
	3	$n-m$ (3-Branes, massive)	$n-m$ (massless)	
Total Massless	2	p	$\boldsymbol{p+m}$	$p+m$
	3	$q+2(n-m)$	$\boldsymbol{q+2(n-m)}$	q
	4	p	$\boldsymbol{p+m}$	$p+m$
H_*	2	p	p	$p+m$
	3	$q+2(n-m)$	$q+(n-m)$	q
	4	p	$p+m$	$p+m$
				$H_*(Y) = IH_*(S)$

$n =$ number of nodes in S,
$p = b_2(X_\epsilon)$,
$q = \mathrm{rk}(H_3(S-\Sigma) \to H_3(S)) = \mathrm{rk}\, IH_3(S)$,
$m = \mathrm{rk}\,\mathrm{coker}(H_4(X_\epsilon) \to H_4(S))$.

Problem posed by T. Hübsch (suggested by work of Strominger):

Construct a homology theory $\mathcal{H}$ defined at least on conifolds S, such that

(SH1) $\mathcal{H}_*(S) = H_*(S)$ (ordinary homology) if the singular set of S is empty,

(SH2) $\mathcal{H}_*(S)$ satisfies Poincaré duality, and

(SH3) $\mathcal{H}_3(S)$ is an extension of $H_3(S)$ by $\ker(H_3(S - \Sigma) \to H_3(S))$.

Abdul Rahman: Approach via MacPherson-Vilonen Zig-Zag analysis of perverse sheaves.

Backbone of Construction: SPATIAL HOMOLOGY TRUNCATION.

$$\boxed{\begin{array}{l}\text{Postnikov}\\ \text{decomposition}\end{array}} \longleftrightarrow \boxed{\begin{array}{l}\text{Moore space}\\ \text{decomposition}\end{array}}$$

Eckmann-Hilton
dual

- $p_n(X) : X \to P_n(X)$ stage-n Postnikov approximation for X:

  $$p_n(X)_* : \pi_r(X) \to \pi_r(P_n(X))$$

  is an isomorphism for $r \leq n$ and
  $\pi_r(P_n(X)) = 0$ for $r > n$.

- If Z is a space with $\pi_r(Z) = 0$ for $r > n$ then any map $g : X \to Z$ factors up to homotopy uniquely through $P_n(X)$.

In particular: Given any map $f : X \to Y$, there exists, uniquely up to homotopy, a map

$$p_n(f) : P_n(X) \to P_n(Y)$$

such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p_n(X)}\downarrow & & \downarrow{\scriptstyle p_n(Y)} \\ P_n(X) & \xrightarrow{p_n(f)} & P_n(Y) \end{array}$$

homotopy commutes.

Observation:
This functorial property of Postnikov approximations does **not** dualize to homology decompositions!

Example.

- $X := S^2 \cup_2 e^3$ a Moore space $M(\mathbb{Z}/_2, 2)$.
- $Y := X \vee S^3$.
- $X_{\leq 2} = X$, $Y_{\leq 2} = X$ Moore approximations.

Claim: whatever maps $i : X_{\leq 2} \to X$ and $j : Y_{\leq 2} \to Y$ such that $i_* : H_r(X_{\leq 2}) \to H_r(X)$ and $j_* : H_r(Y_{\leq 2}) \to H_r(Y)$ are isomorphisms for $r \leq 2$ one takes, there is always a map $f : X \to Y$ that cannot be compressed into the stage-2 Moore approximations:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X_{\leq 2} \arrow[u, "i"] \arrow[r, dashed, "\nexists f_{\leq 2}"] & Y_{\leq 2} \arrow[u, "j"'] \end{tikzcd}$$

commutative up to homotopy.

Take $f : S^2 \cup_2 e^3 \to \frac{S^2 \cup_2 e^3}{S^2} = S^3 \hookrightarrow X \vee S^3$.

- Point of view adopted here: lack of functoriality of Moore approximations due to wrong category theoretic setup.

- Solution: consider CW-complexes endowed with extra structure and cellular maps that preserve that extra structure.

- Will see that such morphisms can then be compressed into homology truncations.

- Every CW-complex can indeed be endowed with the requisite extra structure (in general not canonically).

- Given a cellular map, it is *not* always possible to adjust the extra structure on the source and on the target of the map so that the map preserves the structures.

Concepts.

Let n be a positive integer.

Def. A CW-complex K is called *n-segmented* if it contains a subcomplex $K_{<n} \subset K$ such that

$$H_r(K_{<n}) = 0 \text{ for } r \geq n \tag{1}$$

and

$$i_* : H_r(K_{<n}) \xrightarrow{\cong} H_r(K) \text{ for } r < n, \tag{2}$$

where i is the inclusion of $K_{<n}$ into K.

Lemma. Let K be an n-dimensional CW-complex. If its group of n-cycles has a basis of cells then K is n-segmented.

Let $n \geq 3$ be an integer.

Def. A (*homological*) n-*truncation structure* is a quadruple $(K, K/n, h, K_{<n})$, where

1. K is a simply connected CW-complex,

2. K/n is an n-dimensional CW-complex with $(K/n)^{n-1} = K^{n-1}$ and such that the group of n-cycles of K/n has a basis of cells,

3. $h : K/n \to K^n$ is the identity on K^{n-1} and a cellular homotopy equivalence rel K^{n-1},

4. $K_{<n} \subset K/n$ is a subcomplex with properties (1) and (2) with respect to K/n and such that $(K_{<n})^{n-1} = K^{n-1}$.

Prop. Every simply connected CW-complex K can be completed to an n-truncation structure $(K, K/n, h, K_{<n})$.

Proof is based on methods due to **P. Hilton**.
Not 3-segm. $K = S^2 \cup_4 e^3 \cup_6 e^3 \overset{h}{\simeq}$

$$S^2 \cup_2 e^3 \cup_0 e^3 = K/3 \text{ (3-segmented)}$$

Def. A *morphism*

$$(K, K/n, h_K, K_{<n}) \longrightarrow (L, L/n, h_L, L_{<n})$$

of homological n-truncation structures is a commutative diagram

$$\begin{tikzcd}
K \arrow[d, "f"'] & K^n \arrow[l, hook'] \arrow[d, "f|"'] & K/n \arrow[l, "h_K", "\simeq"'] \arrow[d, "f/n"'] & K_{<n} \arrow[l, hook'] \arrow[d, "f_{<n}"'] \\
L & L^n \arrow[l, hook'] & L/n \arrow[l, "h_L", "\simeq"'] & L_{<n} \arrow[l, hook']
\end{tikzcd}$$

in **CW**. Get category $\mathbf{CW}_{\supset<n}$ and associated homotopy category $\mathbf{HoCW}_{\supset<n}$.

Def. Category $\mathbf{CW}^n_{\supset\partial}$ of *n-boundary-split CW-complexes*:

- Objects are pairs (K, Y), where
  - K is a simply connected CW-complex,
  - $Y \subset C_n(K)$ is a subgroup $Y = s(\mathrm{Im}\,\partial_n)$ given by some splitting

    $$s : \mathrm{Im}\,\partial_n \to C_n(K)$$

    of the boundary operator
    $\partial_n : C_n(K) \to \mathrm{Im}\,\partial_n(\subset C_{n-1}(K))$.

- Morphisms $(K, Y_K) \to (L, Y_L)$ are cellular maps $f : K \to L$ such that $f_*(Y_K) \subset Y_L$.

Will construct covariant assignment

$$\tau_{<n} : \mathbf{CW}^n_{\supset\partial} \longrightarrow \mathbf{HoCW}_{\supset<n}$$

of objects and morphisms.

- Given $(K,Y) \in \mathbf{CW}^n_{\supset\partial}$.

- By the proposition, (K,Y) can be completed to an n-truncation structure $(K, K/n, h, K_{<n})$ in $\mathbf{CW}_{\supset<n}$ such that

  $$h_* i_* C_n(K_{<n}) = Y,$$

  where $i_* : C_n(K_{<n}) \to C_n(K/n)$ is the monomorphism induced by the inclusion $i : K_{<n} \hookrightarrow K/n$.

- Choose such a completion and set

  $$\tau_{<n}(K,Y) = (K, K/n, h, K_{<n}).$$

Compression Theorem. Any morphism $f : (K, Y_K) \to (L, Y_L)$ in $\mathbf{CW}^n_{\supset\partial}$ can be completed to a morphism $\tau_{<n}(K, Y_K) \to \tau_{<n}(L, Y_L)$ in $\mathbf{HoCW}_{\supset<n}$.

Set $\tau_{<n}(f) = (f, f^n, f/n, f_{<n})$.

Instructive to return to the example
$f : K = S^2 \cup_2 e^3 \to (S^2 \cup_2 e^3) \vee S^3 = L$,
$\nexists f_{<3} : K_{<3} \to L_{<3}$.
Compr. Thm. $\Rightarrow$ f cannot be promoted to a morphism $f : (K, Y_K) \to (L, Y_L)$.

Indeed:
$f_* : C_3(K) = \mathbb{Z}e^3 \to \mathbb{Z}e^3 \oplus \mathbb{Z}S^3 = C_3(L)$
$f_*(e^3) = S^3$.
$Y_K = C_3(K)$ unique, $Y_L = \mathbb{Z}(e^3 + mS^3)$, $m \in \mathbb{Z}$.
$\Rightarrow f_*(Y_K) = \mathbb{Z}S^3 \not\subset Y_L$.

• Compressibility criteria for homotopies
$\rightsquigarrow$ Obstruction theory for
$\tau_{<n}(g \circ f) = \tau_{<n}(g) \circ \tau_{<n}(f)$ in $\mathbf{HoCW}_{\supset <n}$.
(Vanish over $\mathbb{Q}$ if $H_2 = 0$.)

• $f : (K, Y_K) \to (L, Y_L)$ homotopy equ. $\Rightarrow f_{<n} : K_{<n} \to L_{<n}$ homotopy equ. (no obstruction).

• Particularly benign cases: (s.c.) spaces with vanishing odd-dimensional homology (s.c. 4-manifolds; nonsingular toric varieties,...)

• Continuity properties for benign K:

$$\begin{tikzcd} & G(t_{<n}K) \arrow[d] \\ \mathrm{Homeo}_{CW}(K) \arrow[ur, dotted, "\tilde{t}_{<n}"] \arrow[r, "t_{<n}"] & G[t_{<n}K] \end{tikzcd}$$

The map $t_{<n}$ is a group homomorphism. The map $\tilde{t}_{<n}$ is an H-map, but not in general a monoid homomorphism.

• Fiberwise spatial homology truncation in certain situations.

The Intersection Space in the Isolated Singularities Case.

Let X be an n-dimensional, compact, oriented pseudomanifold with isolated singularities $x_1, \ldots, x_w$ ("conifold") and simply connected links $L_i = \text{Link}(x_i)$.

- Set cut-off value to $k = n - 1 - \bar{p}(n)$.
- Fix completions (L_i, Y_i) of L_i so that every (L_i, Y_i) is an object in $\mathbf{CW}^k_{\supset\partial}$.
- Applying truncation, obtain

  $\tau_{<k}(L_i, Y_i) = (L_i, L_i/k, h_i, (L_i)_{<k}) \in \mathbf{HoCW}_{\supset<k}$.
- Let $f_i : (L_i)_{<k} \longrightarrow L_i$ be the composition

$$(L_i)_{<k} \hookrightarrow L_i/k \overset{h_i}{\simeq} L_i^k \hookrightarrow L_i.$$

- $M := X - \bigsqcup_i \overset{\circ}{\text{cone}}(L_i) \simeq X - \text{Sing}$.

- $\partial M = \bigsqcup_i L_i =: L$.

- $L_{<k} := \bigsqcup_i (L_i)_{<k}$

- Define a map

$$g : L_{<k} \longrightarrow M$$

by composing

$$L_{<k} \xrightarrow{f} \partial M \overset{j}{\hookrightarrow} M,$$

where $f = \bigsqcup_i f_i$.

- The **intersection space** is the homotopy cofiber of the map g:

$$\boxed{I^{\bar{p}}X = \text{cone(g)} = M \cup_g c(L_{<k}).}$$

THEOREM. (Rational Coefficients; $\bar{p}$, $\bar{q}$ complementary perversities.)

1. Generalized Poincaré Duality:

$$\widetilde{H}_i(I^{\bar{p}}X)^* \cong \widetilde{H}_{n-i}(I^{\bar{q}}X).$$

2. $IH^{\bar{p}}_\bullet(X)$ and $\widetilde{H}_\bullet(I^{\bar{p}}X)$ are "reflectively" related:

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_{k-1}(I^{\bar{p}}X)^* \arrow[r] \arrow[d, "\cong"] & IH^{\bar{p}}_{k-1}(X)^* \arrow[r] \arrow[d, "\cong"] & H_{k-1}(L)^* \arrow[r, "\beta_+^*"] \arrow[d, "\cong"] & {} \\
\cdots \arrow[r] & \widetilde{H}_{n-k+1}(I^{\bar{q}}X) \arrow[r] & IH^{\bar{q}}_{n-k+1}(X) \arrow[r] & H_{n-k}(L) \arrow[r, "\delta_-"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
 & IH^{\bar{p}}_k(X)^* \arrow[dr, hook, "\alpha'^*_-"] \arrow[d, "\cong"] & & \\
H_k(j)^* \arrow[ur, two heads, "\alpha'^*_+"] \arrow[dr, hook, "\alpha^*_+"'] \arrow[d, "d'_M"', "\cong"] & IH^{\bar{q}}_{n-k}(X) \arrow[dr, hook, "\gamma'_+"] & H_k(M)^* \arrow[r, "\beta^*_-"] \arrow[d, "d_M"', "\cong"] & {} \\
H_{n-k}(M) \arrow[ur, two heads, "\gamma'_-"] \arrow[dr, hook, "\gamma_-"'] & \widetilde{H}_k(I^{\bar{p}}X)^* \arrow[ur, two heads, "\alpha^*_-"'] \arrow[d, "d"', "\cong"] & H_{n-k}(j) \arrow[r, "\delta_+"] & {} \\
 & \widetilde{H}_{n-k}(I^{\bar{q}}X) \arrow[ur, two heads, "\gamma_+"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
H_k(L)^* \arrow[r] \arrow[d, "d_L"', "\cong"] & IH^{\bar{p}}_{k+1}(X)^* \arrow[r] \arrow[d, "\cong"] & \widetilde{H}_{k+1}(I^{\bar{p}}X)^* \arrow[r] \arrow[d, "\cong"] & \cdots \\
H_{n-k-1}(L) \arrow[r] & IH^{\bar{q}}_{n-k-1}(X) \arrow[r] & \widetilde{H}_{n-k-1}(I^{\bar{q}}X) \arrow[r] & \cdots
\end{tikzcd}$$

RETURNING TO STRING THEORY.

Given conifold transition $X \rightsquigarrow S \rightsquigarrow Y$.
COR. $H_3(IS)$ is an extension of $H_3(S)$ by $\ker(H_3(S - \mathrm{Sing}) \to H_3(S))$.

Proof: $(M, \partial M) = S-$open nbhd. of nodes.

$$\begin{tikzcd}
 & H_3(M) \arrow[dr, hook, "\alpha_-"] & & & \\
\text{Exact: } H_3(\partial M) \arrow[ur, "j_*"] \arrow[rr, "\alpha_- j_*"'] & & H_3(IS) \arrow[r, "\alpha_+"] & H_3(j) \arrow[r] & 0.
\end{tikzcd}$$

$$0 \to \mathrm{Im}(\alpha_- j_*) \longrightarrow H_3(IS) \xrightarrow{\alpha_+} H_3(S) \to 0,$$

$\mathrm{Im}\, j_* \cong \mathrm{Im}(\alpha_- j_*)$,

$$H_3(\partial M) \xrightarrow{j_*} H_3(M) \longrightarrow H_3(M, \partial M),$$

$$\begin{tikzcd}
H_3(M) \arrow[r] \arrow[d, "\cong"'] & H_3(M, \partial M) \arrow[d, "\cong"] \\
H_3(S - \mathrm{Sing}) \arrow[r] & H_3(S)
\end{tikzcd}$$

□

An Example.

- Set $N^4 = S^2 \times T^2$.

- Drill out a small open 4-ball: $N_0 = N - \operatorname{int} D^4$, $\partial N_0 = S^3$.

- Set $M^8 = N_0 \times S^2 \times S^2$.

- $L = \partial M = S^3 \times S^2 \times S^2$.

- The pseudomanifold
$$X^8 = M \cup_L \operatorname{cone}(L)$$
has one singular point of even codimension.

- $IH_\bullet^{\bar{m}}(X) = IH_\bullet^{\bar{n}}(X)$, and for the intersection spaces $I^{\bar{m}}X = I^{\bar{n}}X$.

- Cut-off value $k = 4$, $L_{<4} = S^3 \vee S^2 \vee S^2$.

- Intersection space
$$IX = \frac{N_0 \times S^2 \times S^2}{S^3 \vee S^2 \vee S^2}.$$

- Generating cycles in $H_\bullet(N)$:
$a = [S^2 \times \cdot \times \cdot]$, $b = [\cdot \times S^1 \times \cdot]$, $c = [\cdot \times \cdot \times S^1]$.

- Generating cycles in $H_\bullet(L)$:
$x = [S^3 \times \cdot \times \cdot]$, $y = [\cdot \times S^2 \times \cdot]$, $z = [\cdot \times \cdot \times S^2]$.

$$IH_4(X) = \mathbb{Q}\langle ay, bcy, az, bcz \rangle.$$

$$H_4(M) = \mathbb{Q}\langle ay, bcy, az, bcz, \boxed{yz} \rangle.$$

$$H_4(j) = H_4(X) = \mathbb{Q}\langle ay, bcy, az, bcz, \boxed{abc} \rangle,$$

$$\widetilde{H}_4(IX) = \mathbb{Q}\langle ay, bcy, az, bcz, \boxed{abc, yz} \rangle.$$

Cap Products.

- $\widehat{X}$:= cone(j) = $X/(x_1 \sim x_2 \sim \cdots \sim x_w)$ ("denormalization" of X).

- Canonical maps

$$M \xrightarrow{b} I^{\bar{p}}X \xrightarrow{c} \widehat{X}$$

such that

$$\begin{tikzcd}
H_r(M) \arrow[r, "b_*"] \arrow[dr, "a_*"'] & H_r(I^{\bar{p}}X) \arrow[d, "c_*"] \\
& H_r(\widehat{X})
\end{tikzcd}$$

commutes.

Proposition. Suppose $n = \dim X \equiv 2 \mod 4$. Then there exists a cap-product

$$\widetilde{H}^{2l}(I^{\bar{m}}X) \otimes \widetilde{H}_i(\hat{X}) \xrightarrow{\cap} \widetilde{H}_{i-2l}(I^{\bar{m}}X)$$

such that

$$\begin{tikzcd}
\widetilde{H}^{2l}(I^{\bar{m}}X) \otimes \widetilde{H}_i(\hat{X}) \arrow[r, "\cap"] & \widetilde{H}_{i-2l}(I^{\bar{m}}X) \arrow[d, "c_*"] \\
\widetilde{H}^{2l}(\hat{X}) \otimes \widetilde{H}_i(\hat{X}) \arrow[u, "c^* \otimes id"] \arrow[r, "\cap"] & \widetilde{H}_{i-2l}(\hat{X})
\end{tikzcd}$$

commutes.

Proposition. Suppose $n = \dim X \equiv 1 \mod 4$. Then there exists a cap-product

$$\widetilde{H}^{2l}(I^{\bar{m}}X) \otimes \widetilde{H}_i(\hat{X}) \xrightarrow{\cap} \widetilde{H}_{i-2l}(I^{\bar{n}}X)$$

such that

$$\begin{tikzcd}
\widetilde{H}^{2l}(I^{\bar{m}}X) \otimes \widetilde{H}_i(\hat{X}) \arrow[r, "\cap"] & \widetilde{H}_{i-2l}(I^{\bar{n}}X) \arrow[d, "c_*"] \\
\widetilde{H}^{2l}(\hat{X}) \otimes \widetilde{H}_i(\hat{X}) \arrow[u, "c^* \otimes id"] \arrow[r, "\cap"] & \widetilde{H}_{i-2l}(\hat{X})
\end{tikzcd}$$

commutes.

(Similar statements for $n \equiv 0, 3 \mod 4$.)

L-Theory.

- Let $\mathbb{L}^\bullet$ be the symmetric L-spectrum with homotopy groups

$$\pi_i(\mathbb{L}^\bullet) = L^i(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i \equiv 0(4) \text{ (sign.)} \\ \mathbb{Z}/_2, & i \equiv 1(4) \text{ (de Rham)} \\ 0, & i \equiv 2,3(4). \end{cases}$$

- A compact, oriented n-manifold-with-boundary $(M, \partial M)$ possesses a canonical $\mathbb{L}^\bullet$-orientation $[M, \partial M]_{\mathbb{L}} \in H_n(M, \partial M; \mathbb{L}^\bullet)$.

- given rationally by the homology L-class of M:

$$[M, \partial M]_{\mathbb{L}} \otimes 1 = \mathcal{L}_*(M, \partial M) = \mathcal{L}^*(M) \cap [M, \partial M]$$

$$\in H_n(M, \partial M; \mathbb{L}^\bullet) \otimes \mathbb{Q} = \bigoplus_{i \geq 0} H_{n-4i}(M, \partial M; \mathbb{Q}).$$

There is defined a cap-product

$$\cap : H^i(M;\mathbb{L}^\bullet)\otimes H_n(M,\partial M;\mathbb{L}^\bullet) \longrightarrow H_{n-i}(M,\partial M;\mathbb{L}^\bullet)$$

such that

$$-\cap [M,\partial M]_{\mathbb{L}} : H^i(M;\mathbb{L}^\bullet) \longrightarrow H_{n-i}(M,\partial M;\mathbb{L}^\bullet)$$

is an isomorphism (**Poincaré duality**).

THEOREM. Let X be an n-dimensional pseudomanifold with isolated singularities. Capping with the $\mathbb{L}^\bullet$-homology fundamental class $[\hat{X}]_{\mathbb{L}} \in \widetilde{H}_n(\hat{X};\mathbb{L}^\bullet)$ induces rationally an isomorphism

$$[\hat{X}]_{\mathbb{L}}\otimes 1\cap - : \widetilde{H}^0(I^{\bar{m}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} \xrightarrow{\cong} \widetilde{H}_n(I^{\bar{m}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q}$$

for $n \equiv 2 \mod 4$ and $n \equiv 4 \mod 8$ such that

$$\begin{array}{ccc}
\widetilde{H}^0(I^{\bar{m}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} & \xrightarrow[{[\hat{X}]_{\mathbb{L}}\otimes 1\cap -}]{\cong} & \widetilde{H}_n(I^{\bar{m}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} \\
{\scriptstyle c^*}\uparrow & & \downarrow{\scriptstyle c_*} \\
\widetilde{H}^0(\hat{X};\mathbb{L}^\bullet)\otimes\mathbb{Q} & \xrightarrow[{[\hat{X}]_{\mathbb{L}}\otimes 1\cap -}]{} & \widetilde{H}_n(\hat{X};\mathbb{L}^\bullet)\otimes\mathbb{Q}
\end{array}$$

commutes.

For $n \equiv 1 \mod 4$:

$$\begin{tikzcd}
\widetilde{H}^0(I^{\bar{m}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} \arrow[r, "\cong", "{[\hat{X}]_{\mathbb{L}}\otimes 1\cap -}"'] & \widetilde{H}_n(I^{\bar{n}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} \arrow[d, "c_*"] \\
\widetilde{H}^0(\hat{X};\mathbb{L}^\bullet)\otimes\mathbb{Q} \arrow[u, "c^*"] \arrow[r, "{[\hat{X}]_{\mathbb{L}}\otimes 1\cap -}"'] & \widetilde{H}_n(\hat{X};\mathbb{L}^\bullet)\otimes\mathbb{Q}
\end{tikzcd}$$

For $n \equiv 3 \mod 4$:

$$\begin{tikzcd}
\widetilde{H}^0(I^{\bar{n}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} \arrow[r, "\cong", "{[\hat{X}]_{\mathbb{L}}\otimes 1\cap -}"'] & \widetilde{H}_n(I^{\bar{m}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} \arrow[d, "c_*"] \\
\widetilde{H}^0(\hat{X};\mathbb{L}^\bullet)\otimes\mathbb{Q} \arrow[u, "c^*"] \arrow[r, "{[\hat{X}]_{\mathbb{L}}\otimes 1\cap -}"'] & \widetilde{H}_n(\hat{X};\mathbb{L}^\bullet)\otimes\mathbb{Q}
\end{tikzcd}$$

An Example.

Consider the pseudomanifold

$$X^{12} = D^4 \times \mathbb{P}^4 \cup_{S^3 \times \mathbb{P}^4} c(S^3 \times \mathbb{P}^4).$$

- Cutoff-value $k = 6$.
- $L_{<6} = (S^3 \times \mathbb{P}^4)^5$. (5-skeleton)
- $I^{\bar{m}}X = \text{cofiber}((S^3 \times \mathbb{P}^4)^5 \hookrightarrow D^4 \times \mathbb{P}^4)$.

$$\begin{array}{rccc}
\widetilde{H}_{12}(I^{\bar{m}}X; \mathbb{L}^\bullet) \otimes \mathbb{Q} = & \mathbb{Q}[pt] \times [\mathbb{P}^4] & \oplus & \mathbb{Q}\mu \times [\mathbb{P}^0] \\
& \updownarrow_{PD.} & & \updownarrow_{PD.} \\
\widetilde{H}^0(I^{\bar{m}}X; \mathbb{L}^\bullet) \otimes \mathbb{Q} = & \mathbb{Q}d \times 1 & \oplus & \mathbb{Q}1 \times g^4.
\end{array}$$

($\mu = [D^4, S^3] \in H_4(D^4, S^3)$,
$d \in H^4(D^4, S^3)$ gen. s.t. $d \cap \mu = [pt] \in H_0(D^4)$,
$g = -c_1(\text{taut. line bundle}) \in H^2(\mathbb{P}^4)$.)

Beyond Isolated Singularities.

- Let X be an n-dimensional, compact, stratified pseudomanifold with **two strata**

$$X = X_n \supset X_{n-c}.$$

- The singular set $\Sigma = X_{n-c}$ is thus an $(n-c)$-dimensional closed manifold.

- Assume that X has a **trivial link bundle**, that is, a neighborhood of Σ in X looks like $\Sigma \times \overset{\circ}{\text{cone}}(L)$, where L is a $(c-1)$-dimensional closed manifold, the link of Σ.

- Assume that L is simply connected.

- Idea: construct $I^{\bar{p}}X$ by performing **fiber-wise truncation**.

- Set cut-off to $k = c - 1 - \bar{p}(c)$.

- Fix completion (L, Y) of L so that $(L, Y) \in \mathbf{CW}^k_{\supset \partial}$.

- Applying truncation, obtain

  $$\tau_{<k}(L, Y) = (L, L/k, h, L_{<k}) \in \mathbf{HoCW}_{\supset <k}$$

  and a map

  $$f : L_{<k} \longrightarrow L$$

- Manifold $M^n := X - (\Sigma \times \overset{\circ}{\text{cone}}(L))$.

- $\partial M = \Sigma \times L$.

- Let

$$g : \Sigma \times L_{<k} \longrightarrow M$$

be the composition

$$\Sigma \times L_{<k} \xrightarrow{id_\Sigma \times f} \Sigma \times L = \partial M \overset{j}{\hookrightarrow} M.$$

- The intersection space is the homotopy cofiber of the map g:

$$\boxed{I^{\bar{p}}X = \text{cone(g)} = M \cup_g c(\Sigma \times L_{<k}).}$$

THEOREM. There exists a generalized Poincaré duality isomorphism

$$D : \widetilde{H}^{n-r}(I^{\bar{p}}X) \xrightarrow{\cong} \widetilde{H}_r(I^{\bar{q}}X)$$

such that both

$$\begin{tikzcd}
\widetilde{H}^{n-r}(I^{\bar{p}}X) \arrow[r] \arrow[d, "D"', "\cong"] & H^{n-r}(M) \arrow[d, "\cong"', "-\cap[M,\partial M]"] \\
\widetilde{H}_r(I^{\bar{q}}X) \arrow[r] & H_r(M, \partial M)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^{n-r-1}(\Sigma \times L_{<k}) \arrow[r, "\delta^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & \widetilde{H}^{n-r}(I^{\bar{p}}X) \arrow[d, "\cong"', "D"] \\
H_r(\partial M, \Sigma \times L_{<c-k}) \arrow[r] & \widetilde{H}_r(I^{\bar{q}}X)
\end{tikzcd}$$

commute.

An Example.

- $L := S^3 \times S^4$, $M^{14} := D^3 \times S^2 \times S^2 \times L$.

- Pseudomanifold

$$X^{14} = M \cup_{\partial M} S^2 \times S^2 \times S^2 \times \text{cone}(L).$$

- Singular set $\Sigma = S^2 \times S^2 \times S^2 \times \{c\}$,
  Link $= L$.

- Cut-off value $k = 4$.

- $L_{<4} = S^3 \times pt$.

- Intersection space

$$I^{\bar{m}}X \simeq \frac{D^3 \times S^2 \times S^2 \times S^3 \times S^4}{S^2 \times S^2 \times S^2 \times S^3 \times pt}.$$

- If A, B are cycles in a 2-sphere and C is a cycle in the 3-sphere then

$$D^3 \times A \times B \times C \times pt \cup_{S^2 \times A \times B \times C \times pt}$$

$$\text{cone}(S^2 \times A \times B \times C \times pt)$$

is a cycle in the space $I^{\bar{m}}X$.

- We shall denote the homology class of such a cycle briefly by $[D^3 \times A \times B \times C \times pt]^{\wedge}$.

Dual cycles are next to each other in the same row.

	$\widetilde{H}_*(I^{\bar{m}}X)$	$\widetilde{H}_{14-*}(I^{\bar{m}}X)$
$* = 0$	0	0
$* = 1$	0	0
$* = 2$	0	0
$* = 3$	$[D^3 \times pt \times pt \times pt \times pt]^\wedge$	$[pt \times S^2 \times S^2 \times S^3 \times S^4]$
$* = 4$	$[pt \times pt \times pt \times pt \times S^4]$	$[D^3 \times S^2 \times S^2 \times S^3 \times pt]^\wedge$
$* = 5$	$[D^3 \times S^2 \times pt \times pt \times pt]^\wedge$ $[D^3 \times pt \times S^2 \times pt \times pt]^\wedge$	$[pt \times pt \times S^2 \times S^3 \times S^4]$ $[pt \times S^2 \times pt \times S^3 \times S^4]$
$* = 6$	$[pt \times S^2 \times pt \times pt \times S^4]$ $[pt \times pt \times S^2 \times pt \times S^4]$ $[D^3 \times pt \times pt \times S^3 \times pt]^\wedge$	$[D^3 \times pt \times S^2 \times S^3 \times pt]^\wedge$ $[D^3 \times S^2 \times pt \times S^3 \times pt]^\wedge$ $[pt \times S^2 \times S^2 \times pt \times S^4]$
$* = 7$	$[pt \times pt \times pt \times S^3 \times S^4]$ $[D^3 \times S^2 \times S^2 \times pt \times pt]^\wedge$	$[D^3 \times S^2 \times S^2 \times pt \times pt]^\wedge$ $[pt \times pt \times pt \times S^3 \times S^4]$