COMPUTING TWISTED L-CLASSES OF NON-WITT SPACES.

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ABSTRACT. In previous joint work with Cappell and Shaneson, we have established an Atiyah-Lusztig-Meyer-type multiplicative characteristic class formula for the twisted signature (and the twisted L-classes) of a stratified Witt space, e.g. a space with only even-codimensional strata. Twisted signatures arise naturally in geometric mapping situations and the ability to calculate them is an integral part of programs that aim to understand topological invariants of singular spaces. The talk will discuss recent developments on this front, in particular our recent result that the above mentioned characteristic class formula holds even when the space does not satisfy the Witt condition. It constitutes one of the first applications of a new homology theory, signature homology, introduced by Augusto Minatta.

Contents

1. Chern-Hirzebruch-Serre

Theorem 1.1. *(Chern-Hirzebruch-Serre.) If $F \to E \to B$ is a fiber bundle of closed, connected, coherently oriented manifolds, then the signature is multiplicative,*

$$\sigma(E) = \sigma(F)\sigma(B),$$

provided $\pi_1(B)$ acts trivially on $H^{\mathrm{mid}}(F;\mathbb{R})$. (E.g. B simply connected.)

2. Atiyah's Formula

Question: What if the representation is nontrivial?
Suppose $\dim F = 2k$. The above representation gives rise to a flat vector bundle $\mathbf{H}^k(F) \to B$ with fiber $H^k(F)$.

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Assume k even. Then the cup-product form on $H^k(F)$ is symmetric. Can construct an orthogonal splitting

$$\mathbf{H}^k(F) = \mathbf{H}^+ \oplus \mathbf{H}^-,$$

such that the form is positive definite on $\mathbf{H}^+$ and negative definite on $\mathbf{H}^-$. ($\mathbf{H}^+$ and $\mathbf{H}^-$ need not be flat anymore.) In the real K-theory of B, $KO^0(B)$, we can thus form the *K-theory signature*

$$[\mathbf{H}^k(F)]_K = [\mathbf{H}^+] - [\mathbf{H}^-] \in KO^0(B).$$

Theorem 2.1. *(Atiyah) Let E, B, F be smooth, closed, oriented manifolds. Then*

$$\sigma(E) = \langle \widetilde{ch}[\boldsymbol{H}^k(F)]_K \cup L(B), [B] \rangle,$$

where $L(B) \in H^{4*}(B;\mathbb{Q})$ is the Hirzebruch L-class and $\widetilde{ch}$ is the Chern character modified by precomposing with the second Adams operation.

The proof uses the Atiyah-Singer index theorem.

3. Meyer's Formula

Forget the bundle: Start with B, and a local coeff. system $\mathcal{S}/B$, not necessarily of geometric origin, together with symmetric, nondegenerate $\mathcal{S} \otimes \mathcal{S} \to \mathbb{R}_B$ ("Poincaré local system"). Then have twisted signature $\sigma(B;\mathcal{S})$.

Theorem 3.1. *(W. Meyer)*

$$\sigma(B;\mathcal{S}) = \langle \widetilde{ch}[\mathcal{S}]_K \cup L(B), [B] \rangle.$$

4. Motivation Stratified Maps: The work of Cappell-Shaneson

X^n, Y^m oriented, closed, Whitney stratified spaces of even dimension with <u>only even-codimensional strata</u>. Using **intersection homology**, have

- signature $\sigma(X)$,
- Goresky-MacPherson L-class $L(X) \in H_*(X;\mathbb{Q})$.

Definition 4.1. *Stratified map $f : Y \to X$*: Preimages of pure strata are unions of pure strata, restrictions to preimages of pure strata are topological fiber bundles.

If $\mathcal{S}$ is a Poincaré local system on the top stratum $X - \Sigma$, then can form Goresky-MacPherson-Deligne extension $\mathbf{IC}^\bullet_{\bar{m}}(X;\mathcal{S})$ (the twisted intersection chain complex of sheaves). If X has only even-codimensional strata, then this is self-dual. So can let $\sigma(X;\mathcal{S}) := \sigma(\mathbf{IC}^\bullet_{\bar{m}}(X;\mathcal{S}))$.

Theorem 4.1. *(Cappell-Shaneson.) Let $f : Y^m \longrightarrow X^n$ be a stratified map of oriented, compact, Whitney stratified spaces with only even-codimensional strata, $m - n$ even. Then*

$$\sigma(Y) = \sigma(X;\mathcal{S}^f_{X-\Sigma}) + \sum_{\textit{pure strata } Z \textit{ in } X} \sigma(\overline{Z};\mathcal{S}^f_Z),$$

where $\mathcal{S}^f_Z$ is a Poincaré local system over Z. If $Z = X - \Sigma$, then its stalk is $IH^{\bar{m}}_{\mathrm{mid}}(f^{-1}(pt))$, $pt \in X - \Sigma$. More generally, a similar formula holds for the pushforward of the L-class.

Question: How to compute these twisted terms further?

5. Witt Spaces: Work of Banagl-Cappell-Shaneson

Definition 5.1. X is *Witt*, if $IH^{\bar{m}}_{\mathrm{mid}}(Link(x)) = 0$ for points x in strata of odd codimension.

(Example: $X^7 = \Sigma\mathbb{C}P^3$.)

Given Poincaré local system $\mathcal{S}/X - \Sigma$.

2 Assumptions: (1) X is Witt,
(2) $\mathcal{S}$ is constant on links.

Theorem 5.1. *(Banagl-Cappell-Shaneson.) The twisted L-class $L(X;\mathcal{S}) \in H_*(X;\mathbb{Q})$ can be computed by*

$$L(X;\mathcal{S}) = \widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X).$$

In particular,

$$\sigma(X;\mathcal{S}) = \langle \widetilde{\mathrm{ch}}[\mathcal{S}]_K, L(X)\rangle.$$

GOAL OF THIS TALK: Eliminate assumption (1).
Assumption (2) cannot be eliminated because the formula will fail, as can be shown by constructing examples of 4-dimensional orbifolds with isolated singularities.

6. Non-Witt Spaces

Theorem 6.1. *(Banagl.) X closed, oriented, Whitney stratified pseudomanifold such that $\sigma(X)$ is defined. $\mathcal{S}/X - \Sigma$ Poincaré local system constant on links. Then*

$$L(X;\mathcal{S}) = \widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X).$$

Looking at this theorem, have to discuss:
How are $\sigma(X), L(X), L(X;\mathcal{S})$ defined? (These are not in general the Goresky-MacPherson L-classes, unless X is a Witt space.)
Define category $SD(X)$: full subcategory of derived category D(X) satisfying axioms: top stratum normalization, lower bound, $\bar{n}$-stalk vanishing condition, self-duality.
Theorem (B.): Can be described equivalently by Lagrangian subsheaves along strata of odd codimension.

Examples:
- $X^6 = S^1 \times \Sigma\mathbb{C}P^2$: $SD(X^6) = \varnothing$.
- $X^4 = S^1 \times \Sigma T^2$: $SD(X^4) \neq \varnothing$.

Given $\mathbf{S}^\bullet \in SD(X)$, have its signature $\sigma(\mathbf{S}^\bullet)$.

Thom-Pontrjagin construction $\rightsquigarrow L(\mathbf{S}^\bullet) \in H_*(X;\mathbb{Q})$.

Theorem 6.2. *(B.) $L(\boldsymbol{S}^\bullet)$ is independent of the choice of $\boldsymbol{S}^\bullet \in SD(X)$. Thus have well-defined $\sigma(X), L(X)$.*

6.1. Tool: Signature Homology Theory - Minatta/Kreck.

$S_*(Y) \approx$ bordism theory of stratified singular spaces X + object in $SD(X)$:

$$[(X, \mathbf{S}^\bullet/X, X \xrightarrow{f} Y)]$$

• **Coefficients:**
Groups introduced in Banagl's PhD-thesis.

$$S_n(pt) = \begin{cases} \mathbb{Z}, & n = 4k \text{ by } \sigma, \\ 0, & \text{otherw.} \end{cases}$$

Proof. Only n=4k.
σ onto: ($\mathbb{C}P^{2k}$, trivial sheaf) represents an element in $S_{4k}(pt)$.
σ injective: Suppose $\sigma(X, \mathbf{S}^\bullet) = 0$. $W = cone(X)$ is in general non-Witt, but since $\sigma = 0$, there exists Lagrangian structure at the cone point, so have $\mathbf{W}^\bullet \in SD(\operatorname{int} W)$. Then $(W, \mathbf{W}^\bullet)$ is a nullbordism for $(X, \mathbf{S}^\bullet)$. $\square$

[Compare Witt bordism: $\Omega_{4k}^{Witt} \cong Witt(\mathbb{Q}) = \mathbb{Z} \oplus \mathbb{Z}/_2 \oplus$ (copies of $\mathbb{Z}/_4$ or $\mathbb{Z}/_2 \oplus \mathbb{Z}/_2$).]

• **Several constructions:**
1. Top (Minatta): X = oriented, topological stratifolds (M. Kreck). His assumptions imply that X is a topological stratified pseudomanifold, but it's equipped with extra structure. This extra structure makes them easy to glue.

2. PL (Banagl): X = oriented PL-pseudomanifold + choice of triangulation, stratified simplicially. Advantage: monodromy difficulties regarding Lagrangian structures disappear.

3. Baas-Sullivan construction (Minatta/Kreck): Unitary bordism with Baas-Sullivan singularities $\{x_n\}_{n \neq 2}$, where $\{x_1, x_2, \ldots\}$ is a basis sequence for unitary bordism $\Omega_*^U = \mathbb{Z}[x_1, x_2, \ldots]$, $\deg x_n = 2n$, having $\sigma(x_2) = 1$, $\sigma(x_n) = 0$ for $n \neq 2$.

• **Situation at odd primes:**

$$\begin{tikzcd}
\Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\frac{1}{2}] \arrow[rr, "\simeq"] \arrow[dr, "\text{Minatta}"', "\simeq"] & & ko_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \\
& S_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \arrow[ur, "\simeq"] & \\
\Omega_*^{SO}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \arrow[uu] \arrow[ur, "\text{natural}"'] \arrow[rr, two heads, "\text{natural}"'] & & \Omega_*^{Witt}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \arrow[ul, "\simeq"'] \arrow[uu, "\text{Siegel}", "\simeq"']
\end{tikzcd}$$

6.2. **Sketch of Main Theorem's Proof.** Let $(Y, \mathcal{S})$ be a space + Poincaré local system on it.

1. Verify that assigning to a triple $(X, \mathbf{S}^\bullet, X \xrightarrow{f} Y)$ (representing an element of $S_n(Y)$) the integer

$$\langle \widetilde{\mathrm{ch}}[f^*\mathcal{S}]_K, L(X) \rangle$$

is a bordism invariant on the bordism group $S_n(Y)$: Only need $\partial_* L(W) = L(\partial W)$,

$$H_{*+1}(W, \partial W) \xrightarrow{\partial_*} H_*(W).$$

2. Verify that assigning to a triple $(X, \mathbf{S}^\bullet, X \xrightarrow{f} Y)$ the signature

$$\sigma(X; f^*\mathcal{S})$$

is a bordism invariant on $S_n(Y)$. Here, $\sigma(X; f^*\mathcal{S})$ is defined as the signature of any complex of sheaves in $SD(X; f^*\mathcal{S})$, which is nonempty since $\mathbf{S}^\bullet \in SD(X; \mathbb{R})$, and since the monodromy obstructions for Lagrangian structures vanish provided we are using S_*^{PL}.

3. Consider the element

$$[(X, \mathbf{S}^\bullet, X \xrightarrow{\mathrm{id}} X)] \otimes 1 \in S_n(X) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Since the canonical map

$$\Omega_n^{SO}(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \twoheadrightarrow S_n(X) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

is onto, $\exists$ manifold M^n, a continuous map $f : M \to X$, and $k \in \mathbb{Z}$ such that

$$k \cdot [(X, \mathbf{S}^\bullet, X \xrightarrow{\mathrm{id}} X)] = [(M, \mathbb{R}_M, M \xrightarrow{f} X)] \in S_n(X).$$

Thus

$$\begin{array}{rcll} k\sigma(X; \mathcal{S}) & = & \sigma(M; f^*\mathcal{S}) & \text{(bordism invariance, step 2.)} \\ & = & \langle \widetilde{\mathrm{ch}}[f^*\mathcal{S}]_K, L(M) \rangle & \text{(by Atiyah/Meyer)} \\ & = & k\langle \widetilde{\mathrm{ch}}[\mathcal{S}]_K, L(X) \rangle & \text{(bordism invariance, step 1.)} \end{array}$$