FIRST CASES OF INTERSECTION SPACES IN STRATIFICATION DEPTH 2

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Abstract. Previous constructions of intersection spaces for stratified pseudomanifolds all required the stratification depth to be at most 1. Here, we construct intersection spaces for certain simple stratifications of depth 2, involving different singularity links.

1. Introduction

In [Ban10], we introduced a method that associates to certain classes of stratified pseudomanifolds X CW complexes

$$I^{\bar{p}}X,$$

the *intersection spaces of* X, where $\bar{p}$ is a perversity in the sense of Goresky and MacPherson's intersection homology, such that the *ordinary* (reduced) cohomology $\widetilde{H}^*(I^{\bar{p}}X;\mathbb{Q})$ satisfies generalized Poincaré duality when X is closed and oriented. The resulting cohomology theory $X \rightsquigarrow HI^*_{\bar{p}}(X) = H^*(I^{\bar{p}}X)$ is not isomorphic to intersection cohomology $IH^*_{\bar{p}}(X)$, since the former has a $\bar{p}$-internal cup product while the latter does not, in general. For example, the singular Calabi-Yau quintic

$$X = \{z \in \mathbb{P}^4 \mid z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0\}$$

has intersection cohomology ranks

$$\operatorname{rk} IH^2(X) = 25,\ \operatorname{rk} IH^3(X) = 2,\ \operatorname{rk} IH^4(X) = 25,$$

whereas

$$\operatorname{rk} HI^2(X) = 1,\ \operatorname{rk} HI^3(X) = 204,\ \operatorname{rk} HI^4(X) = 1.$$

The expository article [BM12] of the present volume provides a gentle introduction to intersection spaces and surveys results obtained by Maxim and the author in [BM11] on the stability of HI^* under nearby smooth deformations of a singular projective hypersurface. Given a spectrum E in the sense of stable homotopy theory, one may form $EI^*_{\bar{p}}(X) = E^*(I^{\bar{p}}X)$. This, then, yields an approach to defining intersection versions of generalized cohomology theories such as K-theory. The theory HI^* also addresses questions in type II string theory related to the existence of massless D-branes arising in the course of a Calabi-Yau conifold transition. These questions are answered by IH^* for IIA theory, and by HI^* for IIB theory; see Chapter 3 of [Ban10]. A de Rham-type description of HI^* has been developed in [Ban11a], which has been applied in [Ban11b] to obtain spectral sequence degeneration results for flat bundles

Date: January, 2012.

2010 *Mathematics Subject Classification.* 55N33, 57P10.

Key words and phrases. Stratified spaces, pseudomanifolds, intersection homology, Poincaré duality.

The author was in part supported by a research grant of the Deutsche Forschungsgemeinschaft.

and equivariant cohomology of isometric group actions.

Up to the present point, intersection spaces have only been constructed for singular spaces that possess a stratification of depth at most 1, although a construction method for greater depths has been proposed in [Ban10], pp. 186 – 189. In the present note, we implement this method for certain stratifications of depth 2. We consider only the two middle perversities $\bar{m}$ and $\bar{n}$. Let X be an n-dimensional, compact, oriented PL pseudomanifold (without boundary) with $n = 2k > 0$ even. Suppose X can be endowed with a PL stratification of the form $X = X_n \supset X_1 \supset X_0$ with X_1 a circle and X_0 a point, such that the respective links L, L_0 of the two singular strata $X_1 - X_0, X_0$ are simply connected. For the link L of the odd-(co)dimensional stratum $X_1 - X_0$, we require the following strong version of the Witt condition: X satisfies the *strong Witt condition*, if L possesses a CW structure such that the cellular boundary operator $C_{k-1}(L) \to C_{k-2}(L)$ is injective. This condition implies of course that the middle homology $H_{k-1}(L)$ of the manifold L vanishes, which is the classical Witt condition of [Sie83] when rational coefficients are used. The strong Witt condition is obviously satisfied if L has no middle-dimensional cells. Closely related versions of this condition have been considered in the literature before. Weinberger [Wei99] calls an n-manifold *antisimple*, if its chain complex is chain homotopy equivalent to a complex of projective modules P_* with $P_i = 0$ for $i = [n/2]$. Hausmann considers manifolds that have a handlebody without middle-dimensional handles, which is stronger than our condition; see [Hau80, p.334, p.336]. For X satisfying the strong Witt condition, we follow the method of [Ban10] to construct the two middle-perversity intersection spaces $I^{\bar{m}}X$ and $I^{\bar{n}}X$. As expected, they turn out to be equal, and we put $IX = I^{\bar{m}}X = I^{\bar{n}}X$. The main theorem (Theorem 6.2) asserts that there exists a Poincaré duality isomorphism

$$D : \widetilde{H}^{n-r}(IX;\mathbb{Q}) \xrightarrow{\cong} \widetilde{H}_r(IX;\mathbb{Q})$$

that is compatible with Poincaré-Lefschetz duality for the exterior of the singular set.

The basic paradigm for the construction of intersection spaces is to replace links by their spatial homology truncations (Moore approximations), where the truncation degree is determined by the perversity function. We review spatial homology truncation in Section 3. The simple connectivity assumption on the links is adopted to ensure the existence of homology truncations, and is in practice not always necessary. Roughly, we proceed as follows: We first disassemble the boundary of a regular neighborhood of the singular set, so that we can build a nice homotopy theoretic model of it. This involves certain simple kinds of homotopy colimits, whose properties we collect in Section 2. In the disassembled state, the pieces are the link bundle over $X_1 - X_0$, the space obtained from the link L_0 of X_0 by removing cone neighborhoods of its two singular points, and a space $\ddot{L}$, PL homeomorphic to two copies of L, where the two other pieces are glued. The gluing involves maps from $\ddot{L}$ to the other two pieces. We then apply spatial homology truncation to truncate all these pieces (more precisely, the bundle over $X_1 - X_0$ is truncated in a fiberwise fashion), as well as the maps relating them to each other. Then the truncated pieces are reassembled again, using the truncated maps, and IX is the homotopy cofiber of the map from the reassembly to the complement of the open regular neighborhood of the singular set.

Notation and Conventions: If X and Y are topological spaces, $A \subset X$ a subspace, and $f : A \to Y$ a continuous map, then $Y \cup_f X$ denotes the space obtained from the disjoint union of X and Y by attaching X along A to Y using the map f, that is, $Y \cup_f X = (Y \sqcup X)/(a \sim f(a)$ for all $a \in A)$. Our convention for the mapping cylinder $Y \cup_f X \times I$ of a map $f : X \to Y$ is that the attaching is carried out at time 1, that is, the points of $X \times \{1\} \subset X \times I$ are attached to Y using f. The homology $H_*(f)$ of the map f is defined to be $H_*(f) = H_*(Y \cup_f X \times I, X \times \{0\})$. For products in cohomology and homology, we will use the conventions of Spanier's book [Spa66]. In particular, for an inclusion $i : A \subset X$ of spaces and elements $\xi \in H^p(X)$, $x \in H_n(X, A)$, the formula $\partial_*(\xi \cap x) = i^*\xi \cap \partial_* x$ holds for the connecting homomorphism ∂_* (no sign). For the compatibility between cap- and cross-product, one has the sign

$$(\xi \times \eta) \cap (x \times y) = (-1)^{p(n-q)}(\xi \cap x) \times (\eta \cap y),$$

where $\xi \in H^p(X)$, $\eta \in H^q(Y)$, $x \in H_m(X)$, and $y \in H_n(Y)$.

2. Required Properties of Homotopy Pushouts

In order to form the intersection space of a given pseudomanifold, one has to glue together pieces obtained at various stages of homology (Moore) towers. The gluing is accomplished via homotopy pushouts, whose fundamentals we shall collect in the present section. It is not possible to glue through ordinary pushouts, since the output of spatial homology truncation is only well-defined up to homotopy.

A *3-diagram* Γ *of spaces* is a diagram of the form

$$X \xleftarrow{f} A \xrightarrow{g} Y,$$

where A, X, Y are topological spaces and f, g are continuous maps. If A, X, Y are CW complexes and f, g are cellular, then we call Γ a *CW-3-diagram.* The *realization* $|\Gamma|$ of Γ is the pushout of f and g, that is,

$$|\Gamma| = (X \sqcup Y)/(f(a) \sim g(a), \text{ for all } a \in A).$$

If Γ is a CW-3-diagram and g is the inclusion of a subcomplex, then $|\Gamma|$ is a CW complex, [May99]. In particular, the mapping cylinder $\operatorname{cyl}(f)$ is a CW complex in a natural way. A *morphism* $\Gamma \to \Gamma'$ of 3-diagrams is a commutative diagram

(1)
$$\begin{tikzcd} X \arrow[d] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y' \end{tikzcd}$$

in the category of topological spaces. If Γ and Γ' are both CW-3-diagrams, then we call the morphism *cellular*, if all vertical arrows are cellular maps. The universal property of the pushout implies that a morphism $\Gamma \to \Gamma'$ induces a map $|\Gamma| \to |\Gamma'|$ between realizations. If $\Gamma \to \Gamma'$ is cellular, with g, g' subcomplex inclusions, then $|\Gamma| \to |\Gamma'|$ is cellular. A homotopy theoretic weakening of a morphism is the notion of an *h-morphism* $\Gamma \to_h \Gamma'$. This is again a diagram of the above form (1), but the two squares are required to commute only up to homotopy. An h-morphism does not induce a map between realizations. The remedy is to use the *homotopy pushout*, or double mapping cylinder. This is a special case of the notion of a *homotopy colimit.* To a 3-diagram Γ we associate another 3-diagram $H(\Gamma)$ given by

$$X \cup_f A \times I = \mathrm{cyl}(f) \overset{\text{at } 0}{\hookleftarrow} A \overset{\text{at } 0}{\hookrightarrow} \mathrm{cyl}(g) = Y \cup_g A \times I.$$

If Γ is a CW-3-diagram, then $\mathrm{cyl}(f)$ and $\mathrm{cyl}(g)$ are CW complexes and hence $H(\Gamma)$ is again a CW-3-diagram. We define the homotopy pushout, or homotopy colimit, of Γ to be

$$\mathrm{hocolim}(\Gamma) = |H(\Gamma)|.$$

If Γ is CW, then, as the two maps in $H(\Gamma)$ are subcomplex inclusions, $|H(\Gamma)|$ is a CW complex. Sometimes, especially in large diagrams, we will omit parentheses and briefly write $H\Gamma$ for $H(\Gamma)$, $C_*|H\Gamma|$ for the chain groups $C_*(|H\Gamma|)$, and $H_*|H\Gamma|$ for the homology groups $H_*(|H\Gamma|)$. The morphism $H(\Gamma) \to \Gamma$ given by

$$\begin{tikzcd} X \cup_f A \times I \arrow[d, "r"] & A \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}_A"] & Y \cup_g A \times I \arrow[d, "r"] \\ X & A \arrow[l, "f"'] \arrow[r, "g"] & Y, \end{tikzcd}$$

where the maps r are the canonical mapping cylinder retractions, induces a canonical map

$$\mathrm{hocolim}(\Gamma) \longrightarrow |\Gamma|.$$

A morphism $\Gamma \to \Gamma'$, given by

$$\begin{tikzcd} X \arrow[d, "\xi"] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\alpha"] & Y \arrow[d, "\eta"] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y', \end{tikzcd}$$

induces a morphism $H(\Gamma) \to H(\Gamma')$, given by

$$\begin{tikzcd} X \cup_f A \times I \arrow[d, "\xi \cup (\alpha \times \mathrm{id}_I)"] & A \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] \arrow[d, "\alpha"] & Y \cup_g A \times I \arrow[d, "\eta \cup (\alpha \times \mathrm{id}_I)"] \\ X' \cup_{f'} A' \times I & A' \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] & Y' \cup_{g'} A' \times I, \end{tikzcd}$$

which in turn induces a map $\mathrm{hocolim}(\Gamma) \to \mathrm{hocolim}(\Gamma')$. If $\Gamma \to \Gamma'$ is cellular, then $H(\Gamma) \to H(\Gamma')$ is cellular. Since the horizontal arrows are subcomplex inclusions, $\mathrm{hocolim}(\Gamma) \to \mathrm{hocolim}(\Gamma')$ is thus also cellular. If α, ξ and η are homeomorphisms, then $\xi \cup (\alpha \times \mathrm{id}_I)$ and $\eta \cup (\alpha \times \mathrm{id}_I)$ are homeomorphisms and hence $\mathrm{hocolim}(\Gamma) \to \mathrm{hocolim}(\Gamma')$ is a homeomorphism. An h-morphism $\Gamma \to_h \Gamma'$ together with a choice of homotopies between clockwise and counterclockwise compositions will induce a map on the homotopy pushout,

$$\mathrm{hocolim}(\Gamma) \longrightarrow |\Gamma'|.$$

Indeed, let

$$\begin{tikzcd} X \arrow[d, "\xi"] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\alpha"] & Y \arrow[d, "\eta"] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y' \end{tikzcd}$$

be the given h-morphism. Let $F : A \times I \to X'$ be a homotopy between $F_0 = f'\alpha$ and $F_1 = \xi f$. Let $G : A \times I \to Y'$ be a homotopy between $G_0 = g'\alpha$ and $G_1 = \eta g$. Then

$$\begin{tikzcd} X \cup_f A \times I \arrow[d, "\xi \cup_f F"] & A \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] \arrow[d, "\alpha"] & Y \cup_g A \times I \arrow[d, "\eta \cup_g G"] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y', \end{tikzcd}$$

commutes (on the nose) and thus defines a morphism $H(\Gamma) \to \Gamma'$. This morphism induces a continuous map on realizations $\operatorname{hocolim}(\Gamma) = |H(\Gamma)| \to |\Gamma'|$.

A pair (X, A) of (compactly generated) topological spaces is an *NDR pair*, if the inclusion $A \subset X$ is a closed cofibration. A relative CW complex, for instance, is an NDR pair.

Proposition 2.1. *If*

(2)
$$\begin{tikzcd} Y \arrow[d, "\phi_Y", "\simeq"'] & A \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[d, "\phi_A", "\simeq"'] & X \arrow[d, "\phi_X", "\simeq"'] \\ Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X' \end{tikzcd}$$

is a commutative diagram of continuous maps such that i and i' are inclusions of NDR pairs and ϕ_Y, ϕ_A, ϕ_X are homotopy equivalences, then

$$\phi_Y \cup \phi_X : Y \cup_f X \longrightarrow Y' \cup_{f'} X'$$

is a homotopy equivalence.

This is Theorem 1.13 in Section 1 of [FHT01], where a proof can be found. For our purposes, for example when cellular approximation is required, we need to weaken the assumptions of the above proposition by requiring the left square of the diagram to be merely homotopy commutative. A similar conclusion will then hold if the pushouts are replaced by homotopy pushouts.

Proposition 2.2. *If the right hand square of diagram (2) commutes and the left hand square commutes up to homotopy, i and i' are inclusions of NDR pairs and ϕ_Y, ϕ_A, ϕ_X are homotopy equivalences, then the homotopy pushouts of the first and second row are homotopy equivalent,*

$$Y \cup_f A \times I \cup_i X \simeq Y' \cup_{f'} A' \times I \cup_{i'} X'.$$

(In fact, both of these homotopy pushouts are equivalent to $Y' \cup_{f'} X'$.)

Proof. Let $H : A \times I \to Y'$ be a homotopy between $H_0 = f'\phi_A$ and $H_1 = \phi_Y f$. We claim that the map

$$\phi_Y \cup_f H : Y \cup_f A \times I \longrightarrow Y'$$

is a homotopy equivalence. To see this, consider the following homotopy $\{F_s\}_{0 \leq s \leq 1}$,

$$F_s : Y \cup_f A \times I \longrightarrow Y'.$$

For given s, F_s consists of ϕ_Y on Y. On $A \times [s, 1] \subset A \times I$, use $H(a, t)$, $a \in A$, $s \leq t \leq 1$. On the remaining part $A \times [0, s] \subset A \times I$, use $H(a, s)$ (constant in t). Then

$F_0 = \phi_Y \cup_f H$ and F_1 is ϕ_Y on Y and $F_1(a,t) = H(a,1) = \phi_Y f(a)$ for all t. We may think of F_1 as the composition of the mapping cylinder retraction

$$r : Y \cup_f A \times I \longrightarrow Y$$

induced by projection, and $\phi_Y : Y \to Y'$. Since both of these maps are homotopy equivalences, so is F_1. Thus $F_0 = \phi_Y \cup_f H$ is homotopic to a homotopy equivalence, thus itself a homotopy equivalence. Applying Proposition 2.1 to the (on the nose) commutative diagram

$$\begin{tikzcd}
Y \cup_f A \times I \arrow[d, "\phi_Y \cup_f H", "\simeq"'] & A \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "i"] \arrow[d, "\phi_A", "\simeq"'] & X \arrow[d, "\phi_X", "\simeq"'] \\
Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X'
\end{tikzcd}$$

yields the result that

$$\phi_Y \cup_f H \cup \phi_X : Y \cup_f A \times I \cup X \longrightarrow Y' \cup_{f'} X'$$

is a homotopy equivalence. Applying Proposition 2.1 to the commutative diagram

$$\begin{tikzcd}
Y' \cup_{f'} A' \times I \arrow[d, "r'", "\simeq"'] & A' \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "i'"] \arrow[d, equal] & X' \arrow[d, equal] \\
Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X'
\end{tikzcd}$$

where r' is the mapping cylinder retraction, yields an equivalence

$$Y' \cup_{f'} A' \times I \cup X' \xrightarrow{\simeq} Y' \cup_{f'} X'.$$

Both equivalences together show that

$$Y \cup_f A \times I \cup_i X \simeq Y' \cup_{f'} A' \times I \cup_{i'} X'.$$

$\square$

Proposition 2.3. *If a manifold M is decomposed as $M = M_+ \cup_{M_0} M_-$, with M_0 a compact codimension one submanifold and $M_\pm$ codimension 0 submanifolds with common boundary $\partial M_+ = M_0 = M_+ \cap M_- = \partial M_-$ so that $M = |\Gamma|$ with $\Gamma = (M_+ \leftarrow M_0 \to M_-)$, then there is a homeomorphism $|\Gamma| \cong |H(\Gamma)|$.*

Proof. The codimension one submanifold $M_0 = \partial M_+$ has a collar in M_+ and a collar in M_-, as $M_0 = \partial M_-$. Using this bicollar, a homeomorphism to the double mapping cylinder can be readily constructed. $\square$

3. Spatial Homology Truncation

The goal of spatial homology truncation is to associate to a CW complex K and an integer k a complex $t_{<k}K$ together with a cellular map $t_{<k}K \to K$, which induces an isomorphism $H_r(t_{<k}K) \to H_r(K)$ in degrees $r < k$, whereas $H_r(t_{<k}K) = 0$ for $r \geq k$.

Definition 3.1. A CW complex K is called *k-segmented* if it contains a subcomplex $K_{<k} \subset K$ such that $H_r(K_{<k}) = 0$ for $r \geq k$ and

$$i_* : H_r(K_{<k}) \xrightarrow{\cong} H_r(K) \text{ for } r < k,$$

where i is the inclusion of $K_{<k}$ into K.

Not every k-dimensional complex is k-segmented, but if K is simply connected, then K is homotopy equivalent to a k-segmented complex by [Ban10, Prop. 1.6, p. 12]. If the group of k-cycles of a k-dimensional CW complex K has a basis of cells, then K is k-segmented. Spatial homology truncation should also apply to maps $f : K \to L$. However, counterexamples in [Ban10] show that in general there need not exist a truncated map $t_{<k} f : t_{<k} K \to t_{<k} L$, which fits with the structural maps into a homotopy commutative square, see pages 3–5 and p. 39 of *loc. cit.* This problem can be addressed by introducing the following category.

Definition 3.2. The category $\mathbf{CW}_{k\supset\partial}$ of *k-boundary-split CW complexes* consists of the following objects and morphisms: Objects are pairs (K, Y), where K is a simply connected CW complex and $Y \subset C_k(K;\mathbb{Z})$ is a subgroup of the k-th cellular chain group of K that arises as the image $Y = s(\operatorname{im}\partial)$ of some splitting $s : \operatorname{im}\partial \to C_k(K;\mathbb{Z})$ of the boundary map $\partial : C_k(K;\mathbb{Z}) \to \operatorname{im}\partial (\subset C_{k-1}(K;\mathbb{Z}))$. (Given K, such a splitting always exists, since $\operatorname{im}\partial$ is free abelian.) A morphism $(K, Y_K) \to (L, Y_L)$ is a cellular map $f : K \to L$ such that $f_*(Y_K) \subset Y_L$.

Let $\mathbf{HoCW}_{k-1}$ denote the category whose objects are CW complexes and whose morphisms are rel $(k-1)$-skeleton homotopy classes of cellular maps. Let

$$t_{<\infty} : \mathbf{CW}_{k\supset\partial} \longrightarrow \mathbf{HoCW}_{k-1}$$

be the natural projection functor, that is, $t_{<\infty}(K, Y_K) = K$ for an object (K, Y_K) in $\mathbf{CW}_{k\supset\partial}$, and $t_{<\infty}(f) = [f]$ for a morphism $f : (K, Y_K) \to (L, Y_L)$ in $\mathbf{CW}_{k\supset\partial}$. The following theorem is part of Theorem 1.41 in [Ban10].

Theorem 3.3. *Let $k \geq 3$ be an integer. There is a covariant assignment $t_{<k}$: $\mathbf{CW}_{k\supset\partial} \longrightarrow \mathbf{HoCW}_{k-1}$ of objects and morphisms together with a natural transformation* $\mathrm{emb}_k : t_{<k} \to t_{<\infty}$ *such that for an object (K, Y) of $\mathbf{CW}_{k\supset\partial}$, one has $H_r(t_{<k}(K, Y)) = 0$ for $r \geq k$, and*

$$\mathrm{emb}_k(K, Y)_* : H_r(t_{<k}(K, Y)) \xrightarrow{\cong} H_r(K)$$

is an isomorphism for $r < k$.

This means in particular that given a morphism f, one has squares

$$\begin{array}{ccc} t_{<k}(K, Y_K) & \xrightarrow{\mathrm{emb}_k(K,Y_K)} & t_{<\infty}(K, Y_K) \\ {\scriptstyle t_{<k}(f)}\downarrow & & \downarrow{\scriptstyle t_{<\infty}(f)} \\ t_{<k}(L, Y_L) & \xrightarrow{\mathrm{emb}_k(L,Y_L)} & t_{<\infty}(L, Y_L) \end{array}$$

that commute in $\mathbf{HoCW}_{k-1}$. If $k \leq 2$, then the situation is much simpler and the category $\mathbf{CW}_{k\supset\partial}$ is not needed at all. For $k = 1$, there is a covariant truncation functor $t_{<1} : \mathbf{CW}^0 \to \mathbf{HoCW}$, where $\mathbf{CW}^0$ is the category of path-connected CW complexes and cellular maps. For $k = 2$, there is a covariant truncation functor $t_{<2}$: $\mathbf{CW}^1 \to \mathbf{HoCW}$, where $\mathbf{CW}^1$ is the category of simply connected CW complexes and cellular maps. See [Ban10, Section 1.1.5]. We call a space T together with a structural map $e : T \to K$ a *cohomological k-truncation* of K, if $H^r(T) = 0$ for $r \geq k$, and $e^* : H^r(K) \to H^r(T)$ is an isomorphism for $r < k$.

4. Homological Tools

Let j be a positive integer.

Definition 4.1. A CW complex K satisfies *condition* (INJ_j) if and only if the cellular chain boundary operator $\partial_j : C_j(K) \to C_{j-1}(K)$ is injective.

The condition is in particular satisfied if K has no j-cells. It implies of course that $H_j(K) = 0$. Let $Z_j(K) \subset C_j(K)$ denote the subgroup of j-cycles.

Lemma 4.2. *If K satisfies condition (INJ_j), then the following statements hold (for (1) and (2) assume that K is simply connected):*
(1) There is a unique completion of K to an object $(K, Y_j) \in \mathbf{CW}_{j\supset\partial}$, namely $Y_j = C_j(K)$.
(2) There is a unique completion of K to an object $(K, Y_{j+1}) \in \mathbf{CW}_{j+1\supset\partial}$, namely $Y_{j+1} = 0$.
(3) K is j-segmented and $(j+1)$-segmented.
(4) $t_{<j}(K, Y_j) = t_{<j+1}(K, Y_{j+1}) = K^j$.
(5) $t_{<j+1}(K, Y_{j+1})$ is an (integral) cohomological $(j+1)$-truncation.

Proof. The injectivity of $\partial_j : C_j(K) \to C_{j-1}(K)$ means that $Z_j(K) = 0$. Hence, for the decomposition $Z_j(K) \oplus Y_j = C_j(K)$ to hold, we must take $Y_j = C_j(K)$. The injectivity of ∂_j also implies that $\partial_{j+1} = 0 : C_{j+1}(K) \to C_j(K)$ and thus $Z_{j+1}(K) = C_{j+1}(K)$. Hence, for the decomposition $Z_{j+1}(K) \oplus Y_{j+1} = C_{j+1}(K)$ to hold, we must take $Y_{j+1} = 0$. This proves (1) and (2). The $(j+1)$-skeleton of K has the form

$$K^{j+1} = K^{j-1} \cup \bigcup_\alpha y^j_\alpha \cup \bigcup_\beta z^{j+1}_\beta,$$

where the y^j_α are the j-cells and the z^{j+1}_β the $(j+1)$-cells of K. Since $\{z^{j+1}_\beta\}$ is a basis for $Z_{j+1}(K)$, Lemma 1.2 of [Ban10] implies that K^{j+1}, and thus K, is $(j+1)$-segmented. Furthermore, Proposition 1.3 of *loc. cit.* shows that the truncating subcomplex $t_{<j+1}(K, Y_{j+1} = 0) \subset K^{j+1}$ is unique (if we insist on not changing the j-skeleton) and given by $t_{<j+1}(K, Y_{j+1}) = K^j$ because K has no $(j+1)$-cells that are not cycles. Similarly, the empty set is a basis for $Z_j(K) = 0$, so we may apply Lemma 1.2 of [Ban10] to conclude that K^j, and thus K, is j-segmented, proving (3). By Proposition 1.3 *loc. cit.*, the truncating subcomplex $t_{<j}(K, Y_j) \subset K^j$ is unique (if we insist on not changing the $(j-1)$-skeleton) and given by

$$t_{<j}(K, Y_j) = K^{j-1} \cup \bigcup_\alpha y^j_\alpha = K^j,$$

since $\{y^j_\alpha\}$ is the set of j-cells of K that are not cycles. This proves statement (4). Statement (5) follows from Remark 1.42 of [Ban10], observing that $\mathrm{Ext}(H_j(K), \mathbb{Z}) = 0$ is a consequence of (INJ_j). □

To a CW-3-diagram we wish to associate certain Mayer-Vietoris type sequences that compute the homology of their homotopy pushouts. Furthermore, to cellular morphisms of such diagrams we wish to associate long exact sequences of these Mayer-Vietoris sequences. This is carried out in the rest of this section through a progression of ever more general statements culminating in Proposition 4.5. The reader may want to consult [Wal99, Chapter 0] for a general setup of n-ads of CW complexes, but we only need $n = 3$, i.e. triads.

Lemma 4.3. *Let $(Q; Q_+, Q_-)$ be a CW-triad so that $Q = Q_+ \cup Q_-$ and $Q_0 = Q_+ \cap Q_-$ is a subcomplex of Q_+ and of Q_-. Let $i : Q_0 \hookrightarrow Q$ be the inclusion map and $q_*^- : C_*(Q) \to C_*(Q)/C_*(Q_+)$, $q_*^+ : C_*(Q) \to C_*(Q)/C_*(Q_-)$ the natural projections. Then:*
(1) The inclusions $Q_- \subset Q$, $Q_+ \subset Q$ induce isomorphisms

$$\frac{C_*(Q_-)}{C_*(Q_0)} \xrightarrow{\cong} \frac{C_*(Q)}{C_*(Q_+)}, \quad \frac{C_*(Q_+)}{C_*(Q_0)} \xrightarrow{\cong} \frac{C_*(Q)}{C_*(Q_-)}.$$

(2) The sequence

$$0 \to C_*(Q_0) \xrightarrow{i_*} C_*(Q) \xrightarrow{(q_*^-, -q_*^+)} \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \to 0$$

is exact.

Proof. (1) Since $Q_+ \cup Q_- = Q$, the claim follows from the short exact sequences

$$0 \to \frac{C_*(Q_+)}{C_*(Q_+ \cap Q_-)} \longrightarrow \frac{C_*(Q)}{C_*(Q_-)} \longrightarrow \frac{C_*(Q)}{C_*(Q_+ \cup Q_-)} \to 0,$$

$$0 \to \frac{C_*(Q_-)}{C_*(Q_+ \cap Q_-)} \longrightarrow \frac{C_*(Q)}{C_*(Q_+)} \longrightarrow \frac{C_*(Q)}{C_*(Q_+ \cup Q_-)} \to 0$$

of the triad $(Q; Q_+, Q_-)$; see [Wal99, p. 5] for these sequences.

(2) The injectivity of i_* is clear. Let $[a] \in C_*(Q)/C_*(Q_+)$, $[b] \in C_*(Q)/C_*(Q_-)$. By (1), there exist chains $\alpha \in C_*(Q_-)$, $\beta \in C_*(Q_+)$ with $q_*^-(\alpha) = [a]$, $q_*^+(\beta) = [b]$. Since $q_*^-(\beta) = 0$ and $q_*^+(\alpha) = 0$, we have

$$\begin{aligned} (q_*^-, -q_*^+)(\alpha - \beta) &= (q_*^-(\alpha - \beta), -q_*^+(\alpha - \beta)) \\ &= (q_*^-(\alpha), -q_*^+(-\beta)) \\ &= ([a], [b]). \end{aligned}$$

Thus $(q_*^-, -q_*^+)$ is surjective. The composition $(q_*^-, -q_*^+) \circ i_*$ is zero because $Q_0 \subset Q_+$, $Q_0 \subset Q_-$. Let $q \in C_*(Q)$ be a chain such that $q_*^-(q) = 0$, $q_*^+(q) = 0$. This implies that $q \in C_*(Q_+) \cap C_*(Q_-) = C_*(Q_+ \cap Q_-) = C_*(Q_0)$, proving exactness at $C_*(Q)$. □

Let $(Q; Q_+, Q_-)$ be a CW-triad with $Q = Q_+ \cup Q_-$ and set $Q_0 = Q_- \cap Q_+$. Let $(R; R_+, R_-)$ be a CW-triad with $R = R_+ \cup R_-$ and set $R_0 = R_- \cap R_+$. Let Γ be the CW-3-diagram

$$Q_+ \hookleftarrow Q_0 \hookrightarrow Q_-$$

and let Θ be the CW-3-diagram

$$R_+ \hookleftarrow R_0 \hookrightarrow R_-.$$

Suppose that Γ is a CW sub-3-diagram of Θ, that is, there is a commutative diagram

$$\begin{tikzcd} Q_+ \arrow[d] & Q_0 \arrow[l] \arrow[r] \arrow[d] & Q_- \arrow[d] \\ R_+ & R_0 \arrow[l] \arrow[r] & R_- \end{tikzcd}$$

of subcomplex inclusions. Assume furthermore that the equations

$$R_+ \cap Q = Q_+, \; R_- \cap Q = Q_-$$

hold. These equations imply

$$R_0 \cap Q = (R_+ \cap R_-) \cap Q = (R_+ \cap Q) \cap (R_- \cap Q) = Q_+ \cap Q_- = Q_0.$$

Since $Q_+ = Q \cap R_+$, the triad $(R; R_+, Q)$ has an associated exact sequence

$$0 \to \frac{C_*(Q)}{C_*(Q_+)} \longrightarrow \frac{C_*(R)}{C_*(R_+)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_+)} \to 0.$$

Similarly, since $Q_- = Q \cap R_-$, the triad $(R; R_-, Q)$ has an associated exact sequence

$$0 \to \frac{C_*(Q)}{C_*(Q_-)} \longrightarrow \frac{C_*(R)}{C_*(R_-)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_-)} \to 0.$$

These two sequences add to give an exact sequence

$$0 \to \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \longrightarrow \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \to 0.$$

Lemma 4.3 applied to the triads $(Q; Q_+, Q_-)$ and $(R; R_+, R_-)$ delivers exact sequences

$$0 \to C_*(Q_0) \longrightarrow C_*(Q) \longrightarrow \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \to 0,$$

$$0 \to C_*(R_0) \longrightarrow C_*(R) \longrightarrow \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \to 0.$$

We obtain the following commutative 3×3-diagram with exact columns and exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(Q_0) \arrow[r] \arrow[d] & C_*(Q) \arrow[r] \arrow[d] & \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(R_0) \arrow[r] \arrow[d] & C_*(R) \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \arrow[r] \arrow[d] & 0 \\
 & \frac{C_*(R_0)}{C_*(Q_0)} \arrow[d] & \frac{C_*(R)}{C_*(Q)} \arrow[d] & \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The inclusion $C_*(R_0) \to C_*(R)$ induces a map $C_*(R_0)/C_*(Q_0) \to C_*(R)/C_*(Q)$. The identity on $C_*(R)$ induces quotient maps

$$\frac{C_*(R)}{C_*(Q)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_+)}, \ \frac{C_*(R)}{C_*(Q)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_-)}.$$

We use these to complete the above diagram to a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(Q_0) \arrow[r] \arrow[d] & C_*(Q) \arrow[r] \arrow[d] & \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(R_0) \arrow[r] \arrow[d] & C_*(R) \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{C_*(R_0)}{C_*(Q_0)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which has exact top and middle row, as well as exact columns. By the 3×3-lemma, the bottom row is exact as well. Using the isomorphisms of Lemma 4.3(1), this diagram can be rewritten as

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(Q_0) \arrow[r] \arrow[d] & C_*(Q) \arrow[r] \arrow[d] & \frac{C_*(Q_-)}{C_*(Q_0)} \oplus \frac{C_*(Q_+)}{C_*(Q_0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(R_0) \arrow[r] \arrow[d] & C_*(R) \arrow[r] \arrow[d] & \frac{C_*(R_-)}{C_*(R_0)} \oplus \frac{C_*(R_+)}{C_*(R_0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{C_*(R_0)}{C_*(Q_0)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Given a map $a : (X, Y) \to (X', Y')$ of pairs, we write $H_*(a, a|)$ for

$$H_*(X' \cup_a X \times I, (Y' \cup_{a|} Y \times I) \cup (X \times \{0\})).$$

Lemma 4.4. *Let* Θ *be* any *CW-*3*-diagram*

$$S_+ \xleftarrow{f} S_0 \xrightarrow{g} S_-$$

and let Γ*, given by*

$$P_+ \xleftarrow{f|} P_0 \xrightarrow{g|} P_-,$$

be a cellular subdiagram of Θ. *Then the inclusion morphism*

(3)
$$\begin{tikzcd} P_+ \arrow[d, hook] & P_0 \arrow[l, "f|"'] \arrow[r, "g|"] \arrow[d, hook] & P_- \arrow[d, hook] \\ S_+ & S_0 \arrow[l, "f"'] \arrow[r, "g"] & S_- \end{tikzcd}$$

induces on homology the following commutative diagram with exact Mayer-Vietoris-type rows and exact columns:

(4)
$$\begin{tikzcd} H_r(P_0) \arrow[r] \arrow[d] & H_r|H\Gamma| \arrow[r] \arrow[d] & H_r(g|) \oplus H_r(f|) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(P_0) \arrow[d] \\ H_r(S_0) \arrow[r] \arrow[d] & H_r|H\Theta| \arrow[r] \arrow[d] & H_r(g) \oplus H_r(f) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(S_0) \arrow[d] \\ H_r(S_0, P_0) \arrow[r] \arrow[d, "\partial_*"] & H_r(|H\Theta|, |H\Gamma|) \arrow[r] \arrow[d, "\partial_*"] & H_r(g, g|) \oplus H_r(f, f|) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(S_0, P_0) \\ H_{r-1}(P_0) \arrow[r] & H_{r-1}|H\Gamma| \arrow[r] & H_{r-1}(g|) \oplus H_{r-1}(f|). & \end{tikzcd}$$

Proof. The inclusion $\Gamma \subset \Theta$ induces an inclusion $H(\Gamma) \subset H(\Theta)$:

$$\begin{tikzcd} Q_+ := P_+ \cup_{f|} P_0 \times I \arrow[d, hook] & Q_0 := P_0 \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] \arrow[d, hook] & P_- \cup_{g|} P_0 \times I =: Q_- \arrow[d, hook] \\ R_+ := S_+ \cup_f S_0 \times I & R_0 := S_0 \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] & S_- \cup_g S_0 \times I =: R_-. \end{tikzcd}$$

Both $H(\Gamma)$ and $H(\Theta)$ are CW-3-diagrams. With

$$Q = |H(\Gamma)| = Q_+ \cup_{Q_0} Q_-, \; R = |H(\Theta)| = R_+ \cup_{R_0} R_-,$$

the equations

$$Q_+ \cap Q_- = Q_0, \; R_+ \cap R_- = R_0, \; R_+ \cap Q = Q_+, \; R_- \cap Q = Q_-$$

hold. Thus the previous considerations yield a commutative 3×3-diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(P_0) \arrow[r] \arrow[d] & C_*|H(\Gamma)| \arrow[r] \arrow[d] & \frac{C_*(P_- \cup_{g|} P_0 \times I)}{C_*(P_0)} \oplus \frac{C_*(P_+ \cup_{f|} P_0 \times I)}{C_*(P_0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(S_0) \arrow[r] \arrow[d] & C_*|H(\Theta)| \arrow[r] \arrow[d] & \frac{C_*(S_- \cup_g S_0 \times I)}{C_*(S_0)} \oplus \frac{C_*(S_+ \cup_f S_0 \times I)}{C_*(S_0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{C_*(S_0)}{C_*(P_0)} \arrow[r] \arrow[d] & \frac{C_*|H(\Theta)|}{C_*|H(\Gamma)|} \arrow[r] \arrow[d] & \frac{C_*|H(\Theta)|}{C_*(|H\Gamma| \cup S_+ \cup_f S_0 \times I)} \oplus \frac{C_*|H(\Theta)|}{C_*(|H\Gamma| \cup S_- \cup_g S_0 \times I)} \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $(A; A_+, A_-)$ be the CW-triad

$$A = |H\Theta|, \ A_+ = |H\Gamma| \cup R_+, \ A_- = R_-,$$

which satisfies $A_+ \cup A_- = A$. With $A_0 = A_+ \cap A_-$, we have

$$A_0 = (R_+ \cap R_-) \cup (|H\Gamma| \cap R_-) = S_0 \cup Q_-.$$

The isomorphism

$$\frac{C_*(A)}{C_*(A_+)} \cong \frac{C_*(A_-)}{C_*(A_0)}$$

of Lemma 4.3(1) thus identifies

$$\frac{C_*|H(\Theta)|}{C_*(|H\Gamma| \cup S_+ \cup_f S_0 \times I)} \cong \frac{C_*(S_- \cup_g S_0 \times I)}{C_*(P_- \cup_{g|} P_0 \times I \cup S_0)}.$$

In particular,

$$H_*(|H(\Theta)|, |H\Gamma| \cup S_+ \cup_f S_0 \times I) \cong H_*(g, g|),$$

and similarly

$$H_*(|H(\Theta)|, |H\Gamma| \cup S_- \cup_g S_0 \times I) \cong H_*(f, f|).$$

The above chain-level 3×3-diagram then induces the desired diagram of exact Mayer-Vietoris-type sequences on homology. □

Proposition 4.5. *Let* Γ *be any CW-3-diagram*

$$X_+ \xleftarrow{\xi_+} X_0 \xrightarrow{\xi_-} X_-$$

and let Γ' *be any CW-3-diagram*

$$Y_+ \xleftarrow{\eta_+} Y_0 \xrightarrow{\eta_-} Y_-.$$

Any cellular morphism $\epsilon : \Gamma \to \Gamma'$ *given by*

$$\begin{tikzcd} X_+ \arrow[d, "\epsilon_+"] & X_0 \arrow[l, "\xi_+"'] \arrow[r, "\xi_-"] \arrow[d, "\epsilon_0"] & X_- \arrow[d, "\epsilon_-"] \\ Y_+ & Y_0 \arrow[l, "\eta_+"] \arrow[r, "\eta_-"'] & Y_- \end{tikzcd}$$

induces on homology the following commutative diagram with exact Mayer-Vietoris-type rows and exact columns:

(5)
$$\begin{tikzcd} H_r(X_0) \arrow[r] \arrow[d, "\epsilon_{0*}"'] & H_r|H\Gamma| \arrow[r] \arrow[d, "|H\epsilon|_*"'] & H_r(\xi_-) \oplus H_r(\xi_+) \arrow[r, "\partial_*"] \arrow[d, "\epsilon_{-*}\oplus\epsilon_{+*}"'] & H_{r-1}(X_0) \arrow[d, "\epsilon_{0*}"'] \\ H_r(Y_0) \arrow[r] \arrow[d] & H_r|H\Gamma'| \arrow[r] \arrow[d] & H_r(\eta_-) \oplus H_r(\eta_+) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(Y_0) \arrow[d] \\ H_r(\epsilon_0) \arrow[r] \arrow[d, "\partial_*"] & H_r(|H\epsilon|) \arrow[r] \arrow[d, "\partial_*"] & H_r(\eta_-, \xi_-) \oplus H_r(\eta_+, \xi_+) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(\epsilon_0) \\ H_{r-1}(X_0) \arrow[r] & H_{r-1}|H\Gamma| \arrow[r] & H_{r-1}(\xi_-) \oplus H_{r-1}(\xi_+). & \end{tikzcd}$$

Here, the map $|H\epsilon| : |H\Gamma| \to |H\Gamma'|$ *is induced by* ϵ *as explained in Section 2.*

Proof. Set $S_+ = Y_+ \cup_{\epsilon_+} X_+ \times I$, $S_0 = Y_0 \cup_{\epsilon_0} X_0 \times I$, $S_- = Y_- \cup_{\epsilon_-} X_- \times I$, and define the CW-3-diagram $\Theta = \Gamma' \cup_\epsilon \Gamma \times I$ to be

$$S_+ \xleftarrow{\sigma_+ = \eta_+ \cup \xi_+ \times \mathrm{id}_I} S_0 \xrightarrow{\sigma_- = \eta_- \cup \xi_- \times \mathrm{id}_I} S_-.$$

Then Γ is a cellular subdiagram of Θ by including Γ at the free end of the cylinders:

(6)
$$\begin{tikzcd} X_+ \arrow[d, hook, "\text{at } 0"] & X_0 \arrow[l, "\xi_+"'] \arrow[r, "\xi_-"] \arrow[d, hook, "\text{at } 0"] & X_- \arrow[d, hook, "\text{at } 0"] \\ Y_+ \cup_{\epsilon_+} X_+ \times I & Y_0 \cup_{\epsilon_0} X_0 \times I \arrow[l, "\sigma_+"'] \arrow[r, "\sigma_-"] & Y_- \cup_{\epsilon_-} X_- \times I. \end{tikzcd}$$

The canonical cellular inclusion $\iota : \Gamma' \hookrightarrow \Theta$ given by

$$\begin{tikzcd} Y_+ \arrow[d, hook, "\iota_+"] & Y_0 \arrow[l, "\eta_+"'] \arrow[r, "\eta_-"] \arrow[d, hook, "\iota_0"] & Y_- \arrow[d, hook, "\iota_-"] \\ Y_+ \cup_{\epsilon_+} X_+ \times I & Y_0 \cup_{\epsilon_0} X_0 \times I \arrow[l, "\sigma_+"'] \arrow[r, "\sigma_-"] & Y_- \cup_{\epsilon_-} X_- \times I \end{tikzcd}$$

induces a cellular morphism $H(\iota) : H(\Gamma') \to H(\Theta)$ given by

(7)
$$\begin{tikzcd} Y_+ \cup_{\eta_+} Y_0 \times J \arrow[d, hook, "\iota_+ \cup \iota_0 \times \mathrm{id}_J"] & Y_0 \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] \arrow[d, hook, "\iota_0"] & Y_- \cup_{\eta_-} Y_0 \times J \arrow[d, hook, "\iota_- \cup \iota_0 \times \mathrm{id}_J"] \\ S_+ \cup_{\sigma_+} S_0 \times J & S_0 \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] & S_- \cup_{\sigma_-} S_0 \times J, \end{tikzcd}$$

where we have written J for the second copy of the unit interval in order to distinguish it from the first copy, I. The realization $|H(\iota)| : |H(\Gamma')| \to |H(\Theta)|$ is a cellular map and a homotopy equivalence, since $|H(\Theta)|$ deformation retracts to $|H(\Gamma')|$ by deformation retracting $I = [0,1]$ to 1. More formally, applying Proposition 2.1 to

$$\begin{tikzcd}
Y_\pm \arrow[d, hook, "\iota_\pm", "\simeq"'] & Y_0 \times \{1\} \arrow[l, "\eta_\pm"'] \arrow[r, hook, "\mathrm{incl}"] \arrow[d, hook, "\iota_0", "\simeq"'] & Y_0 \times J \arrow[d, hook, "\iota_0 \times \mathrm{id}_J", "\simeq"'] \\
S_\pm & S_0 \times \{1\} \arrow[l, "\sigma_\pm"'] \arrow[r, hook, "\mathrm{incl}"] & S_0 \times J,
\end{tikzcd}$$

we see that $\iota_\pm \cup \iota_0 \times \mathrm{id}_J$ are homotopy equivalences. Then applying Proposition 2.1 to (7), we deduce that $|H(\iota)|$ is an equivalence. Diagram (6) is of type (3), so that by Lemma 4.4, we have a commutative diagram

$$\begin{tikzcd}
H_r(X_0) \arrow[r] \arrow[d] & H_r|H\Gamma| \arrow[r] \arrow[d] & H_r(\xi_-) \oplus H_r(\xi_+) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(X_0) \arrow[d, "\epsilon_{0*}"'] \\
H_r(S_0) \arrow[r] \arrow[d] & H_r|H\Theta| \arrow[r] \arrow[d] & H_r(\sigma_-) \oplus H_r(\sigma_+) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(S_0) \arrow[d] \\
H_r(S_0, X_0) \arrow[r] \arrow[d, "\partial_*"] & H_r(|H\Theta|, |H\Gamma|) \arrow[r] \arrow[d, "\partial_*"] & H_r(\sigma_-, \xi_-) \oplus H_r(\sigma_+, \xi_+) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(S_0, X_0) \\
H_{r-1}(X_0) \arrow[r] & H_{r-1}|H\Gamma| \arrow[r] & H_{r-1}(\xi_-) \oplus H_{r-1}(\xi_+) &
\end{tikzcd}$$

with exact rows and columns. Using the deformation retraction $I = [0,1] \mapsto 1$ throughout the diagram, we obtain the desired diagram (5). $\square$

In constructing the duality isomorphism D of Theorem 6.2, we shall make use of Lemma 4.6 below, a standard result from linear algebra. The lemma is ultimately really only relevant in the middle dimension, see Remark 6.4 following the proof of the theorem.

Lemma 4.6. *(*[Ban10, Lemma 2.46]*) Let*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram of rational vector spaces with exact rows. Then there exists a map $C \to C'$ completing the diagram commutatively.

5. Construction of the Intersection Spaces

Let X^n be an oriented, compact, PL stratified pseudomanifold of even dimension $n = 2k$ with a PL stratification of the form $X_n = X^n \supset X_1 \supset X_0$, $X_1 \cong S^1$, $X_0 = \{x_0\}$. There are thus three strata. (The case of a depth 1 stratified space X^n with stratification $\hat{X}_n = X^n \supset \hat{X}_1 \cong S^1$, $\hat{X}_0 = \varnothing$, and possibly twisted link bundle (mapping torus) can be treated within the present framework by inserting a point $x_0 \in \hat{X}_1$ as a new stratum $\hat{X}_0 = \{x_0\}$, whose link is the suspension of the link of $\hat{X}_1$.) Let N_0 be a regular neighborhood of x_0 in X. Then $N_0 = \mathrm{cone}(L_0)$, where L_0 is a compact PL stratified pseudomanifold of dimension $n-1$, the link of x_0.

Set $X' = X - \operatorname{int}(N_0)$, a compact pseudomanifold with boundary. This X' has one singular stratum, $X_1' = X_1 \cap X' \cong \Delta^1$, where Δ^1 is a closed interval. Let L be the link of X_1', a closed manifold of dimension $n-2$. To be able to carry out spatial homology truncation, we assume that the links L and L_0 are simply connected. (In specific cases this assumption is not always necessary, since a space may very well have a Moore approximation even if it is not simply connected.) The space L_0 may be singular with singular stratum $L_0 \cap X_1 = L_0 \cap X_1' = \partial\Delta^1 = \{\Delta^0_0, \Delta^0_1\}$ (two points). The link L, being triangulable, certainly has some CW structure.

Assumption: The space L possesses a CW structure such that condition (INJ_{k-1}) is satisfied.

(This is the strong Witt condition from the introduction.) Fix such a CW structure on L from now on. A regular neighborhood of Δ^0_i, $i = 0, 1$, in L_0 is PL homeomorphic to $\operatorname{cone}(L)$. If we remove the interiors of these two cones from L_0, we obtain a compact $(n-1)$-manifold W, which is a bordism between L at Δ^0_0 and L at Δ^0_1. Choose any CW structure on W so that ∂W is a subcomplex (This is possible, since W can be triangulated with ∂W as a simplicial subcomplex.) A normal regular neighborhood of X_1' in X' is PL homeomorphic to a product $\Delta^1 \times \operatorname{cone}(L)$. In more detail, this can be seen as follows: By Theorem 2.1 of [Sto72], a normal regular neighborhood N of X_1' in X' is the total space $N = |\xi|$ of a cone block bundle ξ, with fiber $\operatorname{cone}(L)$ over X_1'. As the base X_1' is PL homeomorphic to Δ^1, Theorem I, 1.1 of [Sto72, Appendix] applies to show that ξ is trivial, that is, there is a cone block bundle isomorphism $\xi \cong X_1' \times \operatorname{cone}(L)$. Thus $N = |\xi| \cong X_1' \times \operatorname{cone}(L) \cong \Delta^1 \times \operatorname{cone}(L)$. Removing from X' the preimage of $X_1' \times \overset{\circ}{\operatorname{cone}}(L)$, where $\overset{\circ}{\operatorname{cone}}(L)$ denotes the open cone, under the trivialization, we get a compact n-manifold M with boundary ∂M. In order to describe ∂M as the realization of a 3-diagram, set $N_L = \operatorname{cl}\partial(N - (N \cap \partial X'))$, where cl is closure in X'. Then N_L is the total space of the link bundle of $X_1 - X_0$, restricted to X_1'. In the terminology of [Sto72], N_L is the *rim* of the cone block bundle ξ. This rim is a compact manifold with boundary ∂N_L which is equal to the boundary of W. Let us denote this common boundary by Λ. Then $\partial M = |\Theta|$, where Θ is the 3-diagram

$$W \xleftarrow{\ \mathrm{incl}\ } \Lambda \xhookrightarrow{\ \mathrm{incl}\ } N_L.$$

Let

$$\phi : (N, N_L, N \cap \partial X') \xrightarrow{\cong} (\Delta^1 \times \operatorname{cone}(L), \Delta^1 \times L, (\partial\Delta^1) \times \operatorname{cone}(L))$$

denote the above trivialization of the regular neighborhood and let Γ_∂ be the 3-diagram

$$W \xleftarrow{\ f'\ } \ddot{L} \xhookrightarrow{\ \mathrm{incl} \times \mathrm{id}\ } \overline{L},$$

where we wrote $\overline{L} = \Delta^1 \times L$, $\ddot{L} = \partial\Delta^1 \times L$, and the map f' is the composition

$$\ddot{L} \xrightarrow{\phi|^{-1}} \Lambda \hookrightarrow W.$$

Then ϕ induces a morphism $\Theta \to \Gamma_\partial$ given by

$$\begin{tikzcd} W \arrow[d, equal] & \Lambda \arrow[l, hook', "\mathrm{incl}"'] \arrow[r, hook, "\mathrm{incl}"] \arrow[d, "\phi|", "\cong"'] & N_L \arrow[d, "\phi|", "\cong"'] \\ W & \ddot{L} \arrow[l, "f'"'] \arrow[r, hook, "\mathrm{incl} \times \mathrm{id}"] & \overline{L}. \end{tikzcd}$$

This morphism induces a homeomorphism $\partial M = |\Theta| \cong |\Gamma_\partial|$, and a homeomorphism $|H(\Theta)| \cong |H(\Gamma_\partial)|$. For example, if the link-type does not change running along $X_1 - X_0$ into x_0, then L_0 is the suspension of L and W is the cylinder $W \cong I \times L$. The boundary of M is a mapping torus with fiber L and we may think of f' as the monodromy of the mapping torus. In the diagram Γ_∂, $\overline{L}$ is equipped with the product CW structure. The map f' is in general not cellular.

We shall proceed to define the middle perversity intersection spaces $I^{\bar{m}}X$ and $I^{\bar{n}}X$. It will turn out that the above strong Witt assumption (INJ_{k-1}) on L implies that $I^{\bar{m}}X = I^{\bar{n}}X$. Roughly, the construction paradigm of intersection spaces says that in order to obtain $I^{\bar{p}}X$, for a given perversity $\bar{p}$, every link $\mathcal{L}$ of a stratum of codimension c must be replaced by its spatial homology $k_{\mathcal{L}}(\bar{p})$-truncation (Moore approximation), where

$$k_{\mathcal{L}}(\bar{p}) = c - 1 - \bar{p}(c).$$

The first step is to replace Γ_∂ by a CW-3-diagram Γ in which f' is replaced by a cellular approximation. Thus, let Γ be the CW-3-diagram

$$W \xleftarrow{f} \ddot{L} \hookrightarrow \overline{L},$$

where f is a cellular approximation of f'. In the h-morphism $\Gamma_\partial \to \Gamma$ defined by

$$\begin{tikzcd} W \arrow[d, equal] & \ddot{L} \arrow[l, "f'"'] \arrow[r, hook] \arrow[d, equal] & \overline{L} \arrow[d, equal] \\ W & \ddot{L} \arrow[l, "f"'] \arrow[r, hook] & \overline{L}, \end{tikzcd}$$

the left hand square commutes up to homotopy and the right hand square commutes. Hence, we may apply Proposition 2.2 to obtain a homotopy equivalence

$$|H(\Gamma_\partial)| \simeq |H(\Gamma)|.$$

By Proposition 2.3,

$$|\Theta| \cong |H(\Theta)|.$$

Composing, we get a homotopy equivalence

$$\partial M = |\Theta| \cong |H(\Theta)| \cong |H(\Gamma_\partial)| \simeq |H(\Gamma)|.$$

The space $|H(\Gamma)|$ will be the homotopy theoretic model of the boundary of M that we will subsequently work with.

Let us first discuss the intersection space for the lower middle perversity $\bar{p} = \bar{m}$. For our X, we must truncate L and W. The truncation degrees are

$$\begin{aligned} k_L(\bar{m}) &= n - 2 - \bar{m}(n-1) = k, \\ k_W(\bar{m}) &= n - 1 - \bar{m}(n) = k. \end{aligned}$$

Thus there is one common cut-off degree for both L and W, namely k.

By Lemma 4.2(2), $(L, Y_L = 0)$ is the unique completion of L to an object in $\mathbf{CW}_{k\supset\partial}$. Note that W is simply connected: Write W' for the space obtained from L_0 by deleting one of the two points in $L_0 \cap X_1$. A neighborhood in L_0 of such a point is PL homeomorphic to the cone on L. By the Seifert-van Kampen theorem,

$$1 = \pi_1(L_0) \cong \pi_1(W') *_{\pi_1(L)} \pi_1(\mathrm{cone}(L)) = \pi_1(W')$$

and so

$$1 = \pi_1(W') \cong \pi_1(W) *_{\pi_1(L)} \pi_1(\mathrm{cone}(L)) = \pi_1(W),$$

using the simple connectivity of L. Let (W, Y_W) be any completion of W to an object in $\mathbf{CW}_{k\supset\partial}$. Let $f_i : L = \Delta^0_i \times L \to W$ be the restriction of f to $\Delta^0_i \times L \subset \partial\Delta^1 \times L = \ddot{L}$, $i = 0, 1$. Since the cellular maps f_i satisfy $f_{i*}(Y_L) \subset Y_W$, they both define morphisms $f_i : (L, Y_L) \to (W, Y_W)$ in $\mathbf{CW}_{k\supset\partial}$. Thus there exist truncation cellular maps $t_{<k}(f_i) : t_{<k}(L, Y_L) \to t_{<k}(W, Y_W)$ such that

$$\begin{tikzcd} L_{<k} \arrow[r, "t_{<k}(f_i)"] \arrow[d, "e_L"'] & W_{<k} \arrow[d, "e_W"] \\ L \arrow[r, "f_i"] & W \end{tikzcd}$$

commutes (a priori) up to homotopy rel $(k-1)$-skeleton, where we have written $L_{<k} = t_{<k}(L, Y_L)$, $W_{<k} = t_{<k}(W, Y_W)$, e_L is a cellular rel $(k-1)$-skeleton representative of the homotopy class $\mathrm{emb}_k(L, Y_L)$, and e_W is a cellular rel $(k-1)$-skeleton representative of $\mathrm{emb}_k(W, Y_W)$. We set $\ddot{L}_{<k} = (\Delta^0_0 \times L_{<k}) \sqcup (\Delta^0_1 \times L_{<k})$,

$$t_{<k}(f) = t_{<k}(f_0) \sqcup t_{<k}(f_1) : \ddot{L}_{<k} \longrightarrow W_{<k},$$

and $e_{\ddot{L}} = e_L \sqcup e_L : \ddot{L}_{<k} \to \ddot{L}$. The diagram

$$\begin{tikzcd} \ddot{L}_{<k} \arrow[r, "t_{<k}(f)"] \arrow[d, "e_{\ddot{L}}"'] & W_{<k} \arrow[d, "e_W"] \\ \ddot{L} \arrow[r, "f"] & W \end{tikzcd} \tag{8}$$

commutes (a priori) up to homotopy rel $(k-1)$-skeleton. By Lemma 4.2(4), $L_{<k} = L^{k-1}$ and thus $\ddot{L}_{<k} = \ddot{L}^{k-1}$. The map $t_{<k}(f)$ factors as

$$\ddot{L}^{k-1} \xrightarrow{f|} W^{k-1} \subset W_{<k}.$$

The map $e_{\ddot{L}}$ is the skeletal inclusion $\ddot{L}^{k-1} \hookrightarrow \ddot{L}$. Since the restriction of e_W to W^{k-1} is the skeletal inclusion $W^{k-1} \hookrightarrow W$, we deduce that the diagram (8) commutes on the nose, not just up to homotopy.

Applying Proposition 4.5 to the cellular morphism

$$\begin{tikzcd} W_{<k} \arrow[d, "e_W"'] & \ddot{L}_{<k} \arrow[l, "t_{<k}f"'] \arrow[r, equal] \arrow[d, hook] & \ddot{L}_{<k} \arrow[d, hook] \\ W & \ddot{L} \arrow[l, "f"'] \arrow[r, equal] & \ddot{L} \end{tikzcd}$$

yields the commutative diagram

$$
\begin{tikzcd}
H_r(\ddot{L}_{<k}) \arrow[r, "(t_{<k}f)_*"] \arrow[d] & H_r(W_{<k}) \arrow[r] \arrow[d, "e_{W*}"'] & H_r(t_{<k}f) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(\ddot{L}_{<k}) \arrow[d] \\
H_r(\ddot{L}) \arrow[r, "f_*"] \arrow[d] & H_r(W) \arrow[r] \arrow[d] & H_r(f) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(\ddot{L}) \arrow[d] \\
H_r(\ddot{L}, \ddot{L}_{<k}) \arrow[r] \arrow[d, "\partial_*"] & H_r(e_W) \arrow[r] \arrow[d, "\partial_*"] & H_r(f, t_{<k}f) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \\
H_{r-1}(\ddot{L}_{<k}) \arrow[r] & H_{r-1}(W_{<k}) \arrow[r] & H_{r-1}(t_{<k}f) &
\end{tikzcd}
\tag{9}
$$

with exact rows and columns.

Lemma 5.1. *The map*

$$H_r(f) \longrightarrow H_r(f, t_{<k}f)$$

is an isomorphism for $r \geq k$, while

$$H_r(f, t_{<k}f) = 0$$

for $r < k$.

Proof. The proof is based on an examination of the above diagram (9) in the three cases $r < k$, $r = k$, and $r > k$. Suppose $r < k$. Then $H_r(W_{<k}) \to H_r(W)$ and $H_{r-1}(W_{<k}) \to H_{r-1}(W)$ are isomorphisms. By exactness of the second column of the diagram, $H_r(e_W) = 0$. Similarly, the exactness of the long sequence of the last column implies that $H_{r-1}(\ddot{L}, \ddot{L}_{<k}) = 0$. By the exactness of the third row, $H_r(f, t_{<k}f) = 0$.

Suppose next that $r = k$. Since L satisfies condition (INJ_{k-1}) and $\ddot{L} \cong L \sqcup L$, we have $H_{k-1}(\ddot{L}_{<k}) \cong H_{k-1}(\ddot{L}) = 0$. Together with $H_k(W_{<k}) = 0$, the exactness of the top row shows that $H_k(t_{<k}f) = 0$. An application of the 5-lemma to the ladder

$$
\begin{tikzcd}
H_{k-1}(\ddot{L}_{<k}) \arrow[r] \arrow[d, "\cong"'] & H_{k-1}(W_{<k}) \arrow[r] \arrow[d, "\cong"'] & H_{k-1}(t_{<k}f) \arrow[r, "\partial_*"] \arrow[d] & H_{k-2}(\ddot{L}_{<k}) \arrow[r] \arrow[d, "\cong"'] & H_{k-2}(W_{<k}) \arrow[d, "\cong"'] \\
H_{k-1}(\ddot{L}) \arrow[r] & H_{k-1}(W) \arrow[r] & H_{k-1}(f) \arrow[r, "\partial_*"] & H_{k-2}(\ddot{L}) \arrow[r] & H_{k-2}(W)
\end{tikzcd}
$$

yields that

$$H_{k-1}(t_{<k}f) \longrightarrow H_{k-1}(f)$$

is an isomorphism. The exact sequence

$$0 = H_k(t_{<k}f) \to H_k(f) \to H_k(f, t_{<k}f) \xrightarrow{\partial_* = 0} H_{k-1}(t_{<k}f) \xrightarrow{\cong} H_{k-1}(f)$$

shows that

$$H_k(f) \longrightarrow H_k(f, t_{<k}f)$$

is an isomorphism.

If $r > k$, then using the exact sequences

$$0 = H_r(W_{<k}) \longrightarrow H_r(t_{<k}f) \longrightarrow H_{r-1}(\ddot{L}_{<k}) = 0$$

and

$$0 = H_{r-1}(W_{<k}) \longrightarrow H_{r-1}(t_{<k}f) \longrightarrow H_{r-2}(\ddot{L}_{<k}) = 0,$$

we obtain

$$H_r(t_{<k}f) = 0,\ H_{r-1}(t_{<k}f) = 0.$$

(If $r = k+1$, then $H_{r-2}(\ddot{L}_{<k}) = 0$ is implied by (INJ_{k-1}).) From the exactness of the sequence

$$0 = H_r(t_{<k}f) \to H_r(f) \to H_r(f, t_{<k}f) \xrightarrow{\partial_*} H_{r-1}(t_{<k}f) = 0$$

we deduce that the middle map is an isomorphism. $\square$

Set $\overline{L}_{<k} = \Delta^1 \times L_{<k}$. The notation $\overline{L}_{<k}$ is potentially ambiguous because it could also be construed to indicate a spatial homology truncation $t_{<k}$ of $\overline{L}$. This ambiguity is deliberate, for $\Delta^1 \times L_{<k}$ is indeed a valid homology truncation of $\overline{L}$: The map $\mathrm{id}_{\Delta^1} \times \mathrm{incl} : \overline{L}_{<k} = \Delta^1 \times L_{<k} \to \Delta^1 \times L = \overline{L}$ induces an isomorphism $H_r(\overline{L}_{<k}) \cong H_r(L_{<k}) \cong H_r(L) \cong H_r(\overline{L})$ for $r < k$, and $H_r(\overline{L}_{<k}) \cong H_r(L_{<k})$ vanishes in degrees $r \geq k$. Let $\Gamma^{\bar{m}}$ be the 3-diagram

$$W_{<k} \xleftarrow{t_{<k}(f)} \ddot{L}_{<k} \xhookrightarrow{\mathrm{incl}\times\mathrm{id}} \overline{L}_{<k}.$$

The diagram of commutative squares

$$\begin{tikzcd} W_{<k} \arrow[d, "e_W"'] & \ddot{L}_{<k} \arrow[l, "t_{<k}(f)"'] \arrow[r, hook, "\mathrm{incl}\times\mathrm{id}"] \arrow[d, hook, "e_{\ddot{L}}=\mathrm{incl}"'] & \overline{L}_{<k} \arrow[d, hook, "\mathrm{id}_{\Delta^1}\times\mathrm{incl}"] \\ W & \ddot{L} \arrow[l, "f"'] \arrow[r, hook, "\mathrm{incl}\times\mathrm{id}"] & \overline{L} \end{tikzcd}$$

defines a cellular morphism $\epsilon : \Gamma^{\bar{m}} \to \Gamma$, which induces a cellular map $|H(\epsilon)| : |H(\Gamma^{\bar{m}})| \to |H(\Gamma)|$.

Definition 5.2. The *lower middle perversity intersection space* $I^{\bar{m}}X$ of X is the homotopy cofiber of the composition

$$|H(\Gamma^{\bar{m}})| \xrightarrow{|H(\epsilon)|} |H(\Gamma)| \simeq \partial M \hookrightarrow M.$$

For the upper middle perversity $\bar{p} = \bar{n}$, we have the cut-off values

$$\begin{aligned} k_L(\bar{n}) &= n-2-\bar{n}(n-1) = k-1, \\ k_W(\bar{n}) &= n-1-\bar{n}(n) = k. \end{aligned}$$

The intersection space $I^{\bar{n}}X$ is defined using the construction principle of Definition 5.2, employing an appropriate diagram $\Gamma^{\bar{n}}$ instead of $\Gamma^{\bar{m}}$. Let us construct this $\Gamma^{\bar{n}}$. Since L satisfies condition (INJ_{k-1}), Lemma 4.2(1) asserts that $(L, Y_L' = C_{k-1}(L))$ is the unique completion of L to an object in $\mathbf{CW}_{k-1\supset\partial}$. Furthermore, by Lemma 4.2(4),

$$t_{<k_L(\bar{n})}(L, Y_L') = t_{<k-1}(L, Y_L') = t_{<k}(L, Y_L = 0) = L^{k-1}.$$

Therefore, a CW-3-diagram $\Gamma^{\bar{n}}$ of the required type

$$t_{<k_W(\bar{n})}(W, Y_W) \leftarrow \partial\Delta^1 \times t_{<k_L(\bar{n})}(L, Y_L') \hookrightarrow \Delta^1 \times t_{<k_L(\bar{n})}(L, Y_L')$$

can be defined by

$$\begin{aligned}
\Gamma^{\bar{n}} &= \Big(t_{<k}(W, Y_W) \leftarrow \partial\Delta^1 \times t_{<k-1}(L, Y'_L) \hookrightarrow \Delta^1 \times t_{<k-1}(L, Y'_L)\Big) \\
&= \Big(W_{<k} \leftarrow \partial\Delta^1 \times t_{<k}(L, Y_L) \hookrightarrow \Delta^1 \times t_{<k}(L, Y_L)\Big) \\
&= \Big(W_{<k} \xleftarrow{t_{<k}(f)} \ddot{L}_{<k} \hookrightarrow \overline{L}_{<k}\Big) \\
&= \Gamma^{\bar{m}}.
\end{aligned}$$

Thus, as expected,

$$I^{\bar{m}}X = I^{\bar{n}}X$$

due to the strong Witt assumption on L. We shall denote this space by IX.

6. The Duality Theorem

Rational homology and cohomology will be used throughout this section. Let $e : |H\Gamma^{\bar{m}}| \to \partial M$ be the composition of $|H\epsilon|$ with the homotopy equivalence $|H\Gamma| \simeq \partial M$.

Proposition 6.1. *Cap product with the fundamental class $[\partial M] \in H_{n-1}(\partial M)$ induces an isomorpism*

$$H^{n-r}|H\Gamma^{\bar{m}}| \xrightarrow{\cong} H_{r-1}(e)$$

such that

$$\begin{tikzcd}
H^{n-r}(\partial M) \arrow[r, "e^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, "\cong"] \\
H_{r-1}(\partial M) \arrow[r] & H_{r-1}(e)
\end{tikzcd}$$

commutes. This isomorphism is determined uniquely by the above commutativity requirement.

Proof. The morphism $\epsilon : \Gamma^{\bar{m}} \to \Gamma$ induces a map of standard Mayer-Vietoris sequences for double mapping cylinders:

$$\begin{tikzcd}
H^{n-r-1}(\ddot{L}) \arrow[r, "\delta^*"] \arrow[d, "\mathrm{restr}"] & H^{n-r}|H\Gamma| \arrow[r] \arrow[d, "|H\epsilon|^*"] & H^{n-r}(W) \oplus H^{n-r}(\overline{L}) \arrow[r, "f^*+\mathrm{restr}"] \arrow[d, "e_W^*\oplus\mathrm{restr}"] & H^{n-r}(\ddot{L}) \arrow[d, "\mathrm{restr}"] \\
H^{n-r-1}(\ddot{L}_{<k}) \arrow[r, "\delta^*"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[r] & H^{n-r}(W_{<k}) \oplus H^{n-r}(\overline{L}_{<k}) \arrow[r] & H^{n-r}(\ddot{L}_{<k})
\end{tikzcd}$$

(The last arrow in the bottom row is $(t_{<k}f)^*+$restr.) Using the homotopy equivalence $|H\Gamma| \simeq \partial M$, this diagram may be rewritten as

(10)

$$\begin{tikzcd}
H^{n-r-1}(\ddot{L}) \arrow[r, "\delta^*"] \arrow[d, "\mathrm{restr}"] & H^{n-r}(\partial M) \arrow[r] \arrow[d, "e^*"] & H^{n-r}(W) \oplus H^{n-r}(\overline{L}) \arrow[r] \arrow[d, "e_W^*\oplus\mathrm{restr}"] & H^{n-r}(\ddot{L}) \arrow[d, "\mathrm{restr}"] \\
H^{n-r-1}(\ddot{L}_{<k}) \arrow[r, "\delta^*"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[r] & H^{n-r}(W_{<k}) \oplus H^{n-r}(\overline{L}_{<k}) \arrow[r] & H^{n-r}(\ddot{L}_{<k}).
\end{tikzcd}$$

An application of Proposition 4.5 to $\epsilon : \Gamma^{\bar{m}} \to \Gamma$ yields a commutative diagram

$$\begin{tikzcd}
H_{r-1}(\ddot{L}_{<k}) \arrow[r] \arrow[d] & H_{r-1}|H\Gamma^{\bar{m}}| \arrow[r] \arrow[d, "|H\epsilon|_*"'] & H_{r-1}(\overline{L}_{<k}, \ddot{L}_{<k}) \oplus H_{r-1}(t_{<k}f) \arrow[r] \arrow[d, "\mathrm{incl}_* \oplus e_{W*}"'] & H_{r-2}(\ddot{L}_{<k}) \arrow[d] \\
H_{r-1}(\ddot{L}) \arrow[r] \arrow[d] & H_{r-1}|H\Gamma| \arrow[r] \arrow[d] & H_{r-1}(\overline{L}, \ddot{L}) \oplus H_{r-1}(f) \arrow[r] \arrow[d] & H_{r-2}(\ddot{L}) \arrow[d] \\
H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \arrow[r] \arrow[d] & H_{r-1}(|H\epsilon|) \arrow[r] \arrow[d] & H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \oplus H_{r-1}(f, t_{<k}f) \arrow[r] \arrow[d] & H_{r-2}(\ddot{L}, \ddot{L}_{<k}) \\
H_{r-2}(\ddot{L}_{<k}) \arrow[r] & H_{r-2}|H\Gamma^{\bar{m}}| \arrow[r] & H_{r-2}(\overline{L}_{<k}, \ddot{L}_{<k}) \oplus H_{r-2}(t_{<k}f) &
\end{tikzcd}$$

with exact rows and columns. Again using $|H\Gamma| \simeq \partial M$, we can in particular extract the following map of Mayer-Vietoris sequences:
(11)

$$\begin{tikzcd}
H_{r-1}(\ddot{L}) \arrow[r] \arrow[d] & H_{r-1}(\partial M) \arrow[r] \arrow[d] & H_{r-1}(\overline{L}, \ddot{L}) \oplus H_{r-1}(f) \arrow[r] \arrow[d] & H_{r-2}(\ddot{L}) \arrow[d] \\
H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \arrow[r] & H_{r-1}(e) \arrow[r] & H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \oplus H_{r-1}(f, t_{<k}f) \arrow[r] & H_{r-2}(\ddot{L}, \ddot{L}_{<k}).
\end{tikzcd}$$

We shall distinguish the cases $r > k$ and $r \leq k$. Suppose that $r > k$. Then, since $n = 2k$, $n - r < k$ and the maps restr and e_W^* in diagram (10) are isomorphisms. By the 5-lemma, $e^* : H^{n-r}(\partial M) \to H^{n-r}|H\Gamma^{\bar{m}}|$ is an isomorphism as well. Let us prove next that $H_{r-1}(\partial M) \to H_{r-1}(e)$ is an isomorphism. Since $r - 1 \geq k$, the maps

$$H_{r-1}(f) \longrightarrow H_{r-1}(f, t_{<k}f) \text{ and } H_r(f) \longrightarrow H_r(f, t_{<k}f)$$

are isomorphisms by Lemma 5.1. Applying the same lemma to

$$\begin{tikzcd}
\ddot{L}_{<k} \arrow[r] \arrow[d] & \overline{L}_{<k} \arrow[d] \\
\ddot{L} \arrow[r] & \overline{L}
\end{tikzcd}$$

instead of

$$\begin{tikzcd}
\ddot{L}_{<k} \arrow[r] \arrow[d] & W_{<k} \arrow[d] \\
\ddot{L} \arrow[r, "f"] & W,
\end{tikzcd}$$

we see that the maps

$$H_{r-1}(\overline{L}, \ddot{L}) \longrightarrow H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L})$$

and

$$H_r(\overline{L}, \ddot{L}) \longrightarrow H_r(\overline{L}, \overline{L}_{<k} \cup \ddot{L})$$

are isomorphisms. The map

$$H_{r-1}(\ddot{L}) \longrightarrow H_{r-1}(\ddot{L}, \ddot{L}_{<k})$$

is an isomorphism, as follows from the exact sequence

$$H_{r-1}(\ddot{L}_{<k}) \longrightarrow H_{r-1}(\ddot{L}) \longrightarrow H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \xrightarrow{\partial_*} H_{r-2}(\ddot{L}_{<k})$$

by observing that $H_{r-1}(\ddot{L}_{<k}) = 0$ and even $H_{r-2}(\ddot{L}_{<k}) = 0$, since in the worst case (when $r-2 = k-1$),

$$H_{k-1}(\ddot{L}_{<k}) = H_{k-1}(L^{k-1}) \oplus H_{k-1}(L^{k-1})$$

and

$$H_{k-1}(L^{k-1}) = \ker(\partial_{k-1} : C_{k-1}(L) \to C_{k-2}(L)) = 0$$

by condition (INJ_{k-1}). The map

$$H_{r-2}(\ddot{L}) \longrightarrow H_{r-2}(\ddot{L}, \ddot{L}_{<k})$$

is injective by $H_{r-2}(\ddot{L}_{<k}) = 0$. In summary, the diagram (11) has the form

$$\begin{tikzcd}
H_r(\overline{L}, \ddot{L}) \oplus H_r(f) \arrow[r, "\cong"] \arrow[d, "\partial_*"'] & H_r(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \oplus H_r(f, t_{<k}f) \arrow[d, "\partial_*"'] \\
H_{r-1}(\ddot{L}) \arrow[r, "\cong"] \arrow[d] & H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \arrow[d] \\
H_{r-1}(\partial M) \arrow[r] \arrow[d] & H_{r-1}(e) \arrow[d] \\
H_{r-1}(\overline{L}, \ddot{L}) \oplus H_{r-1}(f) \arrow[r, "\cong"] \arrow[d, "\partial_*"'] & H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \oplus H_{r-1}(f, t_{<k}f) \arrow[d, "\partial_*"'] \\
H_{r-2}(\ddot{L}) \arrow[r, hook] & H_{r-2}(\ddot{L}, \ddot{L}_{<k}).
\end{tikzcd}$$

This is enough to deduce from a sharp version of the 5-lemma that $H_{r-1}(\partial M) \to H_{r-1}(e)$ is an isomorphism, as claimed. Let

$$H^{n-r}|H\Gamma^{\bar{m}}| \xrightarrow{\cong} H_{r-1}(e)$$

be the unique isomorphism such that the square

$$\begin{tikzcd}
H^{n-r}(\partial M) \arrow[r, "e^*", "\cong"'] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, dotted] \\
H_{r-1}(\partial M) \arrow[r, "\cong"] & H_{r-1}(e)
\end{tikzcd}$$

commutes.

Suppose that $r \le k$. If $r < k$, then $n - r \ge k + 1$ and $H^{n-r-1}(\ddot{L}_{<k}) = 0$. If $r = k$, then

$$H^{n-r-1}(\ddot{L}_{<k}) = \mathrm{Hom}(H_{k-1}(\ddot{L}^{k-1}), \mathbb{Q}) = 0$$

by condition (INJ_{k-1}). Since $H^{n-r}(W_{<k}) = 0$ and $H^{n-r}(\overline{L}_{<k}) = 0$, the exactness of

$$H^{n-r-1}(\ddot{L}_{<k}) \xrightarrow{\delta^*} H^{n-r}|H\Gamma^{\bar{m}}| \longrightarrow H^{n-r}(W_{<k}) \oplus H^{n-r}(\overline{L}_{<k})$$

shows that

$$H^{n-r}|H\Gamma^{\bar{m}}| = 0.$$

We shall show that $H_{r-1}(e) = 0$ also. The exactness of

$$H_{r-1}(\ddot{L}_{<k}) \xrightarrow{\cong} H_{r-1}(\ddot{L}) \xrightarrow{0} H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \xrightarrow{0} H_{r-2}(\ddot{L}_{<k}) \xrightarrow{\cong} H_{r-2}(\ddot{L})$$

implies that $H_{r-1}(\ddot{L}, \ddot{L}_{<k}) = 0$. Since $r-1 < k$, we infer from Lemma 5.1 that $H_{r-1}(f, t_{<k}f) = 0$. Similarly, $H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) = 0$, which can either also be deduced from Lemma 5.1 by taking $W = \overline{L}$, $f = \mathrm{incl} : \ddot{L} \hookrightarrow \overline{L}$, $W_{<k} = \overline{L}_{<k} = I \times L^{k-1}$, $t_{<k}(f) = \mathrm{incl} : (\partial I) \times L^{k-1} \to I \times L^{k-1}$, or directly from the exact sequence

$$0 = H_{r-1}(I \times L, I \times L_{<k}) \longrightarrow H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \longrightarrow H_{r-2}((\partial I) \times L, (\partial I) \times L_{<k}) = 0.$$

The vanishing of $H_{r-1}(e)$ follows from the exactness of

$$H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \longrightarrow H_{r-1}(e) \longrightarrow H_{r-1}(f, t_{<k}f) \oplus H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}).$$

Thus for $r \leq k$, the zero map is the unique isomorphism

$$H^{n-r}|H\Gamma^{\bar{m}}| \xrightarrow{\cong} H_{r-1}(e)$$

such that the commutativity requirement is met. □

To a triple of continuous maps

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\psi\phi"'] & B \arrow[d, "\psi"] \\ & C \end{tikzcd}$$

one can associate the 3-diagrams

$$\Gamma(\phi) = \Big(B \xleftarrow{\phi} A \times \{1\} \hookrightarrow \mathrm{cone}(A)\Big),\ \Gamma(\psi) = \Big(C \xleftarrow{\psi} B \times \{1\} \hookrightarrow \mathrm{cone}(B)\Big),$$

$$\Gamma(\psi\phi) = \Big(C \xleftarrow{\psi\phi} A \times \{1\} \hookrightarrow \mathrm{cone}(A)\Big),$$

and the morphisms $\Gamma(\phi) \to \Gamma(\psi\phi) \to \Gamma(\psi)$ given by

$$\begin{tikzcd} B \arrow[d, "\psi"'] & A \times \{1\} \arrow[l, "\phi"'] \arrow[r, hook] \arrow[d, "\mathrm{id}"'] & \mathrm{cone}(A) \arrow[d, "\mathrm{id}"'] \\ C \arrow[d, "\mathrm{id}"'] & A \times \{1\} \arrow[l, "\psi\phi"'] \arrow[r, hook] \arrow[d, "\phi"'] & \mathrm{cone}(A) \arrow[d, "\mathrm{cone}(\phi)"'] \\ C & B \times \{1\} \arrow[l, "\psi"'] \arrow[r, hook] & \mathrm{cone}(B). \end{tikzcd}$$

These morphisms induce maps $|\Gamma(\phi)| \to |\Gamma(\psi\phi)| \to |\Gamma(\psi)|$. Applying this to the triple

$$\begin{tikzcd} {|H\Gamma^{\bar{m}}|} \arrow[r, "e"] \arrow[dr, "g"'] & \partial M \arrow[d, hook, "j"] \\ & M, \end{tikzcd} \tag{12}$$

and observing $|\Gamma(g)| = IX$ and $|\Gamma(j)| = M/\partial M$, we obtain a map $\gamma : IX \to M/\partial M$. Let $\mu : M \to IX$ denote the canonical inclusion of the target of the map g into the mapping cone IX of this map.

Theorem 6.2. *Let X be an n-dimensional, compact, oriented PL pseudomanifold with n even. Suppose X can be endowed with a PL stratification of the form $X = X_n \supset X_1 \supset X_0$ with $X_1 \cong S^1$ and X_0 a point, such that the links of the two strata are simply connected and X satisfies the strong Witt condition. Then there exists a Poincaré duality isomorphism*

$$D : \widetilde{H}^{n-r}(IX) \xrightarrow{\cong} \widetilde{H}_r(IX)$$

for the reduced (co)homology of the middle perversity intersection space IX of X that extends Poincaré duality for the exterior $(M, \partial M)$ of the singular set, that is, D makes

$$\begin{tikzcd}
\widetilde{H}^{n-r}(IX) \arrow[r, "\mu^*"] \arrow[d, "\cong"', "D"] & H^{n-r}(M) \arrow[d, "\cong"', "-\cap[M,\partial M]"] \\
\widetilde{H}_r(IX) \arrow[r, "\gamma_*"] & H_r(M, \partial M)
\end{tikzcd}$$

commute.

Proof. The isomorphism D will be fitted into an isomorphism between the cohomology exact sequence of the pair $|H\Gamma^{\bar{m}}| \to M$ and the complementary homology exact sequence of the triple (12). Proposition 6.1 provides a commutative square

$$\begin{tikzcd}
H^{n-r}(\partial M) \arrow[r, "e^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, "\cong"] \\
H_{r-1}(\partial M) \arrow[r] & H_{r-1}(e).
\end{tikzcd} \tag{13}$$

The connecting homomorphism $\partial_* : H_n(M, \partial M) \to H_{n-1}(\partial M)$ sends the fundamental class $[M, \partial M]$ to $\partial_*[M, \partial M] = [\partial M]$. Since for $j^* : H^{n-r}(M) \to H^{n-r}(\partial M)$ and $\xi \in H^{n-r}(M)$ we have

$$\partial_*(\xi \cap [M, \partial M]) = j^*\xi \cap \partial_*[M, \partial M]$$

(see [Spa66], Chapter 5, Section 6, 20, page 255), the square

$$\begin{tikzcd}
H^{n-r}(M) \arrow[r, "j^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(\partial M) \arrow[d, "\cong"', "-\cap[\partial M]"] \\
H_r(M, \partial M) \arrow[r, "\partial_*"] & H_{r-1}(\partial M)
\end{tikzcd} \tag{14}$$

commutes. Since $g^* = e^* \circ j^*$ and the connecting homomorphism

$$\partial_* : H_r(M, \partial M) \longrightarrow H_{r-1}(e)$$

of the triple factors as

$$H_r(M, \partial M) \xrightarrow{\partial_*} H_{r-1}(\partial M) \longrightarrow H_{r-1}(e),$$

composing diagram (14) and diagram (13) yields a commutative square

$$\begin{tikzcd}
H^{n-r}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, "\cong"] \\
H_r(M, \partial M) \arrow[r, "\partial_*"] & H_{r-1}(e).
\end{tikzcd}$$

We use these squares in the diagram

$$\begin{tikzcd}
H^{n-r-1}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r-1}|H\Gamma^{\bar{m}}| \arrow[r, "\delta^*"] \arrow[d, "\cong"] & H^{n-r}(g) \arrow[r] & H^{n-r}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, "\cong"] \\
H_{r+1}(M,\partial M) \arrow[r, "\partial_*"] & H_r(e) \arrow[r, "j_*"] & H_r(g) \arrow[r] & H_r(M,\partial M) \arrow[r, "\partial_*"] & H_{r-1}(e),
\end{tikzcd}$$

By Lemma 4.6, there exists a map

$$D : \widetilde{H}^{n-r}(IX) = H^{n-r}(g) \longrightarrow H_r(g) = \widetilde{H}_r(IX)$$

filling in the diagram commutatively. By the 5-lemma, D is an isomorphism. $\square$

Remark 6.3. The simple connectivity conditions on the links L, L_0 only enter in so far as to ensure that the homological truncations $L_{<k}$, $W_{<k}$ exist. Actually, regardless of simple connectivity, the strong Witt condition on L alone guarantees that $L_{<k}$ exists (because then we may take $L_{<k} = L^{k-1}$). The simple connectivity of both L and L_0 is a sufficient condition for the existence of $W_{<k}$, but certainly not a necessary condition. Example 6.5 below illustrates this by considering nonsimply connected links such that $L_{<k}, W_{<k}$ exist with the correct properties. The simple connectivity assumption never enters otherwise in the proof of Proposition 6.1 and Theorem 6.2, so that these results remain true for nonsimply connected L, L_0, provided the truncations exist.

Remark 6.4. The construction of the duality isomorphism D in the proof of Theorem 6.2 uses Lemma 4.6 and thus involves an element of choice. A canonical construction of an isomorphism $D : \widetilde{H}^{n-r}(IX) \to \widetilde{H}_r(IX)$ in all degrees r except the middle, avoiding that lemma, runs as follows: Suppose $r > k$. Then, as was shown in the proof of Proposition 6.1, $e^* : H^{n-r}(\partial M) \to H^{n-r}|H\Gamma^{\bar{m}}|$ and $H_{r-1}(\partial M) \to H_{r-1}(e)$ are isomorphisms. From the exact sequences

$$\begin{aligned} H^{n-(r+1+i)}(\partial M) \xrightarrow[e^*]{\cong} H^{n-(r+1+i)}|H\Gamma^{\bar{m}}| \xrightarrow{\delta^*} H^{n-(r+i)}(e) \\ \longrightarrow H^{n-(r+i)}(\partial M) \xrightarrow[e^*]{\cong} H^{n-(r+i)}|H\Gamma^{\bar{m}}|, \ i = 0, 1, \end{aligned}$$

we deduce that $H^{n-r-1}(e) = 0$ and $H^{n-r}(e) = 0$. The exact triple sequence

$$H^{n-r-1}(e) \xrightarrow{\delta^*} H^{n-r}(M, \partial M) \longrightarrow H^{n-r}(g) \longrightarrow H^{n-r}(e)$$

implies that $H^{n-r}(M, \partial M) \to H^{n-r}(g)$ is an isomorphism. From the sequences

$$\begin{aligned} H_{r+i}(\partial M) \xrightarrow{\cong} H_{r+i}(e) \xrightarrow{\partial_*} H_{r+i-1}|H\Gamma^{\bar{m}}| \\ \longrightarrow H_{r+i-1}(\partial M) \xrightarrow{\cong} H_{r+i-1}(e), \ i = 0, 1, \end{aligned}$$

we infer that $H_r|H\Gamma^{\bar{m}}| = 0$ and $H_{r-1}|H\Gamma^{\bar{m}}| = 0$. Hence $H_r(M) \to H_r(g)$ is an isomorphism by the exactness of

$$H_r|H\Gamma^{\bar{m}}| \xrightarrow{g_*} H_r(M) \longrightarrow H_r(g) \xrightarrow{\partial_*} H_{r-1}|H\Gamma^{\bar{m}}|.$$

Define D to be the unique isomorphism such that the square

$$\begin{tikzcd}
H^{n-r}(M, \partial M) \arrow[r, "\cong"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(g) \arrow[d, dotted, "D"] \\
H_r(M) \arrow[r, "\cong"] & H_r(g)
\end{tikzcd}$$

commutes. Suppose $r < k$. Then, as was established in the proof of Proposition 6.1, $H^{n-(r+1)}|H\Gamma^{\bar{m}}| = 0$ and $H_r(e) = 0$. Therefore, by the exactness of

$$0 = H^{n-(r+1)}|H\Gamma^{\bar{m}}| \xrightarrow{\delta^*} H^{n-r}(g) \longrightarrow H^{n-r}(M) \xrightarrow{g^*} H^{n-r}|H\Gamma^{\bar{m}}| = 0,$$

the map $H^{n-r}(g) \to H^{n-r}(M)$ is an isomorphism. The exact sequence

$$0 = H_r(e) \longrightarrow H_r(g) \longrightarrow H_r(M, \partial M) \xrightarrow{\partial_*} H_{r-1}(e) = 0$$

shows that $H_r(g) \to H_r(M, \partial M)$ is an isomorphism. Define D to be the unique isomorphism such that

$$\begin{tikzcd} H^{n-r}(g) \arrow[r, "\cong"] \arrow[d, dotted, "D"'] & H^{n-r}(M) \arrow[d, "\cong"', "-\cap[M,\partial M]"] \\ H_r(g) \arrow[r, "\cong"] & H_r(M, \partial M) \end{tikzcd}$$

commutes. In the middle dimension $r = k$, we have $H_{k-1}(e) = 0$ and $H_k(\partial M) \to H_k(e)$ is an isomorphism. Hence the commutative diagram with exact rows

$$\begin{tikzcd} H_{k+1}(M, \partial M) \arrow[r, "\partial_*"] \arrow[d, equal] & H_k(\partial M) \arrow[r] \arrow[d, "\cong"'] & H_k(M) \arrow[r] \arrow[d] & H_k(M, \partial M) \arrow[r, "\partial_*"] \arrow[d, equal] & H_{k-1}(\partial M) \arrow[d] \\ H_{k+1}(M, \partial M) \arrow[r, "\partial_*"] & H_k(e) \arrow[r] & H_k(g) \arrow[r] & H_k(M, \partial M) \arrow[r, "\partial_*"] & H_{k-1}(e) = 0 \end{tikzcd}$$

shows that $H_k(g) \to H_k(M, \partial M)$ is a surjection and $H_k(M) \to H_k(g)$ is an injection.

Not every stratified pseudomanifold possesses an intersection space. This is not unexpected in view of the rich internal algebraic structure of HI^*, as opposed to intersection cohomology IH^*. For a given perversity, obstructions to constructing an intersection space arise from certain differentials in the Serre spectral sequences of the link bundles. The techniques introduced in the present paper seem to be useful in studying other depth 2 stratifications as well, or perhaps even higher depth, but will not in general by themselves suffice to construct intersection spaces in more general situations. For example, one might consider a stratification $X_n \supset X_2 \supset X_0$ with X_2 a 2-sphere and X_0 a point. Then, as $X_2 - X_0$ is contractible, the link bundle over $X_2 - X_0$ is trivializable (as it is in this paper) and its total space looks like $\operatorname{int}(D^2) \times L$. The link L_0 of X_0 looks like $L_0 = W \cup_{\partial W} S^1 \times \operatorname{cone}(L)$, where W is a manifold with boundary $\partial W = S^1 \times L$. The exterior ∂M of the singular set looks like $|\Gamma_\partial|$, with Γ_∂ the 3-diagram

$$W \xleftarrow{f} S^1 \times L \hookrightarrow D^2 \times L.$$

The map f is a homeomorphism onto its image ∂W. To form the intersection space, one would have to produce the broken arrow in the 3-diagram

$$\Gamma^{\bar{m}} = \left(W_{<k_W} \dashleftarrow S^1 \times L_{<k_L} \hookrightarrow D^2 \times L_{<k_L} \right)$$

by suitably truncating f. However, as mentioned above, fiberwise truncation of maps is generally obstructed. Appropriate assumptions on links and/or structure groups of the involved bundles will imply that these obstructions vanish. Thus, particular features of the geometry of a given pseudomanifold X enter in an interesting, nontrivial way to enable or disable the existence of intersection spaces for X.

We conclude with a simple 6-dimensional example, illustrating in particular Proposition 6.1.

Example 6.5. Suppose that X is a 6-dimensional pseudomanifold with stratification $X_6 \supset X_1 \supset X_0$, X_1 a circle and X_0 a point. Suppose the link L of $X_1 - X_0$ is the 4-manifold $L = S^1 \times S^3$. Let T be a 2-torus with two disjoint small open discs removed. The boundary of T consists of two circles, $\partial T = S^1_c \sqcup S^1_d$. The 5-manifold $W = T \times S^3$ has boundary $\partial W = S^1_c \times S^3 \sqcup S^1_d \times S^3$. Suppose that the link L_0 of X_0 is

$$L_0 = W \cup_{\partial W} (\mathrm{cone}(S^1_c \times S^3) \sqcup \mathrm{cone}(S^1_d \times S^3))$$

and that the link bundle $\Delta^1 \times L$ is attached to W by the identity maps $\{0\} \times L \xrightarrow{\mathrm{id}} S^1_c \times S^3$ and $\{1\} \times L \xrightarrow{\mathrm{id}} S^1_d \times S^3$. We equip the circle factor of L with the CW structure $S^1 = e^0 \cup e^1$ and the 3-sphere factor with the structure $S^3 = e^0_S \cup e^3_S$. Then L receives the product cell structure. We endow T with the CW structure

$$T = (e^0_0 \cup e^0_1) \cup (a \cup b \cup c \cup d \cup e^1_d) \cup e^2,$$

where a, b, c, d and e^1_d are 1-cells such that a, b, c are all attached as loops to e^0_0, whereas d is attached as a loop to e^0_1 and e^1_d joins the two 0-cells e^0_0 and e^0_1. The 2-cell e^2 is attached by the word $abe^1_d d(e^1_d)^{-1}a^{-1}b^{-1}c^{-1}$. Then $S^1_c \subset T$ and $S^1_d \subset T$ are the subcomplexes $S^1_c = e^0_0 \cup c$, $S^1_d = e^0_1 \cup d$. The space W receives the product cell structure. As $\partial_2 : C_2(T) \to C_1(T)$ maps e^2 to $d - c$, we have $[c] = [d] \in H_1(T)$. This group $H_1(T)$ has rank 3 generated by $[a], [b]$ and $[c] = [d]$. Consequently, the homology of W is given by the following generators:

$H_0(W)$	$[e^0_0 \times e^0_S]$
$H_1(W)$	$[a \times e^0_S], [b \times e^0_S], [c \times e^0_S]$
$H_2(W)$	0
$H_3(W)$	$[e^0_0 \times e^3_S]$
$H_4(W)$	$[a \times e^3_S], [b \times e^3_S], [c \times e^3_S]$
$H_5(W)$	0

Note that the strong Witt condition on L is satisfied, as L has no 2-dimensional cells. The link L_0 is not homeomorphic to the suspension of L, since $H_1(L_0)$ has rank 2, generated by $[a \times e^0_S]$ and $[b \times e^0_S]$, while the suspension has trivial first homology. Thus X cannot be restratified with depth 1. Note also that L is not simply connected, but this presents no problem, since the required spatial homology truncation does exist and is given by the 1-skeleton:

$$L_{<k} = L_{<3} = L_{<2} = L^1 = (e^0 \cup e^1) \times e^0_S = S^1 \times \mathrm{pt}\,.$$

The structural map e_L is the inclusion $e_L : L_{<k} = S^1 \times \mathrm{pt} \hookrightarrow S^1 \times S^3 = L$. The spatial homology truncation of W is

$$W_{<k} = W_{<3} = W_{<2} = T \times e^0_S.$$

Thus $W_{<3}$ is precisely the 2-skeleton W^2 of W and the structural map $e_W : W_{<3} \to W$ is the skeletal inclusion $W^2 \hookrightarrow W$. The map $f : \ddot{L} \to W$ is the inclusion given on the component $\{0\} \times L$ by $\{0\} \times L \xrightarrow{\mathrm{id}} S^1_c \times S^3 \hookrightarrow \partial W \hookrightarrow W$ and on the component $\{1\} \times L$ by $\{1\} \times L \xrightarrow{\mathrm{id}} S^1_d \times S^3 \hookrightarrow \partial W \hookrightarrow W$. Its homological truncation $t_{<k} f = t_{<3} f : \ddot{L}_{<3} \to W_{<3}$ is the inclusion given on the two components of $\ddot{L}_{<3}$ by

$$\{0\} \times (e^0 \cup e^1) \times e^0_S \xrightarrow{\mathrm{id}} (e^0_0 \cup c) \times e^0_S \hookrightarrow T \times e^0_S$$

and

$$\{1\} \times (e^0 \cup e^1) \times e^0_S \xrightarrow{\mathrm{id}} (e^0_1 \cup d) \times e^0_S \hookrightarrow T \times e^0_S.$$

The diagram of inclusions

$$\begin{tikzcd} \ddot{L}_{<3} \arrow[r, hook, "t_{<3}f"] \arrow[d, hook, "e_{\ddot{L}}"'] & W_{<3} \arrow[d, hook, "e_W"] \\ \ddot{L} \arrow[r, hook, "f"] & W \end{tikzcd}$$

commutes. The $\bar{m}$-perverse 3-diagram $\Gamma^{\bar{m}}$ is given by

$$\begin{aligned} \Gamma^{\bar{m}} &= (\, W_{<3} \xleftarrow{\;t_{<3}f\;} \ddot{L}_{<3} \xhookrightarrow{\;\mathrm{incl}\times\mathrm{id}\;} \overline{L}_{<3} \,) \\ &= (\, T \times e^0_S \longleftarrow\!\!\!\supset (\partial I) \times S^1 \times e^0_S \hookrightarrow I \times S^1 \times e^0_S \,), \end{aligned}$$

that is, a handle $I \times S^1$ is attached to T along the two boundary circles of the surface T. Hence $|H\Gamma^{\bar{m}}|$ is the orientable closed surface Σ_2 of genus 2. The map

$$e = |H(\epsilon)| : |H\Gamma^{\bar{m}}| \longrightarrow |H\Gamma| = \partial M$$

is given by

$$\mathrm{id}_{\Sigma_2} \times \mathrm{incl} : \Sigma_2 \times e^0_S \hookrightarrow \Sigma_2 \times S^3.$$

A straightforward calculation yields the following table of generators, illustrating the Poincaré duality isomorphism

$$H^{n-r}|H\Gamma^{\bar{m}}| \xrightarrow{\cong} H_{r-1}(e)$$

of Proposition 6.1.

	$H^{n-r}\|H\Gamma^{\bar{m}}\|$	$H_{r-1}(e)$
$r=0$	0	0
$r=1$	0	0
$r=2$	0	0
$r=3$	0	0
$r=4$	$[\Sigma_2 \times \mathrm{pt}]^*$	$[\mathrm{pt} \times S^3]$
$r=5$	$[a \times \mathrm{pt}]^*$	$[b \times S^3]$
	$[b \times \mathrm{pt}]^*$	$[a \times S^3]$
	$[c \times \mathrm{pt}]^*$	$[z \times S^3]$
	$[z \times \mathrm{pt}]^*$	$[c \times S^3]$
$r=6$	$1 = [\mathrm{pt}]^*$	$[\Sigma_2 \times S^3]$

Here, $[\cdot]$ denotes the homology class of a cycle and $[\cdot]^*$ the image in cohomology of the linear dual of a homology class under the universal coefficient isomorphism. Poincaré duals are listed next to each other in the same row. The cycle z is $z = I \cup_{\partial I} e^1_d$.

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