EQUIVALENT L-CLASSES OF ATIYAH-SINGER-ZAGIER TYPE FOR SINGULAR SPACES

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ABSTRACT. If a finite group G acts on a rational homology manifold, then the orbit space is well-known to be a rational homology manifold again. We consider here actions on spaces that may be much more singular. If the G-space is a Witt pseudomanifold, which includes all arbitrarily singular complex pure-dimensional algebraic varieties, then we prove that the orbit space is again a Witt pseudomanifold. In the compact oriented situation, this implies that the orbit space possesses characteristic L-classes, as defined by Goresky and MacPherson. We then construct Atiyah-Singer-Zagier type equivariant L-classes for such G-pseudomanifolds which serve, as we show by establishing an averaging formula, as a tool to compute the Goresky-MacPherson L-class of the orbit space. The construction of the equivariant class builds on intersection homological transfer properties and on recent joint K-theoretic work with Eric Leichtnam and Paolo Piazza, which established a G-signature theorem on Witt pseudomanifolds.

CONTENTS

1. INTRODUCTION

Let X be a compact oriented pseudomanifold (without boundary). We place the Witt condition on X, which requires that the middle perversity intersection chain complex of sheaves is self-dual. For example every pure-dimensional compact complex algebraic variety is a

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Witt pseudomanifold. Suppose that a finite group G acts on X by orientation preserving transformations. Then for every element $g \in G$, we define an *equivariant L-class*

$$L_*(g,X) \in H_*(X;\mathbb{C})^G.$$

For $g = 1$, this class agrees with the Goresky-MacPherson-Siegel L-class of X ([29], [42]), $L_*(1,X) = L_*(X)$. When X is a rational homology manifold, $L_*(g,X)$ agrees with the Poincaré dual of Zagier's cohomological class $L^*(g,X) \in H^*(X;\mathbb{C})^G$ constructed in [45] and [46]. Suppose that X is a smooth manifold such that G acts smoothly on X, preserving the orientation. For an element $g \in G$, let X^g be the submanifold of points fixed by g and let or_{X^g} denote the orientation local system on X^g. Atiyah and Singer constructed an equivariant class in $H^*(X^g;\mathrm{or}_{X^g} \otimes \mathbb{C})$, explicitly given in terms of characteristic classes of X^g and of the equivariant normal bundle of the inclusion $j : X^g \subset X$, see [5, p. 582]. Let $L'(g,X)$ be the formulation of these classes as given by Zagier in [45, p. 12, (27)], [46]. The G-Signature Theorem of Atiyah and Singer is the relation

$$\mathrm{Sign}(g,X) = \langle L'(g,X), [X^g] \rangle.$$

The inclusion has an associated Gysin homomorphism $j_! : H^*(X^g) \to H^*(X)$ and the image of the Atiyah-Singer-class under $j_!$ is Zagier's class, i.e. $j_! L'(g,X) = L^*(g,X)$, [45, p. 4, (3); p. 21, Thm. 1]. As the Gysin map corresponds under Poincaré duality to covariant homological pushforward $j_* : H_*(X^g) \to H_*(X)$, the Atiyah-Singer-class is related to our homological class by

$$j_*(L'(g,X) \cap [X^g]) = L_*(g,X)$$

in the differentiable case.

As in the smooth situation, an important application of the equivariant classes $L_*(g,X)$ is that they enable the computation of the Goresky-MacPherson-Siegel class $L_*(X/G)$ of the orbit space X/G via an averaging formula. For $L_*(X/G)$ to be defined, one must first know that the orbit space is also a Witt pseudomanifold. We prove this in Corollary 7.3. Let us outline the strategy. Like Zagier, we work with triangulated spaces and simplicial actions. Foundational material on such actions is reviewed in Section 7. For example, subanalytic proper actions admit a G-equivariant triangulation, [9, Prop. 6.7]; see Examples 7.1. We show that if an oriented pseudomanifold X is equipped with an oriented action of a finite group G, then X/G is again an oriented pseudomanifold (Theorem 7.2). It is harder to establish the Witt condition for X/G. The idea is to proceed as in a transfer-based proof of the classical Conner conjecture, which asserts that the orbit space of a finite group action on a $\mathbb{Q}$-acyclic space is again $\mathbb{Q}$-acyclic. Such arguments are for instance used to show that the orbit space of a finite group action on a rational homology manifold is again a rational homology manifold (Bredon [12, V §19]). In fact, the Conner conjecture holds more generally for finite ramified coverings $\pi : \widetilde{Y} \to Y$. For such coverings, we construct an intersection homology transfer $\pi_! : IH_*^{\bar{p}}(Y;\mathbb{Z}) \to IH_*^{\bar{p}}(\widetilde{Y};\mathbb{Z})$ for every $\bar{p}$. Such transfers appeared already in the work of Goresky and MacPherson on the Lefschetz fixed point theorem for intersection homology, [28]. Our construction rests on two pillars: an elegant formalization of ramified coverings due to Larry Smith [43] on the one hand, and on Gajer's intersection Dold-Thom theorem ([25], corrected in [26]) on the other. Smith's formalization involves symmetric products and thus, ultimately, the free abelian topological group $AG(\widetilde{Y})$ generated by the points of $\widetilde{Y}$. The intersection Dold-Thom theorem provides natural isomorphisms $IH_*^{\bar{p}}(Y;\mathbb{Z}) \cong I^{\bar{p}}\pi_*(AG(Y))$ for connected Y, where $I^{\bar{p}}\pi_*(-)$ denotes the intersection homotopy groups of a filtered space. The relationship of intersection homotopy to intersection homology groups is clarified by Chataur, Saralegi-Aranguren and Tanré in [17]. Gajer's simplicial set can also be understood in terms of homotopy theoretic truncations of the Postnikov tower of the links in a stratified space, as has been shown by Chataur, Saralegi-Aranguren and Tanré in [18]. Note that the free abelian topological group must be endowed with a filtration. For this, one uses the Lawson filtration (Section 3). Since the ramified covering $\pi : \widetilde{Y} \to Y$ is placid, it induces a homomorphism $\pi_* : IH_*^{\bar{p}}(\widetilde{Y}) \to IH_*^{\bar{p}}(Y)$. We construct a continuous group homomorphism $\tau : AG(Y) \to AG(\widetilde{Y})$ and prove that it is placid with respect to the Lawson filtrations. The desired transfer $\pi_!$ on intersection homology is then induced by τ under the intersection Dold-Thom theorem. There are of course other ways to define such a transfer, but from this homotopy theoretic perspective, it is then particularly straightforward to deduce up-down and down-up formulae. That is, the composition $\pi_* \circ \pi_!$ is multiplication by the degree of the covering (Proposition 5.8), and if $\pi : \widetilde{Y} = X \to X/G = Y$ is an orbit projection, structured as a $|G|$-fold ramified covering, then $\pi_! \circ \pi_* = \sum_{g \in G} g_*$ (Proposition 5.9). The up-down property, in conjunction with regular neighborhood theory and intersection homology properties of suspensions, is then used in establishing the Witt condition for the base space Y of a ramified covering (Theorem 6.9). In summary, we obtain for orbit spaces:

Theorem. *Let G be a finite group and X an oriented Witt pseudomanifold upon which G acts by orientation preserving simplicial maps. Then the orbit space X/G is an oriented Witt pseudomanifold.*

(This is Corollary 7.3 in the main text.) The equivariant L-class $L_*(g,X)$ is then constructed in Section 9. The method is based on Zagier's, which consists of taking equivariant g-signatures $\operatorname{Sign}(g,X)$ of transverse point-preimages of G-invariant maps from X to spheres. As in [5], the equivariant signatures $\operatorname{Sign}(g,X)$ are complex numbers obtained by evaluating representation theoretic G-signatures $\operatorname{Sign}(G,X) \in R(G)$. For Witt spaces, however, we use the G-invariant intersection form on middle intersection homology. The equivariant signature of Witt pseudomanifolds is discussed in Section 8; see also [9, Section 5]. We prove in particular that the signature of the orbit space is given in terms of equivariant signatures by

$$\operatorname{Sign}(X/G) = \frac{1}{|G|} \sum_{g \in G} \operatorname{Sign}(g,X),$$

see Proposition 8.12.

Using the equivariant L-classes $L_*(g,X)$, one can then calculate the Goresky-MacPherson-Siegel L-class of the orbit space as follows (Theorem 10.1).

Theorem. *Let G be a finite group and X an oriented closed Witt pseudomanifold upon which G acts simplicially, preserving the orientation. Then X/G is a Witt pseudomanifold and its Goresky-MacPherson-Siegel L-class $L_*(X/G)$ is related to the equivariant L-classes by*

$$L_*(X/G) = \frac{1}{|G|} \sum_{g \in G} \pi_* L_*(g,X), \tag{1}$$

where $\pi : X \to X/G$ is the orbit projection.

For a smooth oriented closed G-manifold M, endowed with a G-invariant Riemannian metric, the signature operator D^{sign} is a Dirac-type operator commuting with the action of G. It determines a class $[D^{\mathrm{sign}}] \in K_j^G(M)$, $j \equiv \dim M (2)$, in equivariant analytic K-homology which is independent of the choice of metric. By the Hodge theorem, $\operatorname{Sign}(G,M)$ is equal to the equivariant index of the signature operator, $\operatorname{Sign}(G,M) = \operatorname{ind}_G(D^{\mathrm{sign},+}) \in R(G)$. Let X be a Witt G-pseudomanifold, equipped with an equivariant Thom-Mather stratification and a G-invariant wedge metric. As a consequence of Cheeger's Hodge-theorem on Witt spaces ([19]), $\operatorname{Sign}(G,X)$ is again equal to the equivariant index of the signature operator

D^{sign}. In [9], Leichtnam, Piazza and the author use the associated equivariant analytic K-homology class $[D^{\text{sign}}] \in K^G_j(X)$ to extend the fundamental result of Atiyah-Segal-Singer from smooth G-manifolds to Witt G-pseudomanifolds. The characteristic class formula we obtain there applies in situations where the fixed point sets X^g admit G-tubular neighborhoods that have the structure of G-vector bundles. The formula then expresses $\operatorname{Sign}(g,X)$ in terms of the Goresky-MacPherson-Siegel L-class of the Witt pseudomanifold X^g and in terms of characteristic classes of the normal bundle. The latter are those characteristic classes used by Atiyah-Segal-Singer. The proof uses Kasparov's bivariant KK-theory in an essential way, together with the localization theorem. In the present paper, we do not make assumptions on the normal topology of fixed point sets, and characteristic classes of normal bundles are thus not explicitly considered. An investigation of the relationship of $L_*(g,X)$ to $[D^{\text{sign}}] \in K^G_j(X)$ using a suitable Chern character is beyond the scope of this paper.

For a finite group G acting continuously on a topological Witt space X (satisfying weak regularity properties on the fixed point sets), Cappell, Shaneson and Weinberger indicated the construction of a G-equivariant class $\Delta^G(X) \in \mathrm{KO}^G_*(X) \otimes \mathbb{Z}[\frac{1}{2}]$ and a corresponding G-signature theorem in [16]. Free group actions on PL Witt spaces were considered by Curran in [21], and are here considered in Section 11, where we show that for such actions $L_*(g,X) = 0$ for $g \neq 1$ (Theorem 11.1).

For singular complex algebraic varieties, homological Hirzebruch characteristic classes have been defined by Brasselet, Schürmann and Yokura in [11]. Equivariant generalizations $T_{y*}(X;g) \in H^{\mathrm{BM}}_{2*}(X^g) \otimes \mathbb{C}[y]$ have been constructed by Cappell, Maxim, Schürmann and Shaneson in [15] for finite groups G that act on quasi-projective varieties X by algebraic automorphisms. These classes satisfy the analog of our formula (1), i.e. their average over all group elements computes $T_{y*}(X/G)$ ([15, p. 1726, Thm. 1.1]). (Note that X/G is again a quasi-projective variety.)

In [10], we constructed a G-equivariant L-class $L^G_*(X) \in H^G_*(X;\mathbb{Q})$ in equivariant rational homology for compact, oriented Whitney stratified Witt spaces $X \subset M$ that are invariant under the oriented smooth action of a compact Lie group G on an ambient smooth manifold M. If $X = M^m$ is smooth, then equivariant Poincaré duality is an isomorphism $H^{m-i}_G(X) \cong H^G_i(X)$, where equivariant cohomology $H^*_G(-)$ means ordinary cohomology of the Borel space $(-)_G = EG \times_G (-)$. In this case, $L^G_*(X)$ is the equivariant Poincaré dual of the usual equivariant cohomology class $L^*_G(X) = L^*((TX)_G)$, where $(TX)_G$ is the homotopy quotient vector bundle of the equivariant tangent bundle TX. See L. Tu [44, Chapter 30.3] for such equivariant characteristic classes. The present paper is completely independent of [10]. The class L^G_* of [10] is consistent with the equivariant viewpoint of Ohmoto [38], where equivariant Chern-Schwartz-MacPherson classes were developed for algebraic actions of a complex reductive linear algebraic group on a possibly singular complex algebraic variety X. It is also consistent with the construction of equivariant Todd classes by Brylinski and Zhang in [14].

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2. Filtered Spaces and Placid Maps

Intersection homology and intersection homotopy groups are defined for filtered spaces. We thus review basic notions related to filtered spaces with a particular emphasis on stratified, placid, and cofiltered maps. We establish that a finite-to-one surjective simplicial map between triangulated finite-dimensional polyhedra is placid with respect to the simplicial filtrations (Lemma 2.14).

Definition 2.1. Let X be a nonempty topological space. A *filtration* $\mathcal{X}$ *of* X *of formal dimension* n is a finite nested sequence

$$X_{-1} = \varnothing \subset X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X_n = X$$

of subspaces, $X_{n-1} \neq X$. The pair $(X, \mathcal{X})$ is then called a *filtered space*. The nonempty connected components S of the differences $X_i - X_{i-1}$ are called the *strata of* $\mathcal{X}$ *of formal dimension* i. The *formal codimension of* S *in* X is $\operatorname{codim} S := n - i$. It is convenient to set $X^i := X_{n-i}$.

We do not require the subspaces X_i to be closed in X. The above notion of formal dimension need not be related to any notion of topological dimension and will indeed later be applied to spaces of infinite topological dimension. Although the formal dimension is a property of $\mathcal{X}$ and not of the space X, it is nevertheless often convenient to write $\dim X = n$ when the filtration is understood. Base points $x_0 \in X$ of pointed filtered spaces are always chosen to lie in $X - X_{n-1}$. Let $\mathcal{S}(\mathcal{X})$ denote the set of strata of a filtration $\mathcal{X}$.

Lemma 2.2. *Let $(X, \mathcal{X})$ be a filtered space. If $S \in \mathcal{S}(\mathcal{X})$ is a stratum and i an index such that $S \cap X_i \neq \varnothing$, then $S \subset X_i$.*

Lemma 2.3. *Let $(X, \mathcal{X})$ be a filtered space of formal dimension n and let $S \in \mathcal{S}(\mathcal{X})$ be a stratum. Then $S \subset X_{n-k}$ if and only if* $\operatorname{codim} S \geq k$.

Definition 2.4. A *stratified map* $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ between filtered spaces is a continuous map $f : X \to Y$ such that for each stratum S of $\mathcal{X}$ there exists a stratum T of $\mathcal{Y}$ with $f(S) \subset T$.

Note that since $f(S)$ is nonempty and connected, the stratum T is uniquely determined by S and f. We shall also use the notation $f_*(S) := T$. The above notion of stratified map agrees with Friedman's usage of this term in [23, Def. 2.9.1].

Lemma 2.5. *The composition $g \circ f$ of stratified maps f, g is again a stratified map, and $(g \circ f)_* = g_* \circ f_*$.*

Definition 2.6. A stratified map $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ between filtered spaces is called *equidimensionally stratified*, if $f(X_i - X_{i-1}) \subset Y_i - Y_{i-1}$ for every $i = 0, \ldots, n = \dim X$.

Lemma 2.7. *The composition of equidimensionally stratified maps is again an equidimensionally stratified map.*

Definition 2.8. A stratified map $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ between filtered spaces is called *placid* if

$$\operatorname{codim} f_*(S) \leq \operatorname{codim} S$$

for every stratum $S \in \mathcal{S}(\mathcal{X})$.

This agrees with Friedman's usage of this term ([23, Example 4.1.5]). We note parenthetically that what we call a placid map is called a stratified map in [17].

Lemma 2.9. *The composition of placid maps is again a placid map.*

Definition 2.10. (Gajer [25].) Let $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ be filtered spaces. A continuous map $f: X \to Y$ is called *cofiltered* with respect to $\mathcal{X}$ and $\mathcal{Y}$ if $f^{-1}(Y^k) \subset X^k$ for all $k \geq 0$.

Lemma 2.11. *The composition of cofiltered maps is cofiltered.*

Let us clarify the relationship between the concepts of stratified, placid and cofiltered maps. Simple examples show that a cofiltered map need not be stratified. Lemma 2.3 is helpful in proving the next lemma.

Lemma 2.12. *A stratified cofiltered map is placid.*

Lemma 2.13. *Placid maps are cofiltered.*

Proof. Let $f:(X,\mathcal{X}) \to (Y,\mathcal{Y})$ be a placid map and suppose that $x \in f^{-1}(Y^k)$ is any point. Let S be the unique stratum of $\mathcal{X}$ that contains x. Since f is a stratified map, there is a unique stratum T of $\mathcal{Y}$ with $f(S) \subset T$. We know that $f(x) \in Y^k = Y_{m-k}$, where $m = \dim Y$. In addition, $f(x) \in f(S) \subset T$. Thus, $f(x) \in T \cap Y_{m-k}$. By Lemma 2.2, $T \subset Y_{m-k}$, and by Lemma 2.3, $\operatorname{codim} T \geq k$. Since f is placid, $\operatorname{codim} T \leq \operatorname{codim} S$. Therefore, $\operatorname{codim} S \geq k$. By Lemma 2.3, $S \subset X_{n-k}$, where $n = \dim X$. It follows that $x \in S \subset X_{n-k} = X^k$. □

Together, the above two lemmas show that a continuous map is placid if and only if it is stratified and cofiltered.

All of our perversity functions $\bar{p}$ will be Goresky-MacPherson type perversities ([29]). In particular, they only depend on the codimension of a stratum, i.e. $\bar{p}(S) = \bar{p}(\operatorname{codim} S)$ holds on every stratum S; we will use the same symbol $\bar{p}$ to denote the function on strata and the function on codimensions. A placid map f is "$(\bar{p},\bar{p})$-stratified" for every perversity $\bar{p}$, that is, $\bar{p}(S) - \operatorname{codim} S \leq \bar{p}(f_*(S)) - \operatorname{codim} f_*(S)$ for all strata $S \in \mathcal{S}(\mathcal{X})$. This is a consequence of the Goresky-MacPherson growth condition on perversities. See [23, p. 137, Example 4.1.5].

Lemma 2.14. *A finite-to-one surjective simplicial map between triangulated finite-dimensional polyhedra is placid with respect to the simplicial filtrations.*

Proof. Let K and L be finite-dimensional simplicial complexes with associated polyhedra $X = |K|$ and $Y = |L|$. The simplicial filtrations $\mathcal{X},\mathcal{Y}$ are given by the polyhedra $X_i = |K_i|$, $Y_i = |L_i|$ of the i-dimensional simplicial skeleta $K_i \subset K$, $L_i \subset L$. The strata S of $\mathcal{X}$ are given by the interiors $S = \Delta^\circ$ of the simplices $\Delta \in K$. Similarly for the strata T of $\mathcal{Y}$. Formal dimensions and codimensions agree here with topological dimensions and codimensions. Let $n = \dim X$ and $m = \dim Y$.

Suppose that $f: K \to L$ is a simplicial map which is finite-to-one and surjective. In order to prove that $f:(X,\mathcal{X}) \to (Y,\mathcal{Y})$ is placid, we need to establish first that f is a stratified map. Let $S = \Delta^\circ$ be any stratum of $\mathcal{X}$. Since f is simplicial, the images of the vertices of Δ span a simplex $f\Delta$ of L. Then $T := (f\Delta)^\circ$ is a stratum of $\mathcal{Y}$ with $f(S) \subset T$. This shows that f is a stratified map.

It remains to prove that $\operatorname{codim} f_*(S) \leq \operatorname{codim} S$ for every stratum $S \in \mathcal{S}(\mathcal{X})$. The key point is that the finiteness of f ensures that f cannot decrease the dimension of simplices. Indeed, if f did decrease the dimension of some simplex, then the linearity of f on that simplex together with the pigeonhole principle would imply that f is constant on an edge, contradicting the finiteness of f. (The simplexwise linearity of f implies of course that f cannot increase the dimension of simplices. Thus f preserves the dimension of simplices.)

We observe next that $n = m$, i.e. X and Y have the same dimension: Since f does not decrease the dimension of simplices, the inequality $n \leq m$ holds. The surjectivity of f implies that $n \geq m$, by considering a point y in the interior of an m-simplex of L, and then selecting a point x with $f(x) = y$.

Let $S = \Delta^\circ$ be a stratum of $\mathcal{X}$. We are now in a position to prove that $\operatorname{codim} f_*(S) \leq \operatorname{codim} S$. Since f was already shown to be stratified, there does indeed exist a unique stratum $T = f_*(S)$ of $\mathcal{Y}$ such that $f(S) \subset T$. As we have seen, this stratum is the interior $T = (f\Delta)^\circ$ of the simplex $f\Delta$ of L which is spanned by the images of the vertices of Δ. Since f does not decrease the dimension of simplices, we have $\dim \Delta \leq \dim f\Delta$ and consequently

$$\begin{aligned} \operatorname{codim} T &= m - \dim T = n - \dim f\Delta \leq n - \dim \Delta \\ &= n - \dim S = \operatorname{codim} S. \end{aligned}$$

Thus f is placid with respect to the simplicial stratifications. □

The simplicial inclusion of a vertex in an interval, which is not placid, illustrates the necessity of the surjectivity requirement in the lemma. Placid maps $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ induce homomorphisms $f_* : IH_*^{\bar{p}}(X, \mathcal{X}) \to IH_*^{\bar{p}}(Y, \mathcal{Y})$ on intersection homology for every $\bar{p}$, see e.g. [23, Prop. 4.1.6, p. 138]. Thus, by Lemma 2.14, a finite surjective simplicial map $\pi : \widetilde{Y} \to Y$ (between simplicially stratified complexes), induces a homomorphism

$$\pi_* : IH_*^{\bar{p}}(\widetilde{Y}) \longrightarrow IH_*^{\bar{p}}(Y).$$

3. Free Abelian Topological Groups generated by Polyhedra

The Dold-Thom theorem and its intersection version involve the free abelian topological group $AG(Y)$ generated by the points of a space Y. We review here some aspects of such groups, focusing eventually on filtrations on them, and cofiltered maps between them. Let Y be a compactly generated topological space, for example a CW complex. For every positive integer d, let $SP^d(Y)$ denote the d-fold symmetric product $SP^d(Y) = Y^d/\Sigma_d$, where the symmetric group Σ_d on d letters acts on the d-fold product $Y^d = Y \times \cdots \times Y$ by permuting the coordinates. Algebraically, $AG(Y) = \mathbb{Z}[Y]$ is the free abelian group on the underlying set of Y. For every d, there is an obvious set-theoretic injection $SP^d(Y) \to AG(Y)$ given by viewing points of the symmetric product, which are just finite subsets of points of X with multiplicities adding up to d, as linear combinations with those multiplicities as coefficients. One topologizes $AG(Y)$ as the quotient space of $\bigsqcup_{n,m} SP^n(Y) \times SP^m(Y)$ under the map $\bigsqcup_{n,m} SP^n(Y) \times SP^m(Y) \to AG(Y)$ given by $(\phi, \psi) \mapsto \phi - \psi$. Then $AG(Y)$ is a topological group and a compactly generated space. A good reference is Lima-Filho [34, Section 2.1] (where our $AG(Y)$ is denoted by $\mathcal{Z}(Y)$). The elements of $AG(Y)$ are represented by words w which are finite formal sums $w = \sum n_i y_i$, $n_i \in \mathbb{Z}$, $y_i \in Y$. In general, a given element has many different such representations. To a word w one can associated the function $\phi : Y \to \mathbb{Z}$ given by $\phi(y) = \sum_{\{i \ | y_i = y\}} n_i$. Different words representing the same element of $AG(Y)$ yield the same function ϕ. Thus the elements of $AG(Y)$ are functions $\phi : Y \to \mathbb{Z}$ which vanish at all but finitely many points of Y. The *support* of ϕ is the finite subset $\operatorname{supp}(\phi)$ of Y consisting of all points y such that $\phi(y) \neq 0$. To ϕ, we can associate a word w by $w = \sum_{y \in \operatorname{supp}(\phi)} \phi(y) y$. If a word is of this form, we call it *reduced*. That is, a word $w = \sum n_i y_i$ is reduced if and only if $y_i \neq y_j$ whenever $i \neq j$ and $n_i \neq 0$ for all i. The zero element $\phi = 0$ is represented by the empty word. If w is any word (not necessarily reduced), its support is defined to be the support of the function ϕ associated to w. If $w = \sum n_i y_i$ is a reduced word, then $\operatorname{supp}(w) = \{y_i \mid i\}$.

Now let Y be the polyhedron of a countable simplicial complex, not necessarily connected, and let $Y^+ := Y \sqcup \{*\}$ be the disjoint topological union of Y with an isolated point

$* \notin Y$. The pointed space $(Y^+,*)$ has an associated abelian topological group $AG(Y^+,*)$ constructed by Dold-Thom in [22]. Its underlying algebraic group is $\mathbb{Z}[Y^+ - *] = \mathbb{Z}[Y]$. Their theorem [22, p. 274, Satz 6.10 I for Y] provides an isomorphism $H_*(Y;\mathbb{Z}) = \widetilde{H}_*(Y^+;\mathbb{Z}) \cong \pi_*(AG(Y^+,*),0)$. The polyhedron Y need not be connected here. There is a continuous isomorphism $AG(Y^+,*) \cong AG(Y)$ of topological groups (as follows from [34, p. 570 b; and p. 571, Lemma 2.2 a]). Thus

$$H_*(Y;\mathbb{Z}) \cong \pi_*(AG(Y),0)$$

as stated in [25]. As Y is the polyhedron of a countable simplicial complex, $AG(Y^+,*)$ and $AG(Y)$ have the structure of a CW complex. From now on, the term polyhedron will mean the polyhedron of a countable simplicial complex. There are embeddings $Y = SP^1(Y) \subset SP^d(Y) \hookrightarrow AG(Y)$ for every d. The following universal property is implied by [34, p. 570, d].

Proposition 3.1. *(Lima-Filho.) For a compactly generated space Y, the map $j : Y \to AG(Y)$ satisfies the following universal property: Let G be an abelian topological compactly generated group and let $f : Y \to G$ be a continuous map. Then there exists a unique topological group homomorphism $F : AG(Y) \to G$ such that $F \circ j = f$.*

A continuous map $f : Y \to Y'$ between polyhedra as above induces a well-defined continuous homomorphism $AG(f) : AG(Y) \to AG(Y')$ by setting $AG(f)(\sum n_i y_i) := \sum n_i f(y_i)$, ([25, p. 961]), that is, $AG(f)(\phi)(y') = \sum_{y \in f^{-1}(y')} \phi(y)$. In this way, $AG(-)$ becomes a covariant functor ([34, p. 570, Properties 2.1 a], [36, 6.6]).

Let Y be a space which is the polyhedron of a countable simplicial complex. A filtration $\mathcal{Y}$ of Y induces a filtration $AG(\mathcal{Y})$, the *Lawson filtration*, of $AG(Y)$ as follows: The formal dimension m of $AG(\mathcal{Y})$ is declared to be equal to the formal dimension of $\mathcal{Y}$, and the filtration subspaces are given for $k > 0$ by

$$AG(Y)_{m-k} = AG(Y)^k := \{\phi \in AG(Y) \mid \operatorname{supp}(\phi) \cap Y^k \neq \varnothing\}.$$

If $\phi = 0$ is the zero element of $AG(Y)$, then its support is empty. Thus is does not belong to any $AG(Y)_{m-k}$, $k > 0$. For $k = 0$, we define

$$AG(Y)_m = AG(Y)^0 := \{\phi \in AG(Y) \mid \operatorname{supp}(\phi) \cap Y^0 \neq \varnothing\} \cup \{0\} = AG(Y).$$

If $k > 0$ and $\operatorname{supp}(\phi) \cap Y_{m-k} \neq \varnothing$, then $\operatorname{supp}(\phi) \cap Y_{m-k+1} \neq \varnothing$. This shows that

$$AG(Y)^k = AG(Y)_{m-k} \subset AG(Y)_{m-k+1} = AG(Y)^{k-1}.$$

The next lemma follows by a straightforward verification from definitions.

Lemma 3.2. *Let $(Y,\mathcal{Y})$ be a filtered polyhedron. The canonical embedding*

$$i : (Y,\mathcal{Y}) \hookrightarrow (AG(Y), AG(\mathcal{Y}))$$

is equidimensionally stratified and cofiltered.

Let us investigate when the extension of an equidimensionally stratified and cofiltered map $Y \to AG(X)$ to a continuous homomorphism $AG(Y) \to AG(X)$ is again equidimensionally stratified and cofiltered. This will be so under the following condition.

Definition 3.3. Let X be a polyhedron and Y a topological space. A map $f : Y \to AG(X)$ is called *separated*, if for all $y, y' \in Y$ with $y \neq y'$, we have $\operatorname{supp}(f(y)) \cap \operatorname{supp}(f(y')) = \varnothing$.

Lemma 3.4. *Let $(X,\mathcal{X})$ be a filtered polyhedron and $(Y,\mathcal{Y})$ a filtered compactly generated space such that* $\dim X = \dim Y$. *Let $f : Y \to AG(X)$ be a separated continuous map. Let $F : AG(Y) \to AG(X)$ be the unique extension of f to a continuous group homomorphism. If*

f is equidimensionally stratified and cofiltered (with respect to $\mathcal{Y}$ and the Lawson filtration on $AG(X)$ induced by $\mathcal{X}$), then F is equidimensionally stratified and cofiltered (with respect to the Lawson filtrations).

Proof. Set $n := \dim X = \dim Y$. Then by definition of the Lawson filtration, $n = \dim AG(X) = \dim AG(Y)$. Therefore, given a codimension $k \geq 0$, we must show that

$$F(AG(Y)^k - AG(Y)^{k+1}) \subset AG(X)^k - AG(X)^{k+1}.$$

Since f is equidimensionally stratified and $\dim Y = \dim AG(X)$, we have

$$f(Y^k - Y^{k+1}) \subset AG(X)^k - AG(X)^{k+1}. \tag{2}$$

Let $w = \sum n_i y_i \in AG(Y)^k - AG(Y)^{k+1}$ be a reduced word, $\mathrm{supp}(w) = \{y_i \mid i\}$. Thus

$$\mathrm{supp}(w) \cap Y^k \neq \varnothing \text{ and } \mathrm{supp}(w) \cap Y^{k+1} = \varnothing.$$

Consequently, there exists an index i_0 such that

$$y_{i_0} \in Y^k - Y^{k+1}. \tag{3}$$

We will show that $\mathrm{supp}(F(w)) \cap X^k \neq \varnothing$ and $\mathrm{supp}(F(w)) \cap X^{k+1} = \varnothing$, which will then place $F(w)$ in $AG(X)^k - AG(X)^{k+1}$. In order to do this, we shall determine the support of $F(w)$. The extension F, being a group homomorphism, is given by $F(w) = \sum n_i f(y_i)$. For every i, let $f(y_i) = \sum_j m_{ij} x_{ij}$ be the representation by a reduced word, $m_{ij} \in \mathbb{Z} - \{0\}$, $x_{ij} \in X$. Since for every i, the word is reduced we know that $x_{ij} \neq x_{ij'}$ if $j \neq j'$. Thus the support of $f(y_i)$ is $\mathrm{supp}(f(y_i)) = \{x_{ij} \mid j\}$. The image of w can be expanded as $F(w) = \sum_i \sum_j n_i m_{ij} x_{ij}$. We claim that this word is reduced. Indeed, if $i \neq i'$, then $y_i \neq y_{i'}$ as w is reduced. Therefore, since f is separated,

$$\{x_{ij} \mid j\} \cap \{x_{i'j} \mid j\} = \mathrm{supp}(f(y_i)) \cap \mathrm{supp}(f(y_{i'})) = \varnothing.$$

We conclude that $x_{ij} \neq x_{i'j'}$ whenever $(i,j) \neq (i',j')$. Furthermore, $n_i m_{ij} \neq 0$ for all (i,j). This shows that $\sum_{i,j} n_i m_{ij} x_{ij}$ is a reduced word, which implies that $\mathrm{supp}(F(w)) = \{x_{ij} \mid i,j\}$. By (2) and (3), $f(y_{i_0}) \in AG(X)^k - AG(X)^{k+1}$. So $\mathrm{supp}(f(y_{i_0})) \cap X^k \neq \varnothing$ and $\mathrm{supp}(f(y_{i_0})) \cap X^{k+1} = \varnothing$. Hence, there exists a point $x_{i_0 j_0} \in \mathrm{supp}(f(y_{i_0})) \cap X^k$. Since

$$\mathrm{supp}(f(y_{i_0})) = \{x_{i_0 j} \mid j\} \subset \{x_{ij} \mid i,j\} = \mathrm{supp}\, F(w),$$

the point $x_{i_0 j_0}$ is in $\mathrm{supp}\, F(w) \cap X^k$, which shows that $\mathrm{supp}(F(w)) \cap X^k \neq \varnothing$.

We will verify next that $\mathrm{supp}(F(w)) \cap X^{k+1} = \varnothing$. This will use the assumption that f is cofiltered. We proceed by contradiction: Suppose that $\mathrm{supp}(F(w)) \cap X^{k+1} \neq \varnothing$. Then there exists a pair (i_0, j_0) such that $x_{i_0 j_0} \in X^{k+1}$. Since this point is in the support of the corresponding $f(y_{i_0})$, it follows that $f(y_{i_0}) \in AG(X)^{k+1}$. As f is cofiltered, the inclusion $f^{-1}(AG(X)^{k+1}) \subset Y^{k+1}$ holds, which implies that $y_{i_0} \in Y^{k+1}$. This contradicts $\mathrm{supp}(w) \cap Y^{k+1} = \varnothing$. Therefore, $\mathrm{supp}(F(w)) \cap X^{k+1} = \varnothing$ as claimed. We have shown that F is equidimensionally stratified.

It remains to be shown that F is cofiltered. We must thus establish the inclusion

$$F^{-1}(AG(X)^k) \subset AG(Y)^k$$

for every $k \geq 0$. Given any element $w \in F^{-1}(AG(X)^k)$, let $q \geq 0$ be the unique index with $w \in AG(Y)^q - AG(Y)^{q+1}$. Then, as F is already known to be equidimensionally stratified, we have $F(w) \in AG(X)^q - AG(X)^{q+1}$. Hence $AG(X)^k \cap (AG(X)^q - AG(X)^{q+1}) \neq \varnothing$, which can only happen when $k \leq q$. It follows that $w \in AG(Y)^q \subset AG(Y)^k$, as was to be shown. □

The following lemma will be of use in the context of a group G acting on a filtered polyhedron $(Y, \mathcal{Y})$ by placid homeomorphisms $g : (Y, \mathcal{Y}) \to (Y, \mathcal{Y})$.

Lemma 3.5. *Let $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ be filtered polyhedra of equal dimension $\dim X = \dim Y$. Let $g : (Y,\mathcal{Y}) \to (X,\mathcal{X})$ be an equidimensionally stratified cofiltered map. If g is injective, then the induced morphism $AG(g) : AG(Y) \to AG(X)$ is equidimensionally stratified and cofiltered (with respect to the Lawson filtrations).*

Proof. Let $f : Y \to AG(X)$ be the composition of g with the canonical inclusion $i : X \hookrightarrow AG(X)$. In view of the commutative square

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & X \\ \big\downarrow & \overset{f}{\searrow} & \big\downarrow{\scriptstyle i} \\ AG(Y) & \xrightarrow[AG(g)=F]{} & AG(X), \end{array}$$

Lemma 3.4 will imply that $AG(g) = F$ is equidimensionally stratified and cofiltered once we have shown that f is separated, equidimensionallly stratified and cofiltered. We shall now check these properties, beginning with the separation. Let $y, y' \in Y$ be two points, $y \neq y'$. The support of $f(y)$ is given by the one-point set $\{g(y)\}$. The injectivity of g thus implies

$$\mathrm{supp}(f(y)) \cap \mathrm{supp}(f(y')) = \{g(y)\} \cap \{g(y')\} = \varnothing.$$

This shows that f is separated. By Lemma 3.2, the canonical embedding i is equidimensionally stratified and cofiltered. Since g is, by assumption, equidimensionally stratified and cofiltered, the composition $f = i \circ g$ is equidimensionally stratified and cofiltered (Lemmas 2.7, 2.11). □

Remark 3.6. The injectivity condition in Lemma 3.5 is necessary. For an equidimensionally stratified and cofiltered map g which is not injective, the induced map $AG(g)$ will not generally be equidimensionally stratified. Consider for example the collapse map $g : Y = [0,1] \to S^1 = X$ from the unit interval to the circle with $g(0) = g(1)$. Both source and domain are to have formal dimension 1, and $Y_0 = \{0,1\}$, $X_0 = \{g(0) = g(1)\}$. Then g is equidimensionally stratified and cofiltered. Consider the reduced word $w = y_0 - y_1 \in AG(Y)$, where $y_0 = 0 \in Y$, $y_1 = 1 \in Y$. Then $w \in AG(Y)^1$, but $AG(g)(w) = g(y_1) - g(y_2) = 0$ has empty support and $AG(g)(w) \in AG(X)^0 - AG(X)^1$.

The property of being cofiltered is preserved by $AG(-)$, as was already observed by Gajer.

Lemma 3.7. *(See Gajer* [25, p. 961]*.) Let $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ be filtered polyhedra. If $g : (Y,\mathcal{Y}) \to (X,\mathcal{X})$ is cofiltered, then the induced morphism $AG(g) : AG(Y) \to AG(X)$ is cofiltered (with respect to the Lawson filtrations).*

4. Intersection Simplicial Sets and Intersection Homotopy Groups

Simplicial sets (and more generally simplicial objects) will be denoted by bold face letters such as $\mathbf{S}$. Let $\mathbf{S}(X)$ denote the singular simplicial set of a topological space X. It satisfies the Kan extension condition and if x_0 is a base point in X, then

$$\pi_n(\mathbf{S}(X), x_0) = \pi_n(X, x_0), \tag{4}$$

where x_0 also denotes the simplicial subset of $\mathbf{S}(X)$ generated by $x_0 : \Delta^0 \to X$.

Proposition 4.1. *If $X = G$ is a topological group, then $\mathbf{S}(G)$ is a simplicial group. If G is abelian, then $\mathbf{S}(G)$ is an abelian simplicial group. If $f : G \to H$ is a continuous homomorphism of topological groups, then the induced simplicial map $\mathbf{S}(f) : \mathbf{S}(G) \to \mathbf{S}(H)$ is a simplicial homomorphism of simplicial groups.*

Proof. We must endow every $\mathbf{S}(G)_n$ with a group structure such that the face maps $d_i : \mathbf{S}(G)_n \to \mathbf{S}(G)_{n-1}$ and the degeneracy maps $s_i : \mathbf{S}(G)_n \to \mathbf{S}(G)_{n+1}$ are group homomorphisms. The set $\mathbf{S}(G)_n$ is the set of all continuous maps $\Delta^n \to G$. Given two such maps σ, τ, we define their product $\sigma \cdot \tau$ pointwise by $(\sigma \cdot \tau)(t) := \sigma(t)\tau(t)$ for $t \in \Delta^n$, using the group operation on G. It is then straightforward to verify the required properties. □

A subspace A of a polyhedron P is said to have *polyhedral dimension* less than or equal to k if A is contained in a subpolyhedron $Q \subset P$ with $\dim Q \leq k$. In this case, one writes $\dim A \leq k$. An important property of polyhedral dimension is its monotonicity, that is, if $B \subset A \subset P$, then $\dim B \leq \dim A$ in the sense that for any k with $\dim A \leq k$, one has $\dim B \leq k$. Let $(X, \mathcal{X})$ be a filtered space and let $\bar{p}$ be a (Goresky-MacPherson) perversity. A continuous map $f : P \to X$ is called *$\bar{p}$-allowable* (with respect to $\mathcal{X}$) if for every skeleton X^s of $\mathcal{X}$ the polyhedral dimension of the preimage of the skeleton satisfies

$$\dim f^{-1}(X^s) \leq \dim P - s + \bar{p}(s).$$

Here, the empty set is deemed to have dimension $-\infty$. This definition is used by Gajer, [25, p. 943]. If f is $\bar{p}$-allowable, then it is $\bar{p}$-allowable in the sense of Chataur et al. [17, Def. 2.12, p. 9]. Let $(M, \partial M)$ be a PL manifold. A continuous map $f : M \to X$ is called a *$\bar{p}$-map* (with respect to $\mathcal{X}$), if both f and its restriction to the boundary ∂M are $\bar{p}$-allowable. Thus, if ∂M is empty, then a $\bar{p}$-map is the same thing as a $\bar{p}$-allowable map. A *$\bar{p}$-homotopy* is a $\bar{p}$-map $F : M \times I \to X$. A singular simplex $\sigma : \Delta^i \to X$ is called *$\bar{p}$-full* (Chataur-Saralegi-Tanré [17, p. 10, Def. 3.1]) if σ and all of its faces $d_{j_1} \cdots d_{j_k}(\sigma)$ are $\bar{p}$-allowable. If σ is $\bar{p}$-full, then every degenerate simplex $s_{j_1} \cdots s_{j_k}(\sigma)$ is $\bar{p}$-full, see Gajer [25, p. 945]. Taking $\mathbf{IS}^{\bar{p}}(X, \mathcal{X})_i \subset \mathbf{S}(X)_i$ to be the set of all $\bar{p}$-full singular simplices $\sigma : \Delta^i \to X$, one thus obtains a sub-simplicial set

$$\mathbf{IS}^{\bar{p}}(X, \mathcal{X}) \subset \mathbf{S}(X)$$

of the singular simplicial set of X, the *intersection singular simplicial set* of $(X, \mathcal{X})$. If the perversity $\bar{p}$ is understood, we will often write $\mathbf{IS}(X, \mathcal{X})$ for $\mathbf{IS}^{\bar{p}}(X, \mathcal{X})$, and if the filtration $\mathcal{X}$ on X is understood we will tend to write $\mathbf{IS}(X)$ for $\mathbf{IS}(X, \mathcal{X})$. (In [17], $\mathbf{IS}(X)$ is called the *Gajer $\bar{p}$-space*.) The intersection singular simplicial set $\mathbf{IS}(X)$ satisfies the Kan condition.

If the underlying topological space of a topological group $G = X$ is equipped with a filtration $\mathcal{G} = \mathcal{X}$ of formal dimension n, and the neutral element $1 \in G$ is to serve as a base point (which is typically the case), then 1 must be in $G - G_{n-1}$. Thus every subgroup of G must intersect $G - G_{n-1}$ nontrivially.

Definition 4.2. Let $G = X$ be a topological group. A filtration $\mathcal{G} = \mathcal{X}$ of the underlying topological space of G is called *compatible with the group structure* if $G - G_i$ is an algebraic subgroup of G for every $i < n$, where n is the formal dimension of $\mathcal{G}$.

The examples relevant for us are given by the Lawson filtration of the free abelian topological group on a polyhedron:

Lemma 4.3. *Let $(Y, \mathcal{Y})$ be a filtered polyhedron. Then the Lawson filtration $AG(\mathcal{Y})$ on the free abelian topological group $AG(Y)$ is compatible with the group structure.*

Proof. The proof is straightforward and mainly rests on the fact that $\operatorname{supp}(\phi - \psi) \subset \operatorname{supp}(\phi) \cup \operatorname{supp}(\psi)$ for $\phi, \psi \in AG(Y)$. □

Proposition 4.4. *Let $G = X$ be a topological group whose underlying topological space is endowed with a filtration $\mathcal{G} = \mathcal{X}$. If $\mathcal{G}$ is compatible with the group structure, then the sub-simplicial set $\mathbf{IS}^{\bar{p}}(G, \mathcal{G}) \subset \mathbf{S}(G)$ is a simplicial subgroup of the simplicial group $\mathbf{S}(G)$.*

Proof. Let $\sigma_1 : \Delta^k \to G$ be the unit of $\mathbf{S}(G)_k$, i.e. the constant map $\sigma_1(t) = 1$. We claim that σ_1 is $\bar{p}$-full. Since the faces of σ_1 are again units, the claim follows once we have shown that σ_1 is $\bar{p}$-allowable, that is, $\dim \sigma_1^{-1}(G^s) \leq k - s + \bar{p}(s)$ for all $s \geq 0$. If $s = 0$, then $G^0 = G$ and $\sigma_1^{-1}(G) = \Delta^k$. Therefore, $\dim \sigma_1^{-1}(G) = k = k - 0 + \bar{p}(0)$, as required. If $s > 0$, then $1 \notin G^s$, since $1 \in G^0 - G^1$. Hence in this case, $\sigma_1^{-1}(G^s) = \varnothing$ and $\dim \sigma_1^{-1}(G^s) = \dim \varnothing = -\infty \leq k - s + \bar{p}(s)$, as required. We have shown that the unit of $\mathbf{S}(G)_k$ is in $\mathbf{IS}^{\bar{p}}(G, \mathcal{G})$.

Let $\sigma, \tau \in \mathbf{IS}^{\bar{p}}(G, \mathcal{G})_k$ be elements. We have to show that their product $\sigma \cdot \tau$, taken in the group $\mathbf{S}(G)_k$ as explained in Proposition 4.1, actually lies in $\mathbf{IS}^{\bar{p}}(G, \mathcal{G})_k$. The singular simplices $\sigma, \tau : \Delta^k \to G$ are $\bar{p}$-full, that is, they are $\bar{p}$-allowable and all of their faces are also $\bar{p}$-allowable. Thus for all $s \geq 0$, we have $\max(\dim \sigma^{-1}(G^s), \dim \tau^{-1}(G^s)) \leq k - s + \bar{p}(s)$. We claim that

$$(\sigma \cdot \tau)^{-1}(G^s) \subset \sigma^{-1}(G^s) \cup \tau^{-1}(G^s). \tag{5}$$

Indeed, suppose that $t \in \Delta^k$ is a point with $\sigma(t)\tau(t) = (\sigma \cdot \tau)(t) \in G^s$. If both $\sigma(t)$ and $\tau(t)$ were in $G - G^s$, then their product would be in $G - G^s$, since the latter is a subgroup by compatibility of $\mathcal{G}$. Thus at least one of $\sigma(t), \tau(t)$ must be in G^s. This proves the claim (5). The polyhedral dimension is monotone and satisfies $\dim(A_1 \cup A_2) = \max(\dim A_1,\ \dim A_2)$, see also [17, p. 9]. Thus by (5),

$$\begin{aligned}\dim(\sigma \cdot \tau)^{-1}(G^s) &\leq \dim(\sigma^{-1}(G^s) \cup \tau^{-1}(G^s)) \\ &= \max(\dim \sigma^{-1}(G^s),\ \dim \tau^{-1}(G^s)) \leq k - s + \bar{p}(s).\end{aligned}$$

This shows that $\sigma \cdot \tau : \Delta^k \to G$ is $\bar{p}$-allowable. The $\bar{p}$-allowability of its faces $d_{j_1} \cdots d_{j_m}(\sigma \cdot \tau)$ follows similarly, using that the simplicial face maps d_j in $\mathbf{S}(G)$ are group homomorphisms. This shows that $\sigma \cdot \tau$ is $\bar{p}$-full, and thus an element of $\mathbf{IS}^{\bar{p}}(G, \mathcal{G}))_k$. Finally, one verifies similarly that $\mathbf{IS}^{\bar{p}}(G, \mathcal{G}))_k$ is closed under taking inverses. □

The next statement is then an immediate consequence of Lemma 4.3 and Proposition 4.4.

Proposition 4.5. *Let Y be a polyhedron triangulated by a countable simplicial complex and let $\mathcal{Y}$ be a filtration on Y. Then the sub-simplicial set $\mathbf{IS}^{\bar{p}}(AG(Y), AG(\mathcal{Y})) \subset \mathbf{S}(AG(Y))$ is a simplicial abelian subgroup of the simplicial abelian group $\mathbf{S}(AG(Y))$.*

Proposition 4.6. *A cofiltered map $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ between filtered spaces induces a simplicial map $\mathbf{IS}^{\bar{p}}(f) : \mathbf{IS}^{\bar{p}}(X, \mathcal{X}) \to \mathbf{IS}^{\bar{p}}(Y, \mathcal{Y})$. If X and Y are topological groups, and f is in addition a group homomorphism, then $\mathbf{IS}^{\bar{p}}(f)$ is a simplicial homomorphism of simplicial groups.*

Proof. The first statement has been observed by Gajer [25, p. 946]; see also Chataur et al. [17, p. 11, Prop. 3.5]. If $\sigma : \Delta^i \to X$ is a $\bar{p}$-full singular simplex, one takes $\mathbf{IS}(f)$ to be the restriction of $\mathbf{S}(f)$, that is, $\mathbf{IS}(f)(\sigma) = f \circ \sigma : \Delta^i \to Y$. This works, as the composition $f \circ \sigma$ is again $\bar{p}$-full in $(Y, \mathcal{Y})$ if f is cofiltered. The map $\mathbf{IS}(f)$ is simplicial, since it is the restriction of $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$, which is simplicial. If X and Y are topological groups and $f : X \to Y$ is a continuous homomorphism, then $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ is a homomorphism of simplicial groups by Proposition 4.1. The map $\mathbf{IS}(f)$ is the restriction of $\mathbf{S}(f)$ to $\bar{p}$-full simplices, and thus also a group homomorphism. Here, both $\mathbf{IS}(X)$ and $\mathbf{IS}(Y)$ are simplicial subgroups by Proposition 4.4. □

We recall that base points x_0 of pointed filtered spaces $(X, \mathcal{X}, x_0)$ are required to be in the top stratum, $x_0 \in X^0 - X^1$. The *intersection homotopy groups* of a pointed filtered space

$(X,\mathfrak{X},x_0)$ associated to $\bar{p}$ are given by the simplicial homotopy groups of the simplicial set $\mathbf{IS}^{\bar{p}}(X,\mathfrak{X})$,

$$I^{\bar{p}}\pi_k(X,\mathfrak{X},x_0) := \pi_k(\mathbf{IS}^{\bar{p}}(X,\mathfrak{X}),x_0),$$

see [25, p. 946] and [17, p. 16, Def. 4.1]. This may be viewed as an intersection analog to (4). Since $\mathbf{IS}(X)$ satisfies the Kan condition, the elements of intersection homotopy groups are represented by $\bar{p}$-full singular simplices in X whose boundary maps to x_0. (The homology of $\mathbf{IS}^{\bar{p}}(X,\mathfrak{X})$ will never be considered in this paper, nor will we ever use the simplicial object $AG(\mathbf{IS}^{\bar{p}}(X,\mathfrak{X}))$.)

Lemma 4.7. *Let $f:(X,\mathfrak{X},x_0)\to(Y,\mathfrak{Y},y_0)$ be a cofiltered pointed map between filtered pointed spaces. Then f induces a group homomorphism $f_*: I^{\bar{p}}\pi_*(X,\mathfrak{X},x_0)\to I^{\bar{p}}\pi_*(Y,\mathfrak{Y},y_0)$ for every $\bar{p}$.*

Proof. By Proposition 4.6, f induces a simplicial map $\mathbf{IS}^{\bar{p}}(f):\mathbf{IS}^{\bar{p}}(X,\mathfrak{X})\to\mathbf{IS}^{\bar{p}}(Y,\mathfrak{Y})$. This map induces the desired homomorphism f_* on simplicial homotopy groups. □

Let $(Y,\mathfrak{Y})$ be a filtered polyhedron. The filtration $\mathfrak{Y}$ is called a *PL stratification* if all Y_i are subpolyhedra, the strata are PL manifolds, and $\mathfrak{Y}$ is PL locally cone-like. Examples are the skeletal filtration of a PL triangulation. Gajer's intersection Dold-Thom theorem [25], corrected in [26], provides natural isomorphisms

$$IH_*^{\bar{p}}(Y;\mathbb{Z})\cong I^{\bar{p}}\pi_*(AG(Y),AG(\mathfrak{Y}))$$

for connected PL stratified polyhedra Y.

With a view towards the up-down and down-up properties (Propositions 5.8 and 5.9), we establish the additivity of $\pi_*\mathbf{IS}^{\bar{p}}$. We will focus on endomorphisms, though many of the statements are more generally true for homomorphisms. Let $\mathbf{A}$ be a simplicial abelian group. The set $\mathrm{End}(\mathbf{A})$ of endomorphisms of $\mathbf{A}$ is an abelian group by defining the sum $f+g$ of $f,g\in\mathrm{End}(\mathbf{A})$ to be $(f+g)_n:\mathbf{A}_n\to\mathbf{A}_n$, $(f+g)_n(a)=f_n(a)+g_n(a)$, $a\in\mathbf{A}_n$, using the group law in $\mathbf{A}_n$.

Lemma 4.8. *Given endomorphisms $f,g\in\mathrm{End}(\mathbf{A})$, additivity $\pi_*(f+g)=\pi_*(f)+\pi_*(g)$ holds in $\mathrm{End}(\pi_*(\mathbf{A}))$.*

This follows e.g. from expressing the homotopy group via the Dold-Kan correspondence as the homology group of the Moore chain complex and using the fact that both the Moore and the homology functor are additive. Let $F,G:A\to A'$ be continuous homomorphisms of abelian topological groups. Their sum $F+G:A\to A'$, given by the composition

$$A\xrightarrow{(F,G)}A'\times A'\xrightarrow{+}A',$$

is again a continuous homomorphism. In this way, the set $\mathrm{Hom}(A,A')$ of continuous homomorphisms is an abelian group. In particular, $\mathrm{End}(A)$ is an abelian group.

Lemma 4.9. *Let A be an abelian topological group, for example $A=AG(Y)$. Given $F,G\in$ $\mathrm{End}(A)$, the additive compatibility $\mathbf{S}(F+G)=\mathbf{S}(F)+\mathbf{S}(G)$ holds in $\mathrm{End}(\mathbf{S}(A))$.*

This is a straightforward verification using the various (pointwise) addition laws recalled above, in particular using Proposition 4.1.

Lemma 4.10. *Let $f,g:AG(Y)\to AG(Y)$ be continuous maps. If f and g are cofiltered with respect to the Lawson filtration, then the continuous map $f+g:AG(Y)\to AG(Y)$ given by $(f+g)(\phi)=f(\phi)+g(\phi)$ is also cofiltered. The zero endomorphism $0:AG(Y)\to AG(Y)$ is placid.*

Proof. As f and g are cofiltered, we know that $f^{-1}(AG(Y)^k) \subset AG(Y)^k$, $g^{-1}(AG(Y)^k) \subset AG(Y)^k$. We have to show that $(f+g)^{-1}(AG(Y)^k) \subset AG(Y)^k$. Let $\phi \in AG(Y)$ be an element with $(f+g)(\phi) \in AG(Y)^k$. By definition of the Lawson filtration, membership in $AG(Y)^k$ means that $\operatorname{supp}((f+g)(\phi)) \cap Y^k \neq \varnothing$. Consequently, there is some point $y_0 \in Y^k$ with $f(\phi)(y_0) + g(\phi)(y_0) \neq 0$. Hence $f(\phi)(y_0) \neq 0$ or $g(\phi)(y_0) \neq 0$. If $f(\phi)(y_0) \neq 0$, then $\phi \in f^{-1}(AG(Y)^k) \subset AG(Y)^k$. Similarly, ϕ is also in $AG(Y)^k$ when $g(\phi)(y_0) \neq 0$.

To show that the zero endomorphism 0 on $AG(Y)$ is placid, let us first verify that it is cofiltered. We must show that $0^{-1}(AG(Y)^k) \subset AG(Y)^k$ for all $k \geq 0$. If $k = 0$, then $0 \in AG(Y)^0$ and thus $0^{-1}(AG(Y)^0) = AG(Y) = AG(Y)^0$. If $k > 0$, then $0 \notin AG(Y)^k$ and thus

$$0^{-1}(AG(Y)^k) = \varnothing \subset AG(Y)^k.$$

Thus 0 is cofiltered. Let T be the connected component of 0 in $AG(Y)_m - AG(Y)_{m-1}$, where m is the formal dimension of the filtered polyhedron Y. Then for every stratum S of $AG(Y)$, the image $0(S)$ is contained in T. This shows that 0 is a stratified map. By Lemma 2.12, 0 is placid. □

Remark 4.11. If $f, g : AG(Y) \to AG(Y)$ are stratified with respect to the Lawson filtration, then it is not generally true that $f + g$ is again stratified.

Proposition 4.12. *Let $F, G \in \operatorname{End}(AG(Y))$ be endomorphisms. If F and G are cofiltered with respect to the Lawson filtration, then*

$$\pi_* \mathbf{IS}^{\bar{p}}(F+G) = \pi_* \mathbf{IS}^{\bar{p}}(F) + \pi_* \mathbf{IS}^{\bar{p}}(G) \in \operatorname{End}(\pi_* \mathbf{IS}^{\bar{p}}(AG(Y))).$$

Under the intersection Dold-Thom identification $IH_*^{\bar{p}}(Y;\mathbb{Z}) = I^{\bar{p}}\pi_*(AG(Y)) = \pi_*(\mathbf{IS}^{\bar{p}}(AG(Y)))$, where we take Y to be connected, we thus have $(F+G)_* = F_* + G_* \in \operatorname{End}(IH_*^{\bar{p}}(Y;\mathbb{Z}))$.

Proof. We will write $\mathbf{IS} = \mathbf{IS}^{\bar{p}}$. The sum $F + G$ is cofiltered by Lemma 4.10. Since F, G and $F + G$ are cofiltered, the endomorphisms $\mathbf{IS}(F)$, $\mathbf{IS}(G)$, $\mathbf{IS}(F+G) \in \operatorname{End}(\mathbf{IS}(AG(Y)))$ are well-defined by Proposition 4.6. The simplicial abelian group $\mathbf{IS}(AG(Y))$ is a simplicial subgroup of the simplicial abelian group $\mathbf{S}(AG(Y))$ by Proposition 4.5. Thus there is a commutative diagram

$$\begin{array}{ccc} \mathbf{IS}(AG(Y)) & \xrightarrow{\mathbf{IS}(F)} & \mathbf{IS}(AG(Y)) \\ \big\downarrow & & \big\downarrow \\ \mathbf{S}(AG(Y)) & \xrightarrow[\mathbf{S}(F)]{} & \mathbf{S}(AG(Y)), \end{array}$$

i.e. $\mathbf{IS}(F)$ is the restriction of $\mathbf{S}(F)$ to $\mathbf{IS}(AG(Y))$. There are similar diagrams for $\mathbf{IS}(G), \mathbf{S}(G)$ and for $\mathbf{IS}(F+G), \mathbf{S}(F+G)$. By Lemma 4.9, $\mathbf{S}(F) + \mathbf{S}(G) = \mathbf{S}(F+G)$. Restricting this identity to $\mathbf{IS}(AG(Y))$, we obtain $\mathbf{IS}(F) + \mathbf{IS}(G) = \mathbf{IS}(F+G)$. Taking $\mathbf{A} = \mathbf{IS}(AG(Y))$, $f = \mathbf{IS}(F)$ and $g = \mathbf{IS}(G)$ in Lemma 4.8, it follows that

$$\pi_*(\mathbf{IS}(F+G)) = \pi_*(\mathbf{IS}(F) + \mathbf{IS}(G)) = \pi_*(\mathbf{IS}(F)) + \pi_*(\mathbf{IS}(G)).$$

□

This applies for example to multiplication by an integer, which is relevant for the up-down principle:

Corollary 4.13. *Let d be an integer and let $(Y, \mathfrak{Y})$ be a connected (for intersection Dold-Thom) filtered polyhedron. Let $h_d : IH_j^{\bar{p}}(Y;\mathbb{Z}) \to IH_j^{\bar{p}}(Y;\mathbb{Z})$ and $a_d : AG(Y) \to AG(Y)$ denote* *the group endomorphisms given by multiplication by d. Then, for any $\bar{p}$, under the intersection Dold-Thom identification $IH_j^{\bar{p}}(Y;\mathbb{Z}) = I^{\bar{p}}\pi_j(AG(Y)) = \pi_j(\mathbf{IS}^{\bar{p}}(AG(Y)))$, the placid map a_d induces the endomorphism h_d.*

Here, we used the following basic observation:

Lemma 4.14. *The map $a_d : (AG(Y), AG(\mathcal{Y})) \longrightarrow (AG(Y), AG(\mathcal{Y}))$ is placid.*

Proof. If $d = 0$, then $a_0 = 0$ is placid by Lemma 4.10. If $d \neq 0$, the key point is that a_d does not change supports. □

5. Ramified Covers and Intersection Homology Transfer

A finite ramified covering $\pi : \widetilde{Y} \to Y$ has an associated transfer homomorphism

$$\pi_! : H_*(Y;\mathbb{Z}) \to H_*(\widetilde{Y};\mathbb{Z})$$

such that

$$\pi_* \circ \pi_! : H_*(Y;\mathbb{Z}) \longrightarrow H_*(Y;\mathbb{Z}) \text{ is multiplication by } d, \tag{6}$$

where d is the degree of π and $\pi_* : H_*(\widetilde{Y};\mathbb{Z}) \longrightarrow H_*(Y;\mathbb{Z})$ the covariantly induced homomorphism. In this section, we will show, using the free abelian topological groups generated by $Y, \widetilde{Y}$ in a simplicial setting, that this transfer lifts to intersection homology $IH_*^{\bar{p}}$. There is thus a transfer

$$\pi_! : IH_*^{\bar{p}}(Y;\mathbb{Z}) \longrightarrow IH_*^{\bar{p}}(\widetilde{Y};\mathbb{Z})$$

for every $\bar{p}$, when π is simplicial and $Y, \widetilde{Y}$ PL stratified polyhedra. This transfer will be shown (see Proposition 5.8) to satisfy the above property (6) on intersection homology. Furthermore, it is compatible with the transfer on ordinary homology, that is,

$$\begin{array}{ccc} IH_*^{\bar{p}}(Y;\mathbb{Z}) & \xrightarrow{\pi_!} & IH_*^{\bar{p}}(\widetilde{Y};\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_*(Y;\mathbb{Z}) & \xrightarrow[\pi_!]{} & H_*(\widetilde{Y};\mathbb{Z}) \end{array} \tag{7}$$

commutes, where the vertical maps are the canonical ones.

We begin by recalling the notion of a finite ramified covering of a topological space Y in the elegant formulation of L. Smith, [43].

Definition 5.1. Let d be a positive integer. A *d-fold ramified covering* is a pair (π, μ), where

- $\pi : \widetilde{Y} \to Y$ is a continuous surjective finite-to-one map, and
- $\mu : \widetilde{Y} \to \{1, 2, 3, \ldots\}$ is a map, called the *multiplicity map*,

such that

(1) $\sum_{x \in \pi^{-1}(y)} \mu(x) = d$ for every $y \in Y$, and
(2) the map
$$\tau_\pi : Y \longrightarrow SP^d(\widetilde{Y})$$
defined by sending y into $\pi^{-1}(y)$, where each $x \in \pi^{-1}(y)$ occurs $\mu(x)$ times, is continuous.

We will often write ramified coverings briefly as $\pi : \widetilde{Y} \to Y$, omitting the multiplicity map from the notation.

Example 5.2. Let G be a finite group and X a G-space. Then the orbit projection $\pi : X \to X/G$ admits the structure of a $|G|$-fold ramified covering ([43, Prop. 1.5]).

Ramified coverings can be pulled back under arbitrary continuous maps: The total space is the topological fiber product and the multiplicity map on the fiber product is taken to be μ of the projection to the component $\widetilde{Y}$ ([43, p. 488, Prop. 1.3]). The degree of the pullback is again d. In particular, ramified coverings can be restricted to subspaces of their target. Let ΣX denote the unreduced suspension of a space X.

Lemma 5.3. *Let $\pi : \widetilde{Y} \to Y$ be a ramified covering of degree d with multiplicity map μ. Then the suspension $\Sigma\pi : \Sigma\widetilde{Y} \to \Sigma Y$ is a ramified covering of degree d with multiplicity map μ_Σ given by $\mu_\Sigma(t,x) = \mu(x)$ for $(t,x) \in (-1,1) \times \widetilde{Y}$ and $\mu_\Sigma(\pm 1) = d$, where $\pm 1 \in \Sigma\widetilde{Y}$ are the two suspension points.*

Proof. The map $\Sigma\pi$, which is given by $(\Sigma\pi)(t,x) = (t,\pi(x))$ and $(\Sigma\pi)(\pm 1) = \pm 1$, is surjective and finite-to-one, by the corresponding properties of π. By definition of the multiplicity map μ_Σ, the summation property (1) in Definition 5.1 holds, noting that the fibers of $\Sigma\pi$ are given by $(\Sigma\pi)^{-1}(t,y) = \{t\} \times \pi^{-1}(y)$, $(t,y) \in (-1,1) \times Y$, and $(\Sigma\pi)^{-1}(+1) = \{+1\}$, $(\Sigma\pi)^{-1}(-1) = \{-1\}$. It remains to verify property (2), i.e. that the map $\tau_{\Sigma\pi} : \Sigma Y \to SP^d(\Sigma\widetilde{Y})$ determined by $\Sigma\pi$ and μ_Σ is continuous. Consider the cartesian diagram

$$\begin{array}{ccc} I \times \widetilde{Y} & \xrightarrow{\text{proj}} & \widetilde{Y} \\ {\scriptstyle \overline{\pi}=\mathrm{id}\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ I \times Y & \xrightarrow{\text{proj}} & Y. \end{array}$$

We recalled above that ramified covers can be pulled back. Thus $\overline{\pi}$ is a ramified covering, whose multiplicity function $\overline{\mu} : I \times \widetilde{Y} \to \{1,2,\ldots\}$ is given by $\overline{\mu}(t,x) = \mu(x)$. Therefore, we know the map $\tau_{\overline{\pi}} : I \times Y \longrightarrow SP^d(I \times \widetilde{Y})$, given by

$$\tau_{\overline{\pi}}(t,y) = \sum_{(t,x)\in\overline{\pi}^{-1}(t,y)} \overline{\mu}(t,x)\cdot(t,x) = \sum_{x\in\pi^{-1}(y)} \mu(x)\cdot(t,x),$$

to be continuous. Let $q : I \times Y \longrightarrow \Sigma Y$, $\widetilde{q} : I \times \widetilde{Y} \longrightarrow \Sigma\widetilde{Y}$ denote the quotient maps. The latter quotient map induces a continuous map $SP^d(\widetilde{q}) : SP^d(I \times \widetilde{Y}) \longrightarrow SP^d(\Sigma\widetilde{Y})$. One checks readily that the diagram

$$\begin{array}{ccc} I \times Y & \xrightarrow{\tau_{\overline{\pi}}} & SP^d(I \times \widetilde{Y}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle SP^d(\widetilde{q})} \\ \Sigma Y & \xrightarrow[\tau_{\Sigma\pi}]{} & SP^d(\Sigma\widetilde{Y}) \end{array}$$

commutes. Now by definition of the quotient topology on ΣY, the map $\tau_{\Sigma\pi}$ is continuous if and only if $\tau_{\Sigma\pi} \circ q$ is continuous. By the commutativity of the above diagram, $\tau_{\Sigma\pi} \circ q = SP^d(\widetilde{q}) \circ \tau_{\overline{\pi}}$ and the latter is continuous as the composition of two continuous maps. □

For later use, we record the standard fact that simplicial maps are cellular with respect to the skeletal filtrations. This is a consequence of the simplex-wise linearity of such a map.

Lemma 5.4. *(Cellularity of simplicial maps.) Let X,Y be triangulated polyhedra of the same dimension, equipped with the simplicial filtrations $\mathcal{X},\mathcal{Y}$. If $f : X \to Y$ is a simplicial map, then $f(X^k) \subset Y^k$ for all $k \geq 0$.*

Let $\widetilde{Y}$ and Y be spaces triangulated by countable simplicial complexes and let $\pi : \widetilde{Y} \to Y$ be a simplicial d-fold ramified cover. Via the topological embedding $SP^d(\widetilde{Y}) \hookrightarrow AG(\widetilde{Y})$, we may regard the map τ_π as a continuous map

$$\tau_\pi : Y \longrightarrow AG(\widetilde{Y}),$$

given by $\tau_\pi(y) = \sum_{x\in\pi^{-1}(y)} \mu(x)\cdot x$. Since $\mu(x) \geq 1$ for every $x \in \pi^{-1}(y)$, the support is the fiber of π, $\operatorname{supp}(\tau_\pi(y)) = \pi^{-1}(y)$. The map τ_π is separated in the sense of Definition 3.3, since the fibers over different points are disjoint.

Lemma 5.5. *Let $Y,\widetilde{Y}$ be finite-dimensional triangulated polyhedra. Let $\pi : \widetilde{Y} \to Y$ be a simplicial ramified covering. If $\mathcal{Y},\widetilde{\mathcal{Y}}$ denote the filtrations of $Y,\widetilde{Y}$ given by the simplicial skeleta, then the map*

$$\tau_\pi : (Y,\mathcal{Y}) \to (AG(\widetilde{Y}),AG(\widetilde{\mathcal{Y}}))$$

is equidimensionally stratified and cofiltered.

Proof. We begin by showing that τ_π is equidimensionally stratified. Since π is finite-to-one, surjective and simplicial, the proof of Lemma 2.14 applies to show that $\dim(Y) = \dim(\widetilde{Y}) =: n$. Given $k \geq 0$, we must show that $\tau_\pi(Y_{n-k} - Y_{n-k-1}) \subset AG(\widetilde{Y})_{n-k} - AG(\widetilde{Y})_{n-k-1}$. Let

$$S = \Delta^\circ \subset Y_{n-k} - Y_{n-k-1} = Y^k - Y^{k+1}$$

be a stratum of $\mathcal{Y}$, $\dim\Delta = n-k$. Let $y \in S$ be a point. Since π is simplicial, the preimage of the open simplex has the form $\pi^{-1}(\Delta^\circ) \cong \Delta^\circ \times \pi^{-1}(y)$, see e.g. Milnor-Stasheff [37, p. 236, Lemma 20.5]. The set $\pi^{-1}(y)$ is finite, as π is finite-to-one. Thus $\pi^{-1}(\Delta^\circ) = \Delta_1^{n-k,\circ} \sqcup \cdots \sqcup \Delta_j^{n-k,\circ}$ is a disjoint union of the interiors of $(n-k)$-dimensional simplices in $\widetilde{Y}$. We must show that $\tau_\pi(y) \in AG(\widetilde{Y})^k - AG(\widetilde{Y})^{k+1}$. By definition of the Lawson filtration, this requires us to verify $\operatorname{supp}(\tau_\pi(y)) \cap \widetilde{Y}^k \neq \varnothing$ and $\operatorname{supp}(\tau_\pi(y)) \cap \widetilde{Y}^{k+1} = \varnothing$. The support is given by the fiber

$$\operatorname{supp}(\tau_\pi(y)) = \pi^{-1}(y), \tag{8}$$

which is nonempty, since π is surjective. This nonempty set is contained in

$$\pi^{-1}(\Delta^\circ) = \Delta_1^{n-k,\circ} \sqcup \cdots \sqcup \Delta_j^{n-k,\circ} \subset \widetilde{Y}^k - \widetilde{Y}^{k+1}.$$

This shows that τ_π is equidimensionally stratified.

It remains to be shown that τ_π is cofiltered. We need to verify that $\tau_\pi^{-1}(AG(\widetilde{Y})^k) \subset Y^k$ for every $k \geq 0$. Thus let $y \in Y$ be a point such that $\tau_\pi(y) \in AG(\widetilde{Y})^k$. Then $\operatorname{supp}(\tau_\pi(y)) \cap \widetilde{Y}^k \neq \varnothing$. Since the support is the π-fiber (8), there exists a point $x \in \widetilde{Y}^k$ with $\pi(x) = y$. Now since π is a simplicial map between polyhedra of equal dimension, Lemma 5.4 implies that $\pi(\widetilde{Y}^k) \subset Y^k$. Therefore, $y = \pi(x) \in Y^k$, as was to be shown. We conclude that τ_π is cofiltered. □

Let Y and $\widetilde{Y}$ be finite-dimensional (countably) triangulated polyhedra. As usual, we endow these polyhedra with their simplicial filtrations $\mathcal{Y},\widetilde{\mathcal{Y}}$. Let $\pi : \widetilde{Y} \to Y$ be a simplicial ramified covering. By Lemma 5.5, $\tau_\pi : (Y,\mathcal{Y}) \to (AG(\widetilde{Y}),AG(\widetilde{\mathcal{Y}}))$ is equidimensionally stratified and cofiltered. By the universal property of Proposition 3.1 there exists a unique extension of τ_π to a continuous group homomorphism

$$\tau : AG(Y) \longrightarrow AG(\widetilde{Y}).$$

Since τ_π is separated, equidimensionally stratified and cofiltered, Lemma 3.4 implies that τ is equidimensionally stratified and cofiltered. Explicitly, this map is given by $\tau(\phi)(x) = \phi(\pi(x))\mu(x)$. The function $\tau(\phi)$ is nonzero only at finitely many points of $\widetilde{Y}$, since this is true for ϕ, and π is finite-to-one. On the characteristic function $1_y : Y \to \mathbb{Z}$ of a point $y \in Y$, τ is given by

$$\tau(1_y)(x) = \begin{cases} \mu(x), & x \in \pi^{-1}(y), \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, we may describe τ in the form

$$\sum_{y \in Y} n_y y \mapsto \sum_{y \in Y} n_y \sum_{x \in \pi^{-1}(y)} \mu(x) x.$$

According to Lemma 2.12, τ is placid. By Proposition 4.6, τ induces for every $\bar{p}$ a simplicial group homomorphism

$$\mathbf{IS}^{\bar{p}}(\tau) : \mathbf{IS}^{\bar{p}}(AG(Y), AG(\mathcal{Y})) \longrightarrow \mathbf{IS}^{\bar{p}}(AG(\widetilde{Y}), AG(\widetilde{\mathcal{Y}})).$$

On homotopy groups, this simplicial map induces a homomorphism

$$\tau_* := \pi_*(\mathbf{IS}^{\bar{p}}(\tau)) : \pi_* \mathbf{IS}^{\bar{p}}(AG(Y), AG(\mathcal{Y})) \longrightarrow \pi_* \mathbf{IS}^{\bar{p}}(AG(\widetilde{Y}), AG(\widetilde{\mathcal{Y}})).$$

In this way, we receive a homomorphism

$$\tau_* : I^{\bar{p}}\pi_*(AG(Y), AG(\mathcal{Y})) \longrightarrow I^{\bar{p}}\pi_*(AG(\widetilde{Y}), AG(\widetilde{\mathcal{Y}}))$$

between the intersection homotopy groups of Y and $\widetilde{Y}$. Assume that the polyhedra $Y, \widetilde{Y}$ are connected. Then by Gajer's intersection Dold-Thom theorem, [25] and [26], there are natural isomorphisms

$$IH_*^{\bar{p}}(Y;\mathbb{Z}) \cong I^{\bar{p}}\pi_*(AG(Y), AG(\mathcal{Y})),\ IH_*^{\bar{p}}(\widetilde{Y};\mathbb{Z}) \cong I^{\bar{p}}\pi_*(AG(\widetilde{Y}), AG(\widetilde{\mathcal{Y}})).$$

Under these isomorphisms, τ_* is a homomorphism

$$\pi_! := \tau_* : IH_*^{\bar{p}}(Y;\mathbb{Z}) \longrightarrow IH_*^{\bar{p}}(\widetilde{Y};\mathbb{Z}).$$

This is the desired *intersection homology transfer* of the ramified covering $\pi : \widetilde{Y} \to Y$. The commutativity of Diagram (7) follows from the commutativity of

$$\begin{array}{ccc} \mathbf{IS}^{\bar{p}}(AG(Y)) & \xrightarrow{\mathbf{IS}^{\bar{p}}(\tau)} & \mathbf{IS}^{\bar{p}}(AG(\widetilde{Y})) \\ \big\downarrow & & \big\downarrow \\ \mathbf{S}(AG(Y)) & \xrightarrow[\mathbf{S}(\tau)]{} & \mathbf{S}(AG(\widetilde{Y})). \end{array}$$

If Y has even dimension $2k$, then only the intersection homology in the middle degree k is relevant for the definition of G-signatures in Section 8. In degrees up to the middle, and using the lower middle perversity, the connectivity assumption on Y and $\widetilde{Y}$ is not needed, as we will explain next. If the perversity function $\bar{p}$ is taken to be the lower middle perversity $\bar{m}$, then intersection homology *up to the lower middle degree* is insensitive to suspension.

Lemma 5.6. *(1) If L^{2k} is a $2k$-dimensional pseudomanifold with (unreduced) suspension ΣL^{2k}, then $IH_i^{\bar{m}}(\Sigma L^{2k}) = IH_i^{\bar{m}}(L^{2k})$ for all $i \leq k$.*
(2) If L^{2k+1} is a $(2k+1)$-dimensional pseudomanifold, then $IH_i^{\bar{m}}(\Sigma L^{2k+1}) = IH_i^{\bar{m}}(L^{2k+1})$ for all $i \leq k$.

This is due to the fact that in the lower dimensions specified by the above bounds, allowable chains cannot touch the suspension points. The lemma has the following consequence. Suppose that Y^n is a polyhedral pseudomanifold of dimension n. Even if Y^n is not connected, its suspension ΣY^n is. Thus the intersection Dold-Thom theorem does apply to ΣY^n, yielding an isomorphism $IH_*^{\bar{p}}(\Sigma Y^n) \cong I^{\bar{p}}\pi_*(AG(\Sigma Y^n))$ for every $\bar{p}$. By Lemma 5.6,

$IH_i^{\bar{m}}(Y^n) \cong I^{\bar{m}}\pi_i(AG(\Sigma Y^n))$ for all $i \leq \lfloor \frac{n}{2} \rfloor$, even when Y is not connected. The degree bound is the same for Y and $\widetilde{Y}$, since they have the same dimension n. By Lemma 5.3, the suspension of π is a ramified covering $\Sigma\pi : \Sigma\widetilde{Y} \to \Sigma Y$. It has an associated continuous homomorphism $\tau_\Sigma : AG(\Sigma Y) \to AG(\Sigma\widetilde{Y})$. Thus for $i \leq \lfloor \frac{n}{2} \rfloor$, one obtains a transfer

$$\pi_! : IH_i^{\bar{m}}(Y;\mathbb{Z}) \cong I^{\bar{m}}\pi_i(AG(\Sigma Y)) \xrightarrow{\tau_{\Sigma *}} I^{\bar{m}}\pi_i(AG(\Sigma\widetilde{Y})) \cong IH_i^{\bar{m}}(\widetilde{Y};\mathbb{Z}).$$

If Y happens to be already connected, then this agrees with the transfer as defined previously, since the diagram

$$\begin{array}{ccc} AG(\Sigma Y) & \xrightarrow{\tau_\Sigma} & AG(\Sigma\widetilde{Y}) \\ \uparrow & & \uparrow \\ AG(Y) & \xrightarrow[\tau]{} & AG(\widetilde{Y}) \end{array}$$

commutes, where the vertical homomorphisms are induced by the inclusions $Y \hookrightarrow \Sigma Y$, $\widetilde{Y} \hookrightarrow \Sigma\widetilde{Y}$ at $t = 0$.

The following result affirms the stability of intersection homology up to the middle-degree under an *arbitrary* number of iterated suspensions. Its proof is a straightforward induction on the number of suspensions, using Lemma 5.6.

Lemma 5.7. *If L^{2k} is a $2k$-dimensional pseudomanifold, then for every $s = 0,1,2,3,\ldots$ and for every $i \leq k$,*

$$IH_i^{\bar{m}}(\Sigma^s L^{2k}) = IH_i^{\bar{m}}(L^{2k}).$$

By Lemma 2.14, a finite surjective simplicial map π, in particular a simplicial finite ramified covering $\pi : \widetilde{Y} \to Y$ (between simplicially stratified complexes), is placid and thus induces a homomorphism

$$\pi_* : IH_*^{\bar{p}}(\widetilde{Y}) \longrightarrow IH_*^{\bar{p}}(Y).$$

In particular, by Example 5.2, a simplicial finite orbit projection $\pi : X \to X/G$ induces a map $\pi_* : IH_*^{\bar{p}}(X) \to IH_*^{\bar{p}}(X/G)$. The map $\pi_* : IH_*^{\bar{p}}(\widetilde{Y}) \to IH_*^{\bar{p}}(Y)$ can be described via the intersection Dold-Thom theorem as follows: A continuous map $f : X \to Y$ between polyhedra induces a continuous homomorphism $AG(f) : AG(X) \to AG(Y)$. If f is cofiltered, then $AG(f)$ is cofiltered by Lemma 3.7. Hence in this situation, by Lemma 4.7, $AG(f)$ induces a map $f_* : I^{\bar{p}}\pi_*(AG(X)) \to I^{\bar{p}}\pi_*(AG(Y))$. Now, as a simplicial ramified covering $\pi : \widetilde{Y} \to Y$ is indeed cofiltered, $AG(\pi)$ induces $\pi_* : I^{\bar{p}}\pi_*(AG(\widetilde{Y})) \longrightarrow I^{\bar{p}}\pi_*(AG(Y))$.

Proposition 5.8. *(Up-Down.) If $\pi : \widetilde{Y} \to Y$ is a d-fold simplicial ramified covering of connected polyhedra, then*

$$\pi_* \circ \pi_! : IH_*^{\bar{p}}(Y;\mathbb{Z}) \longrightarrow IH_*^{\bar{p}}(Y;\mathbb{Z})$$

is multiplication by d. Connectivity is not required when $\bar{p} = \bar{m}$ and $* \leq \lfloor (\dim Y)/2 \rfloor$.

Proof. Let $a_d : AG(Y) \to AG(Y)$ denote multiplication by d. We claim that the diagram

$$\begin{array}{ccc} AG(Y) & \xrightarrow{\tau} & AG(\widetilde{Y}) \\ & \searrow^{a_d} & \downarrow AG(\pi) \\ & & AG(Y) \end{array}$$

commutes, i.e. that $(AG(\pi)\circ\tau)(\phi) = d\cdot\phi$. We proceed to verify this formula. On an element $\phi: Y \to \mathbb{Z}$ of $AG(Y)$, we find at a point $y \in Y$,

$$\begin{aligned}((AG(\pi)\circ\tau)(\phi))(y) &= (AG(\pi)(x \mapsto \phi(\pi(x))\mu(x)))(y)\\ &= \sum_{x\in\pi^{-1}(y)} \phi(\pi(x))\mu(x) = \sum_{x\in\pi^{-1}(y)} \phi(y)\mu(x)\\ &= \phi(y)\sum_{x\in\pi^{-1}(y)} \mu(x) = \phi(y)\cdot d,\end{aligned}$$

as claimed. Using Corollary 4.13,

$$\begin{aligned}\pi_*\circ\pi_! &= \pi_*(\mathbf{IS}^{\bar p}(AG(\pi)))\circ\pi_*(\mathbf{IS}^{\bar p}(\tau)) = \pi_*(\mathbf{IS}^{\bar p}(AG(\pi)\circ\tau))\\ &= \pi_*(\mathbf{IS}^{\bar p}(a_d)) = h_d.\end{aligned}$$

When one or both of the polyhedra $Y, \widetilde{Y}$ are not connected, we apply the above argument to the suspension ramified cover $\Sigma\pi$ and obtain that $(\Sigma\pi)_*\circ(\Sigma\pi)_!$ is multiplication by d on $IH_i^{\bar m}(\Sigma Y)$. The result follows by observing that π_* corresponds to $(\Sigma\pi)_*$ under the identifications $IH_i^{\bar m}(Y) = IH_i^{\bar m}(\Sigma Y)$, $IH_i^{\bar m}(\widetilde{Y}) = IH_i^{\bar m}(\Sigma\widetilde{Y})$, $i \le \lfloor(\dim Y)/2\rfloor$, since these are induced by the inclusions $Y \hookrightarrow \Sigma Y$, $\widetilde{Y} \hookrightarrow \Sigma\widetilde{Y}$, and these form a commutative square with π and $\Sigma\pi$. □

When rational (or real) coefficients are used, the up-down formula of Proposition 5.8 implies that $\pi_!$ is a monomorphism and π_* is an epimorphism.

The down-up property of the next proposition is an intersection homology analog of [43, Prop. 2.4, p.493]. Basic material on simplicial group actions is recalled in Section 7. For example, subanalytic proper actions admit a G-equivariant triangulation, [9, Prop. 6.7]; see Examples 7.1.

Proposition 5.9. *(Down-Up.) Let G be a finite group acting simplicially on the connected triangulated polyhedron X with orbit map $\pi: X \to X/G = Y$ structured as a $d := |G|$-fold simplicial ramified covering. Then*

$$\pi_!\circ\pi_*: IH_*^{\bar p}(X;\mathbb{Z}) \to IH_*^{\bar p}(X;\mathbb{Z})$$

is given by

$$\pi_!\circ\pi_* = \sum_{g\in G} g_*,$$

where $g_*: IH_*^{\bar p}(X) \to IH_*^{\bar p}(X)$ is the automorphism induced by $g \in G$. Connectivity is not required when $\bar p = \bar m$ and $* \le \lfloor(\dim X)/2\rfloor$.

Proof. The multiplicity map $\mu: X \to \{1,2,\ldots\}$ is given by $\mu(x) = |G_x|$, the order of the isotropy group at x. Thus, the composition

$$AG(X) \xrightarrow{AG(\pi)} AG(X/G) \xrightarrow{\tau} AG(X)$$

is given by (the linear extension of)

$$\begin{aligned}\tau((AG(\pi))(x)) = \tau(\pi(x)) &= \sum_{x'\in\pi^{-1}(\pi(x))} \mu(x')\, x' = \sum_{x'\in\pi^{-1}(\pi(x))} |G_{x'}|\, x'\\ &= \sum_{x'\in G\cdot x} |\{g\in G \mid gx = x'\}| x' = \sum_{g\in G} gx.\end{aligned}$$

In other words: For every $g \in G$, the (simplicial) homeomorphism $g: X \to X$ induces an automorphism $AG(g): AG(X) \to AG(X)$ of topological abelian groups. By functoriality of $AG(-)$, this defines an action of G on $AG(X)$. The simplicial homeomorphism $g: X \to X$ is equidimensionally stratified and cofiltered with respect to the filtration of X by the simplicial skeleta. By Lemma 3.5, $AG(g)$ is equidimensionally stratified and cofiltered with respect to the Lawson filtration. In particular, $AG(g)$ is placid for every $g \in G$ (Lemma 2.12). The sum $\sum_{g\in G} AG(g) \in \mathrm{End}(AG(X))$ is an endomorphism on the abelian topological group $AG(X)$ and we have

$$\tau \circ AG(\pi) = \sum_{g\in G} AG(g).$$

We note that the sum $\sum_g AG(g)$ is cofiltered by Lemma 4.10. One can also see this as follows: The orbit projection $\pi : X \to X/G$ is cofiltered (in fact even placid) with respect to the simplicial stratifications by Lemma 2.14. According to Lemma 3.7, $AG(\pi)$ is cofiltered. We have seen earlier that τ is placid, in particular cofiltered. Since the composition of cofiltered maps is cofiltered (Lemma 2.11), the map $\tau \circ AG(\pi)$ is cofiltered. This shows that the sum $\sum_{g\in G} AG(g)$ is cofiltered. Since $\pi_* \mathbf{IS}^{\bar{p}}$ is additive by Proposition 4.12, and all involved maps are cofiltered, the claim follows from the calculation

$$\begin{aligned}\pi_! \circ \pi_* &= \pi_*(\mathbf{IS}^{\bar{p}}(\tau)) \circ \pi_*(\mathbf{IS}^{\bar{p}}(AG(\pi))) = \pi_*(\mathbf{IS}^{\bar{p}}(\tau \circ AG(\pi))) \\ &= \pi_*(\mathbf{IS}^{\bar{p}}(\sum_{g\in G} AG(g))) = \sum_{g\in G} \pi_*(\mathbf{IS}^{\bar{p}}(AG(g))) = \sum_{g\in G} g_*.\end{aligned}$$

When X is not connected, one uses the suspension ΣX as follows: The suspension is a G-space in the standard way, i.e. $g \cdot (t,x) = (t, g\cdot x)$. The two suspension points are fixed points. Its orbit space is homeomorphic to $\Sigma(X/G)$ via $(t,x)^* \mapsto (t,x^*)$, where $(-)^*$ denotes the orbit of a point. Under this homeomorphism, the orbit projection $\hat{\pi} : \Sigma X \to (\Sigma X)/G$ identifies with $\Sigma\pi : \Sigma X \to \Sigma(X/G)$. Moreover, the homeomorphism induces an isomorphism on $AG(-)$, under which $\hat{\tau} : AG((\Sigma X)/G) \to AG(\Sigma X)$ corresponds to $\tau_\Sigma : AG(\Sigma(X/G)) \to AG(\Sigma X)$. Since ΣX is connected, we already know that $\hat{\pi}_! \circ \hat{\pi}_* = \sum_g g_*$ on $IH_i^{\bar{m}}(\Sigma X)$. The result follows since for $i \le \lfloor (\dim X)/2 \rfloor$, $\pi_! \circ \pi_*$ corresponds under the isomorphisms $IH_i^{\bar{m}}(X) = IH_i^{\bar{m}}(\Sigma X)$ and $IH_i^{\bar{m}}(X/G) = IH_i^{\bar{m}}(\Sigma(X/G))$ (Lemma 5.6) to $(\Sigma\pi)_! \circ (\Sigma\pi)_*$, which in turn corresponds to $\hat{\pi}_! \circ \hat{\pi}_*$. Also note that the 0-level inclusion $X \hookrightarrow \Sigma X$ is G-equivariant. Thus g_* on $IH_i^{\bar{m}}(X)$ corresponds to g_* on $IH_i^{\bar{m}}(\Sigma X)$. □

Proposition 5.10. *Let G be a finite group acting (simplicially) on the connected polyhedron X with orbit map $\pi : X \to X/G = Y$ structured as a $d := |G|$ fold ramified covering. Then the transfer map $\pi_!$ with $\mathbb{Q}$-coefficients restricts to an isomorphism*

$$\pi_! : IH_*^{\bar{p}}(X/G;\mathbb{Q}) \xrightarrow{\simeq} IH_*^{\bar{p}}(X;\mathbb{Q})^G$$

onto the subspace $IH_*^{\bar{p}}(X;\mathbb{Q})^G \subset IH_*^{\bar{p}}(X;\mathbb{Q})$ of invariant elements. Connectivity is not required when $\bar{p} = \bar{m}$ and $* \le \lfloor (\dim X)/2 \rfloor$.

Proof. We have already pointed out that $\pi_!$ is a monomorphism. We show first that the image of $\pi_!$ is contained in $IH_*(X;\mathbb{Q})^G$. Given any element $v \in IH_*(X/G;\mathbb{Q})$, we put $w := \frac{1}{d}\pi_!(v) \in IH_*(X;\mathbb{Q})$. (Here, rational coefficients are essential.) Then, using the up-down Proposition 5.8, $\pi_*(w) = v$. Using the down-up Proposition 5.9 twice, an element $h \in G$ acts on $\pi_!(v)$ by

$$\begin{aligned}h_*(\pi_!(v)) &= h_*(\pi_!\pi_*(w)) = h_*\left(\sum_{g\in G} g_*(w)\right) = \sum_{g\in G} (hg)_*(w) \\ &= \sum_{g\in G} g_*(w) = \pi_!\pi_*(w) = \pi_!(v).\end{aligned}$$

This shows that $\pi_!(v)$ is G-invariant. Conversely, suppose that $w \in IH_*(X;\mathbb{Q})^G$ is any invariant element. Then $g_*(w) = w$ for all $g \in G$ and thus by the down-up Proposition 5.9,

$$\pi_!\pi_*(w) = \sum_{g \in G} g_*(w) = |G| \cdot w = d \cdot w.$$

Consequently, $w = \pi_!\left(\frac{1}{d}\pi_*(w)\right)$ is in the image of $\pi_!$, using rational coefficients. The above argument applies also when X is not connected, $\bar{p} = \bar{m}$ and the degree i satisfies $i \leq \lfloor(\dim X)/2\rfloor$, since the entire argument is homogeneously internal to that degree i. □

Corollary 5.11. *For connected X, the restriction of $\pi_* : IH^{\bar{p}}_*(X;\mathbb{Q}) \to IH^{\bar{p}}_*(X/G;\mathbb{Q})$ to $IH^{\bar{p}}_*(X;\mathbb{Q})^G$ is an isomorphism*

$$\pi_* : IH^{\bar{p}}_*(X;\mathbb{Q})^G \xrightarrow{\simeq} IH^{\bar{p}}_*(X/G;\mathbb{Q}).$$

Connectivity is not required when $\bar{p} = \bar{m}$ and $* \leq \lfloor(\dim X)/2\rfloor$.

Proof. For improved clarity, we write $j : IH^{\bar{p}}_*(X;\mathbb{Q})^G \subset IH^{\bar{p}}_*(X;\mathbb{Q})$ for the inclusion so that the restriction in question is the composition $\pi_* \circ j$. Let $w \in IH^{\bar{p}}_*(X;\mathbb{Q})^G$ be any element. By Proposition 5.10, there exists a (unique) $v \in IH^{\bar{p}}_*(X/G;\mathbb{Q})$ with $\pi_!(v) = w$. We will show that $\pi_* \circ j$ is injective. Indeed, if $(\pi_* j)(w) = 0$, then by the up-down Proposition 5.8,

$$0 = \pi_* j(w) = \pi_* j \pi_!(v) = d \cdot v,$$

which implies that $v = 0$, and thus $w = 0$. For surjectivity, let v be any element of $IH^{\bar{p}}_*(X/G;\mathbb{Q})$. Then $w := \frac{1}{d}\pi_!(v)$ is an element of $IH^{\bar{p}}_*(X;\mathbb{Q})$ with $\pi_*(w) = v$. By Proposition 5.10, w is G-invariant. □

The above facts are also valid for real coefficients instead of rational ones, indeed they hold over any field of characteristic zero.

6. Duality Properties of Orbit Spaces

If a finite group acts algebraically on a quasi-projective complex algebraic variety, then the orbit space is quasi-projective. In particular, the orbit space can be stratified such that all strata have even codimension and thus it satisfies the Witt condition. We prove in this section that the above observation can be extended from the algebraic setting to the purely topological setting, i.e. the orbit space of the action of a finite group on a Witt space again satisfies the Witt condition. In fact, we will establish a more general statement: If the total space of a ramified covering is a Witt space, then so is its base space (Theorem 6.9). (We will deal with the propagation of the pseudomanifold condition to the orbit space in Section 7.) One may also draw a parallel to the classical Conner conjecture, which asserts that the orbit space of a finite group action on a $\mathbb{Q}$-acyclic space is again $\mathbb{Q}$-acyclic. In fact, this conjecture holds more generally for ramified coverings π and is implied by the existence of an associated transfer homomorphism $\pi_!$ on ordinary homology (L. Smith [43, Cor. 2.4]). Closely related is the well-known fact that if a finite group acts on a locally connected rational homology manifold, then the orbit space is again a rational homology manifold (e.g. Bredon [12]). An important difference to these classical cases is that in order to establish the Witt condition, we need a transfer on intersection homology. Such a transfer is provided by Section 5.

We begin with a lemma that can be deduced from M. Cohen's [20, p. 194, Lemma 2.14]. We provide a direct argument.

Lemma 6.1. *Let $f : X \to Y$ be a PL map between polyhedra and let $y \in Y$ be a point. Then there exists a regular neighborhood $R(y)$ of y in Y such that $f^{-1}R(y)$ is a regular neighborhood of the fiber $f^{-1}(y)$ in X.*

Proof. Let K, L be simplicial complexes with $|K| = Y$, $|L| = X$ and such that $f : L \to K$ is simplicial. Let K' be the subdivision of K obtained by starring at the point $y \in |K|$. Then y is a vertex of K'. There exists a simplicial subdivision L' of L such that $f : L' \to K'$ is simplicial. The single vertex subcomplex $\{y\}$ is full in K'. Therefore, if we define a simplicial map $g : K' \to [0,1]$ by setting $g(y) = 0$ and $g(v) = 1$ for all vertices $v \in K'$, $v \neq y$, then $g^{-1}(0) = \{y\}$. Here, we regard the interval $[0,1]$ as a simplicial complex with one 1-simplex and the two endpoints as the only vertices. Then

$$R(y) := g^{-1}[0, \tfrac{1}{2}],$$

is a regular neighborhood of y in $|K|$. We shall verify that $f^{-1}R(y)$ is a regular neighborhood of the fiber $F := f^{-1}(y)$, which is a subpolyhedron of $|L|$. The fiber F is triangulated by the subcomplex $T := \{\sigma \in L' : f(\sigma) = \{y\}\}$ of L'. This subcomplex is full in L'. Define a simplicial map $g_F : L' \to [0,1]$ to be the composition $g \circ f$. Its vanishing set is $g_F^{-1}(0) = f^{-1}(g^{-1}(0)) = f^{-1}(y) = F$. In particular, any vertex of L' which is not in T cannot be mapped to 0 by g_F and must therefore be mapped to 1, since this is the only other vertex in $[0,1]$ available. (This also shows that T is full in L'.) By regular neighborhood theory, the polyhedron $g_F^{-1}[0, \frac{1}{2}]$ is a regular neighborhood of F, and $g_F^{-1}[0, \frac{1}{2}] = f^{-1}(g^{-1}[0, \frac{1}{2}]) = f^{-1}R(y)$. □

Lemma 6.2. *Let X be a polyhedron and $F = \{x_1, \dots, x_k\}$ a finite subset of X. Then for any regular neighborhood $R(F)$ of F in X, there exists for every i a regular neighborhood $R(x_i)$ of x_i in X such that there is a PL homeomorphism $X \to X$ which carries $R(F)$ onto $R(x_1) \sqcup \cdots \sqcup R(x_k)$ and which is the identity on F.*

This is a consequence of the uniqueness theorem for regular neighborhoods of compact polyhedra in polyhedra (see e.g. Rourke-Sanderson, p. 33, Thm. 3.8). Lemmas 6.1 and 6.2 imply:

Lemma 6.3. *Let $f : X \to Y$ be a finite-to-one PL map and let $y \in Y$ be a point. Then there exist a regular neighborhood $R(y)$ of y in Y and, for every point $x \in f^{-1}(y)$ in the fiber, a regular neighborhood $R(x)$ of x in X such that there is a PL homeomorphism*

$$f^{-1}R(y) \cong \bigsqcup_{x \in f^{-1}(y)} R(x)$$

which is the identity on $f^{-1}(y)$.

If $A \subset X$ is a subset of a topological space X, we will write $\operatorname{Bdy}_X(A)$, or simply $\operatorname{Bdy}(A)$, for the topological boundary (frontier) of A in X. We will denote by $\operatorname{Lk}(p,P)$ the *polyhedral* link of a point p in a polyhedron P. This link is a compact subpolyhedron of P that can be computed as follows: Choose a simplicial complex K which triangulates P and contains p as a vertex. Then $\operatorname{Lk}(p,P)$ is PL homeomorphic to the polyhedron $|\operatorname{Lk}(p,K)|$, where $\operatorname{Lk}(p,K)$ denotes the *simplicial* link of the vertex p in the complex K, see e.g. Armstrong [4, p. 177]. If L is a subcomplex of a simplicial complex K, then $N(L,K)$ denotes the simplicial neighborhood of L in K. The simplicial boundary of $N(L,K)$ is denoted by $\dot{N}(L,K)$.

Lemma 6.4. *Let P be a polyhedron and $p \in P$ a point. If $R(p)$ is any regular neighborhood of p, then there is a PL homeomorphism* $\operatorname{Bdy}R(p) \cong \operatorname{Lk}(p,P)$.

Proof. We can write any given regular neighborhood as $R(p) = |N(p,K')|$, where K is a simplicial complex with $|K| = P$, p is a vertex of K, and K' is a derived of K near p. By M. Cohen [20, p. 194, Lemma 2.12], $|\dot{N}(p,K')| = \operatorname{Bdy}|N(p,K')| = \operatorname{Bdy}R(p)$. By their very definition, the simplicial complexes $\dot{N}(p,K')$ and $\operatorname{Lk}(p,K')$ are equal. The statement of the lemma follows, as the polyhedron of the simplicial link $\mathrm{Lk}(p,K')$ is PL homeomorphic to the polyhedral link $\mathrm{Lk}(p,P)$. ☐

Lemma 6.5. *Let $A \subset K$ be a full subcomplex. Suppose that $g : K \to [0,1]$ is a simplicial map such that $A = g^{-1}(0)$. Let $R(A)$ be the regular neighborhood of $|A|$ in $|K|$ given by $R(A) = g^{-1}[0, \frac{1}{2}]$. Then $\mathrm{Bdy}R(A) = g^{-1}(\frac{1}{2})$.*

Proof. This is a consequence of the fact that $\mathrm{Bdy}R(A)$ is bicollared in $|K| - |A|$, see Cohen [20, p. 203, Thm. 5.3]. ☐

Lemma 6.6. *Let $f : X \to Y$ be a finite-to-one PL map between polyhedra. Then for every point $y \in Y$, there is a simplicial complex K such that $|K| = Y$, y is a vertex of K, $|\mathrm{Lk}(y,K)| \cong \mathrm{Lk}(y,Y)$, and there is a PL homeomorphism*

$$f^{-1}(|\mathrm{Lk}(y,K)|) \cong \bigsqcup_{x \in f^{-1}(y)} \mathrm{Lk}(x,X).$$

Proof. Let $R(y)$ be the regular neighborhood constructed in Lemma 6.1. Thus L', K' are simplicial complexes with $X = |L'|$, $Y = |K'|$, y is a vertex of K', $f : L' \to K'$ is simplicial and $R(y) = g^{-1}[0, \frac{1}{2}]$, where $g : K' \to [0,1]$ is simplicial, $g(y) = 0$ and $g(v) = 1$ for all vertices $v \in K'$, $v \neq y$; we have $g^{-1}(0) = \{y\}$. Let K'' be a derived of K' near y such that $R(y) = |N(y,K'')|$. By Lemma 6.5, $\mathrm{Bdy}R(y) = g^{-1}(\frac{1}{2})$. Hence

$$f^{-1}(\mathrm{Bdy}R(y)) = f^{-1}(g^{-1}(\tfrac{1}{2})) = g_F^{-1}(\tfrac{1}{2}),$$

where $g_F : L' \to [0,1]$ is the simplicial map obtained by composing f and g, as in the proof of Lemma 6.1; $F = f^{-1}(y)$ denotes the fiber over y. In that proof, F was triangulated by a full subcomplex T of L', and $g_F^{-1}(0) = F$, $g_F^{-1}[0, \frac{1}{2}] = f^{-1}R(y) =: R(F)$. Using Lemma 6.5 again, $g_F^{-1}(\frac{1}{2}) = \mathrm{Bdy}R(F)$. This shows that $f^{-1}(\mathrm{Bdy}R(y)) = \mathrm{Bdy}R(F)$. (We note on the side that given a continuous map, the operation of taking preimages does not in general commute with the operation of taking the topological boundary of a subset.) By Lemma 6.3, using that f is finite-to-one, every point $x \in f^{-1}(y)$ in the fiber possesses a regular neighborhood $R(x)$ such that there is a PL homeomorphism $R(F) = f^{-1}R(y) \cong \bigsqcup_{x \in f^{-1}(y)} R(x)$. By disjointness of this finite union of closed subspaces, $\mathrm{Bdy}R(F) \cong \bigsqcup_{x \in f^{-1}(y)} \mathrm{Bdy}R(x)$. So

$$f^{-1}(\mathrm{Bdy}R(y)) \cong \bigsqcup_{x \in f^{-1}(y)} \mathrm{Bdy}R(x).$$

As explained in the proof of Lemma 6.4,

$$\mathrm{Bdy}R(y) = \mathrm{Bdy}|N(y,K'')| = |\dot{N}(y,K'')| = |\mathrm{Lk}(y,K'')| \cong \mathrm{Lk}(y,Y).$$

According to Lemma 6.4, there are PL homeomorphisms $\mathrm{Bdy}R(x) \cong \mathrm{Lk}(x,X)$ for every $x \in F$. The desired statement follows. ☐

Let L, K be simplicial complexes and let $f : L \to K$ be a simplicial map. Let Δ be a simplex of K and let $y \in \Delta^\circ$ be a point in the interior of Δ. We define a subset of the complex L by

$$f^*(\Delta) := \{\Delta' \in L \mid f^{-1}(y) \cap \Delta'^\circ \neq \varnothing\}.$$

Lemma 6.7. *If f is finite-to-one, then the preimage of the open simplex Δ° is given by*

$$f^{-1}(\Delta^\circ) = \bigsqcup_{\Delta' \in f^*(\Delta)} \Delta'^\circ,$$

and $\dim \Delta' = \dim \Delta$ for all $\Delta' \in f^*(\Delta)$. In particular, $f^*(\Delta)$ is independent of the choice of $y \in \Delta^\circ$.

Proof. We show first that if $\Delta' \in f^*(\Delta)$, then $\Delta'^\circ \subset f^{-1}(\Delta^\circ)$. There exists a point $x \in \Delta'^\circ$ such that $f(x) = y$. Let $v_0, \dots, v_r$ be the vertices of Δ', $r = \dim \Delta'$. Then $x = \sum_{i=0}^r t_i v_i$ for positive real numbers t_i with sum 1. Thus

$$y = f(x) = t_0 f(v_0) + \cdots + t_r f(v_r) \in \Delta^\circ. \tag{9}$$

Since f is simplicial, the $f(v_i)$ are vertices of K and the set $f(v_0), \dots, f(v_r)$ spans a simplex σ of K. Note that since f is finite-to-one, $f(v_i) \neq f(v_j)$ whenever $i \neq j$. Therefore, $\dim \sigma = r$. Since all t_i are positive, (9) shows that y is in the interior of σ. But it is also in the interior of Δ. Now, a point of a triangulated polyhedron is interior to precisely one simplex of its triangulation. Hence $\sigma = \Delta$. This shows that every $f(v_i)$ is a vertex of Δ, and that $\dim \Delta = \dim \sigma = r = \dim \Delta'$. Given any point $x' \in \Delta'^\circ$, we can write it as a convex combination $x' = \sum_{i=0}^r s_i v_i$ for positive real numbers s_i with sum 1. Then $f(x') = \sum s_i f(v_i) \in \Delta^\circ$. This shows that $\Delta'^\circ \subset f^{-1}(\Delta^\circ)$.

Conversely, we prove that $f^{-1}(\Delta^\circ) \subset \bigsqcup \Delta'^\circ$: Let $x' \in |L|$ be any point with $f(x') \in \Delta^\circ$. Let Δ' be the unique simplex of L such that $x' \in \Delta'^\circ$. We claim that $\Delta' \in f^*(\Delta)$. Thus we need to show that Δ'° contains a point of $f^{-1}(y)$. Let $v_0, \dots, v_r$ be the vertices of Δ'. Then $x' = \sum t_i v_i$ for positive real numbers t_i with sum 1. Thus $f(x') = \sum t_i f(v_i) \in \Delta^\circ$. This shows, as above, that every $f(v_i)$ is a vertex of Δ and that, since $f(x')$ is an interior point of Δ, the set $f(v_0), \dots, f(v_r)$ constitutes the *complete* list of vertices of Δ, i.e. every vertex of Δ is of the form $f(v_i)$ for some i. Hence, $y \in \Delta^\circ$ can be written as a convex combination $y = \sum s_i f(v_i) \in \Delta^\circ$ with positive s_i. Set $x := \sum s_i v_i$. Then $x \in \Delta'^\circ$ is such that $f(x) = y$. This shows that $\Delta' \in f^*(\Delta)$. □

Given a simplicial complex K (always assumed to be locally finite) and a simplex Δ in K, we denote the simplicial link of Δ in K by $\mathrm{Lk}(\Delta, K)$. This is a (finite) simplicial complex.

Lemma 6.8. *Let $Y, \widetilde{Y}$ be pseudomanifolds triangulated by simplicial complexes L, K; $\widetilde{Y} = |L|$, $Y = |K|$. Let $\pi : \widetilde{Y} \to Y$ be a simplicial ramified covering with respect to L, K. Let Δ be any simplex in K and let $s = \dim \Delta$. Then there is a PL map*

$$\lambda : \bigsqcup_{\Delta' \in \pi^*(\Delta)} \Sigma^s |\mathrm{Lk}(\Delta', L)| \longrightarrow \Sigma^s |\mathrm{Lk}(\Delta, K)|$$

which is a ramified covering.

Proof. Let $y \in \Delta^\circ$ be any point in the interior of Δ. Then by Akin [1, p. 420, c $\to$ a], there is a PL homeomorphism

$$\mathrm{Lk}(y, Y) \cong \Sigma^s |\mathrm{Lk}(\Delta, K)|, \tag{10}$$

see also Armstrong [4, p. 178]. We noted earlier that the restriction of a d-fold ramified cover over any subspace of its target is again a d-fold ramified covering. Hence the restriction of π defines a ramified covering $\pi| : \pi^{-1}(|\mathrm{Lk}(y, K'')|) \to |\mathrm{Lk}(y, K'')|$, where K' is a simplicial subdivision of K such that y is a vertex of K' and K'' is a derived of K' near y. Composition with the PL homeomorphism $\pi^{-1}(|\mathrm{Lk}(y, K'')|) \cong \bigsqcup_{x \in \pi^{-1}(y)} \mathrm{Lk}(x, \widetilde{Y})$ provided by Lemma 6.6, yields a PL map $\lambda_1 : \bigsqcup_{x \in \pi^{-1}(y)} \mathrm{Lk}(x, \widetilde{Y}) \to |\mathrm{Lk}(y, K'')|$. Under the homeomorphism to the disjoint union, the ramification data, i.e. the multiplicity function, can be transferred from $\pi^{-1}(|\mathrm{Lk}(y, K'')|)$ to $\bigsqcup_{x \in \pi^{-1}(y)} \mathrm{Lk}(x, \widetilde{Y})$. Then λ_1 is again a ramified covering. The pullback of λ_1 under a PL homeomorphism $\mathrm{Lk}(y, Y) \cong |\mathrm{Lk}(y, K'')|$ is then a ramified covering

$$\lambda_0 : \bigsqcup_{x \in \pi^{-1}(y)} \mathrm{Lk}(x, \widetilde{Y}) \longrightarrow \mathrm{Lk}(y, Y).$$

Given $x \in \widetilde{Y}$, let Δ'_x be the unique simplex of L such that $x \in \Delta'^{\circ}_x$. Then $\pi^*(\Delta) = \{\Delta'_x \mid x \in \pi^{-1}(y)\}$. By Lemma 6.7, $\pi^{-1}(\Delta^\circ) = \bigsqcup_{x\in\pi^{-1}(y)} \Delta'^{\circ}_x$, and $\dim \Delta'_x = \dim \Delta = s$ for all $x \in \pi^{-1}(y)$. Again by Akin [1, p. 420, c $\to$ a],

$$\mathrm{Lk}(x, \widetilde{Y}) \cong \Sigma^s |\mathrm{Lk}(\Delta'_x, L)| \tag{11}$$

for every $x \in f^{-1}(y)$, since x is interior to Δ'_x. Under the PL homeomorphisms (10) and (11), the ramified cover λ_0 can be written as a ramified cover

$$\lambda : \bigsqcup_{x\in\pi^{-1}(y)} \Sigma^s |\mathrm{Lk}(\Delta'_x, L)| \longrightarrow \Sigma^s |\mathrm{Lk}(\Delta, K)|.$$

□

Theorem 6.9. *Let $\widetilde{Y}$ and Y be triangulated oriented pseudomanifolds. Let $\pi : \widetilde{Y} \to Y$ be a simplicial ramified covering (of finite degree). If $\widetilde{Y}$ satisfies the Witt condition, then so does Y.*

Proof. If the Witt condition holds in some stratification, then it holds in every stratification by the Proposition in [30, Section 2.4]. We equip $\widetilde{Y}$ and Y with the simplicial stratifications $\widetilde{\mathfrak{Y}}$ and $\mathfrak{Y}$. The filtration subspaces $\widetilde{Y}_i$ and Y_i are thus given by the i-dimensional simplicial skeleta L_i, K_i of the triangulations $|L| = \widetilde{Y}$, $|K| = Y$. Note that the link of any simplex in a simplicial pseudomanifold is a (compact) pseudomanifold, see e.g. Siegel [42, p. 1070, I.2].

Let $\Delta \in K$ be a simplex whose simplicial link $\mathrm{Lk}(\Delta, K)$ has even dimension $2k$. We have to prove that $IH^{\bar{m}}_k(|\mathrm{Lk}(\Delta, K)|; \mathbb{Q}) = 0$. By Lemma 6.8, the restriction of π induces a ramified covering

$$\lambda : \bigsqcup_{\Delta'\in\pi^*(\Delta)} \Sigma^s |\mathrm{Lk}(\Delta', L)| \longrightarrow \Sigma^s |\mathrm{Lk}(\Delta, K)|,$$

where $s = \dim \Delta$. Let B denote the target of λ and $\widetilde{B}$ its source. By Lemma 6.7, $\dim \Delta' = \dim \Delta$ for every $\Delta' \in \pi^*(\Delta)$. Since π is finite-to-one and surjective, $\dim \widetilde{Y} = \dim Y$ by (the proof of) Lemma 2.14. Therefore, $\dim Lk(\Delta', L) = \dim Lk(\Delta, K) = 2k$ and B and $\widetilde{B}$ are PL pseudomanifolds of dimension $\dim B = s + 2k = \dim \widetilde{B}$. Neither $\widetilde{B}$ nor B (when $s = 0$) needs to be connected. Since our description of the transfer uses Gajer's intersection homology Dold-Thom theorem, which seems to be available only for connected polyhedra, it is perhaps not immediately clear whether λ possesses a transfer homomorphism on intersection homology. In any case, if necessary, this technical issue can be overcome by the following method: The suspension of λ is a map $\Sigma\lambda : \Sigma\widetilde{B} \to \Sigma B$ between *connected* PL pseudomanifolds, which is again a PL ramified cover by Lemma 5.3. Therefore, it has an associated transfer $(\Sigma\lambda)_! : IH^{\bar{m}}_*(\Sigma B) \to IH^{\bar{m}}_*(\Sigma\widetilde{B})$, whose general construction was provided in Section 5. Using Lemma 2.14 with respect to appropriate simplicial subdivisions, the placid map $\Sigma\lambda$ induces covariantly a homomorphism $(\Sigma\lambda)_* : IH^{\bar{m}}_*(\Sigma\widetilde{B}) \to IH^{\bar{m}}_*(\Sigma B)$. By Proposition 5.8, the composition

$$\begin{array}{ccc} IH^{\bar{m}}_k(\Sigma B; \mathbb{Q}) & \xrightarrow{(\Sigma\lambda)_!} & IH^{\bar{m}}_k(\Sigma\widetilde{B}; \mathbb{Q}) \\ & \searrow^{d} & \downarrow (\Sigma\lambda)_* \\ & & IH^{\bar{m}}_k(\Sigma B; \mathbb{Q}) \end{array}$$

is multiplication by the degree $d \geq 1$ of the ramified covering $\Sigma\lambda$ (which equals the degree of λ). (Using the coefficient isomorphism $IH^{\bar{m}}_*(X; \mathbb{Z}) \otimes \mathbb{Q} = IH^{\bar{m}}_*(X; \mathbb{Q})$, we can pass from integral information to rational one.) By Lemma 5.6, $IH^{\bar{m}}_k(\Sigma B) = IH^{\bar{m}}_k(B)$, $IH^{\bar{m}}_k(\Sigma\widetilde{B}) = IH^{\bar{m}}_k(\widetilde{B})$.

So the above commutative diagram can be desuspended to $\widetilde{B}$ and B and we obtain a commutative diagram

$$\begin{array}{ccc} IH^{\bar{m}}_k(\Sigma^s|\mathrm{Lk}(\Delta,K)|;\mathbb{Q}) & \xrightarrow{(\Sigma\lambda)_!} & \bigoplus_{\Delta'\in\pi^*(\Delta)} IH^{\bar{m}}_k(\Sigma^s|\mathrm{Lk}(\Delta',L)|;\mathbb{Q}) \\ & \searrow^{d} & \downarrow (\Sigma\lambda)_* \\ & & IH^{\bar{m}}_k(\Sigma^s|\mathrm{Lk}(\Delta,K)|;\mathbb{Q}). \end{array}$$

Now, on a rational vector space, multiplication by $d \neq 0$ is an isomorphism with inverse given by multiplication by $1/d$. Since $\dim \mathrm{Lk}(\Delta',L) = 2k$, Lemma 5.7 implies that

$$IH^{\bar{m}}_k(\Sigma^s|\mathrm{Lk}(\Delta',L)|;\mathbb{Q}) = IH^{\bar{m}}_k(|\mathrm{Lk}(\Delta',L)|;\mathbb{Q}).$$

Since $\widetilde{Y}$ satisfies the Witt condition by assumption, we have $IH^{\bar{m}}_k(|\mathrm{Lk}(\Delta',L)|;\mathbb{Q}) = 0$ for every $\Delta' \in \pi^*(\Delta)$. This shows that the isomorphism $(\Sigma\lambda)_* \circ (\Sigma\lambda)_!$ factors through the zero vector space and is thus zero. This implies that $IH^{\bar{m}}_k(\Sigma^s|\mathrm{Lk}(\Delta,K)|;\mathbb{Q}) = 0$. By Lemma 5.7,

$$IH^{\bar{m}}_k(|\mathrm{Lk}(\Delta,K)|;\mathbb{Q}) = IH^{\bar{m}}_k(\Sigma^s|\mathrm{Lk}(\Delta,K)|;\mathbb{Q}) = 0,$$

as was to be shown.

□

7. Simplicial Actions and Pseudomanifolds

We recall here some basic material on simplicial group actions. The main result of this section is Theorem 7.2, which states that when a finite group acts simplicially and orientedly on an oriented pseudomanifold, then the orbit space is again an oriented pseudomanifold.

Let K be an (abstract) simplicial complex with associated polyhedron $X = |K|$. A discrete group G acts on X *simplicially* (with respect to K) if every transformation $g : X \to X$ is given by a simplicial map $g : K \to K$, $g \in G$. A simplicial action induces a simplicial action on the barycentric subdivision of K. The action on K is called *regular*, if for each subgroup H of G the following condition holds: If $g_0,\dots,g_n$ are elements of H and $(v_0,\dots,v_n)$ and $(g_0v_0,\dots,g_nv_n)$ are both simplices of K, then there exists an element $g \in H$ such that $g(v_i) = g_i(v_i)$ for all i. (Here the vertices $v_0,\dots,v_n$ need not be distinct.) This condition implies that if v and gv belong to the same simplex, then $gv = v$. Thus if $g(\sigma) = \sigma$ for some simplex σ, then g acts as the identity on every vertex of σ, and thus on every point of σ. Any simplicial action can be turned into a regular one by passing to the second barycentric subdivision. Thus, without loss of generality, we can and will assume from now on that every simplicial action is regular. See Bredon [13, p. 114ff] and Illman [32] for foundational material on simplicial and regular actions.

Then a simplicial complex K/G is given by taking the vertices of K/G to be the G-orbits $v^* := Gv$ of the vertices v of K, and by taking the simplices of K/G to be tuples of the form $(v^*_0,\dots,v^*_n)$, where $(v_0,\dots,v_n)$ is a simplex of K. (This does *not* mean that given a simplex $(v^*_0,\dots,v^*_n)$ of K/G, then $(v_0,\dots,v_n)$ is a simplex of K. In this situation, one knows only that for each i, there exists a vertex $w_i \in v^*_i$, such that $(w_0,\dots,w_n)$ is a simplex of K.) By regularity, if two simplices of K lie over the same simplex of K/G, then they can be moved to each other by a single element of G. That is, if $(v_0,\dots,v_n)$ and $(w_0,\dots,w_n)$ are in K such that $(v^*_0,\dots,v^*_n) = (w^*_0,\dots,w^*_n)$, then there exists a $g \in G$ with $g(v_0,\dots,v_n) = (w_0,\dots,w_n)$. This notation means that there is a permutation s of $\{0,\dots,n\}$ such that $gv_i = w_{s(i)}$. The assignment $v \mapsto v^*$ defines a simplicial map $K \to K/G$, whose associated continuous map $X = |K| \to |K/G|$ can be identified canonically with the orbit projection $X \to X/G = |K|/G$.

This shows that the orbit projection $\pi : X \to X/G$ is a simplicial map with respect to the above triangulations. We will also write K^* for the complex K/G.

Examples 7.1. In [9, Prop. 6.7], Leichtnam, Piazza and the author proved that subanalytic proper actions admit a G-equivariant triangulation. The proof uses the subanalytic triangulation theorem [35, Cor. 3.5] of Matumoto and Shiota, and Illman's general equivariant triangulation theorem [33, p. 497, Thm. 5.5]. Recall that a topological group G is called *subanalytic* if it is contained in some real analytic manifold M as a subanalytic subset. Every finite group is a subanalytic group. We assume that if a subanalytic group $G \subset M$ acts on a subanalytic set $X \subset N$, then it does so subanalytically, i.e. the graph of the action $G \times X \to X$ is subanalytic in $M \times N \times N$. Now let X be a locally compact subanalytic set and let G be a subanalytic proper transformation group of X. Then, according to [9, Prop. 6.7], X admits a G-equivariant triangulation.

The pseudomanifold property is preserved by orientation preserving (regular) simplicial actions:

Theorem 7.2. *Let K be a simplicial complex which is an oriented pseudomanifold. Suppose that K is equipped with a (regular) simplicial action of a finite group G which preserves the orientation. Then the orbit complex K/G is a pseudomanifold, and it is oriented.*

Proof. We recall that a simplicial complex K is, by definition, an *n-dimensional pseudomanifold* (without boundary), if

(PM1) every simplex of K is a face of some n-simplex of K, and
(PM2) every $(n-1)$-simplex of K is a face of precisely two n-simplices of K.

(See e.g. [29, p. 137], [6, p. 74, Examples 4.1.3].) We must verify (PM1) and (PM2) for the complex $K^* = K/G$.

We establish first the following auxiliary claim: If $(v_0,\dots,v_n)$ is an n-dimensional simplex of K, then its image simplex $(v_0^*,\dots,v_n^*)$ in K^* also has dimension n. Indeed, suppose that the image simplex had dimension strictly less than n. Then $v_i^* = v_j^*$ for some indices $i \neq j$. Thus there exists $g \in G$ such that $gv_i = v_j$. Consequently, v_i and gv_i are vertices of the same simplex of K. Thus the regularity of the action implies that $gv_i = v_i$. This yields the contradiction $v_i = v_j$. Therefore, $(v_0^*,\dots,v_n^*)$ has dimension n, as claimed.

Let us now prove property (PM1) for K^*. Let $(v_0^*,\dots,v_d^*)$ be any simplex in K^*, where $(v_0,\dots,v_d)$ is a simplex in K with $v_i \neq v_j$ for all $i \neq j$. Since K is a pseudomanifold, there is an n-simplex of the form $(v_0,\dots,v_d,v_{d+1},\dots,v_n)$ in K. Then the simplex $(v_0^*,\dots,v_d^*)$ is the face of the simplex $(v_0^*,\dots,v_d^*,v_{d+1}^*,\dots,v_n^*)$ in K^*. Note that by the auxiliary claim, the simplex $(v_0^*,\dots,v_n^*)$ is indeed n-dimensional. This verifies property (PM1) for K^*.

We turn to (PM2) for K^*. Let $(v_0^*,\dots,v_{n-1}^*)$ be any $(n-1)$-simplex in K^*, where the tuple $(v_0,\dots,v_{n-1})$ is an $(n-1)$-simplex in K. Since K is a pseudomanifold, there exist vertices v_+, v_-, $v_+ \neq v_-$, in K such that $(v_0,\dots,v_{n-1},v_+)$ and $(v_0,\dots,v_{n-1},v_-)$ are n-simplices in K. Furthermore, there is no vertex $v \in K$ such that $(v_0,\dots,v_{n-1},v)$ is an n-simplex in K and $v \notin \{v_+,v_-\}$. Then $(v_0^*,\dots,v_{n-1}^*)$ is a face of both $(v_0^*,\dots,v_{n-1}^*,v_+^*)$ and $(v_0^*,\dots,v_{n-1}^*,v_-^*)$. These are simplices of K^*, and we will show that they are two *different* simplices. (Each one is indeed n-dimensional by the auxiliary claim.) Suppose, by contradiction, that $v_+^* = v_-^*$. Then we have two simplices $(v_0,\dots,v_{n-1},v_+)$ and $(v_0,\dots,v_{n-1},v_-)$ in K that lie over the same simplex $(v_0^*,\dots,v_{n-1}^*,v_+^*) = (v_0^*,\dots,v_{n-1}^*,v_-^*)$ of K^*. Thus the regularity of the action implies that there is an element $g \in G$ such that $g(v_0,\dots,v_{n-1},v_+) = (v_0,\dots,v_{n-1},v_-)$. Hence there exists a permutation s of $\{0,\dots,n-1,-\}$ such that $gv_j = v_{s(j)}$, $\forall j \in \{0,\dots,n-1\}$, and $gv_+ =$

$v_{s(-)}$. For any $j \in \{0,\dots,n-1\}$, v_j and $v_{s(j)}$ are vertices of the simplex $(v_0,\dots,v_{n-1},v_-)$ in K. Therefore, since $v_{s(j)} = gv_j$, the vertices v_j and gv_j belong to the same simplex of K. Thus by regularity of the action, $gv_j = v_j$ for all $0 \le j \le n-1$. It follows that $gv_+ = v_-$. (The permutation s is thus the identity.) So g acts locally near points in the interior of the simplex $(v_0,\dots,v_{n-1})$ as a reflection across the hyperplane spanned by $v_0,\dots,v_{n-1}$. Let $\Sigma \subset |K| = X$ denote the polyhedron of the $(n-2)$-skeleton of K. Since K is an oriented pseudomanifold, $X-\Sigma$ is an oriented manifold (without boundary) which is open and dense in X. Let $A \subset X$ be the union of the simplices $(v_0,\dots,v_{n-1},v_+)$ and $(v_0,\dots,v_{n-1},v_-)$. Then the interior U of A is a g-invariant open neighborhood of the interior of $(v_0,\dots,v_{n-1})$, and U is contained in the oriented manifold $X-\Sigma$. The orientation of $X-\Sigma$ induces an orientation of U. The transformation g acts on U as the reflection across a hyperplane and such a reflection reverses the orientation. This contradicts the assumption that G acts orientation preservingly. It follows that $v_+^* \neq v_-^*$. This shows that $(v_0^*,\dots,v_{n-1}^*)$ is the face of at least two different n-simplices in K^*. It remains to show that it is not the face of any other, third, n-simplex. Thus, suppose that $(v_0^*,\dots,v_{n-1}^*,w^*)$ is an n-simplex of K^*; we must show that $w^* \in \{v_+^*, v_-^*\}$. Then there exists an n-simplex $(w_0,\dots,w_n)$ in K with $w_j^* = v_j^*$ for all $j = 0,\dots,n-1$, and $w_n^* = w^*$. Thus $(w_0,\dots,w_{n-1})$ and $(v_0,\dots,v_{n-1})$ are two $(n-1)$-simplices in K that both lie over the simplex $(w_0^*,\dots,w_{n-1}^*) = (v_0^*,\dots,v_{n-1}^*)$ in K^*. The regularity of the action implies that there exists a $g \in G$ such that $g(w_0,\dots,w_{n-1}) = (v_0,\dots,v_{n-1})$. Consider the translate $g(w_0,\dots,w_{n-1},w_n)$. This is a simplex in K, since g acts simplicially. It is n-dimensional, since $(w_0,\dots,w_n)$ is; and we have $g(w_0,\dots,w_{n-1},w_n) = (v_0,\dots,v_{n-1},gw_n)$. Thus $gw_n \notin \{v_0,\dots,v_{n-1}\}$. Since K is a pseudomanifold, we must have $gw_n \in \{v_+,v_-\}$, say $gw_n = v_+$. Then $w^* = w_n^* = (gw_n)^* = v_+^*$, Similarly, $w^* = v_-^*$ if $gw_n = v_-$. This verifies property (PM2) for K^*. We have shown that K^* is an n-dimensional pseudomanifold.

To prove that K^* is oriented, we need to orient the manifold U^* given by the complement of the polyhedron Σ^* of the $(n-2)$-skeleton of K^*. Let U be the manifold given by the complement of the polyhedron Σ of the $(n-2)$-skeleton of K. Then U is oriented by assumption. Since the orbit projection $\pi : K \to K^*$ is simplicial and finite-to-one, we have $\pi^{-1}(\Sigma^*) = \Sigma$ and $\pi^{-1}(U^*) = U$. Thus U is a G-invariant set and π restricts to a map $\pi : U \to U^*$, which is the orbit projection $U \to U/G = U^*$. Now whenever a finite group acts in an orientation preserving way on an oriented manifold, then the orbit space is an orientable rational homology manifold. So U/G, and hence U^* is orientable. □

In this theorem, the assumption of orientedness of the action is essential. Simple counterexamples, such as the complex conjugation action of $\mathbb{Z}/_2$ on the unit circle, readily show that the pseudomanifold property is not generally preserved by nonoriented actions. If one allows strata of codimension 1 (i.e. if one drops requirement (PM2)) and only retains the density condition, expressed simplicially in (PM1), then Popper shows in [40, Thm. 3.4] that the orbit space of a continuous action of a compact Lie group on a topological pseudomanifold whose orbits have conical slices is again a topological pseudomanifold. For our purposes, the classical notion of pseudomanifold as understood by Goresky-MacPherson [29] is the relevant one.

Since the orbit projection associated to the action of a finite group is a ramified covering (Example 5.2), we obtain, using Theorem 7.2, the following corollary to Theorem 6.9:

Corollary 7.3. *Let G be a finite group and X an oriented triangulated Witt pseudomanifold upon which G acts by orientation preserving simplicial maps. Then the orbit space X/G is an oriented Witt pseudomanifold.*

8. G-SIGNATURES OF SINGULAR SPACES

Let G be a finite group which acts (from the left) simplicially on an oriented pseudomanifold X of even dimension $n = 2m$. We assume that this action preserves the orientation of X and that X satisfies the Witt condition. To define a G-signature for X, we proceed along the lines of Atiyah-Singer [5, §6], except that the singularities in X require us to work with intersection homology instead of ordinary homology. Let $IH_*(X)$ denote the intersection homology of X, with real coefficients, taken with respect to the lower or upper middle perversity. The Witt condition ensures that the canonical map from lower to upper middle perversity intersection homology is an isomorphism, and that the bilinear intersection form

$$B : IH_m(X) \times IH_m(X) \longrightarrow \mathbb{R}$$

is nondegenerate. This form is symmetric when m is even and skew-symmetric when m is odd. For every $g \in G$, the map $g : X \to X$ is a simplicial isomorphism and thus induces an automorphism $g_* : IH_m(X) \to IH_m(X)$, which endows $IH_m(X)$ with a left G-module structure. When X is a rational homology manifold, the action of G is usually (e.g. in [5] and [45]) considered on *co*homology $H^m(X)$ rather than homology. Since cohomology is contravariant, $H^m(X)$ is made into a *left* G-module by $g \cdot x := (g^*)^{-1}(x)$ (see also Hirzebruch-Zagier [31, p. 30] and Gilmer [27, p. 106]). Then the Poincaré duality isomorphism $H^m(X) \cong H_m(X)$ is an isomorphism of left G-modules in view of

$$g^{-1*}(\xi) \cap [X] = g_* g_*^{-1}(g^{-1*}(\xi) \cap [X]) = g_*(\xi \cap g_*^{-1}[X]) = g_*(\xi \cap [X]).$$

In any case, the traces of g_* and g_*^{-1} are equal for a real representation of a finite group, but in the complex case, which is relevant when m is odd, the two traces are only conjugate to each other. We also note that the Poincaré duality isomorphism takes the cohomological intersection form to the homological form B. Consideration of the G-signature involves only the perversity $\bar{m}$-intersection homology in the middle degree m, and thus no connectivity requirement on X is necessary when applying the results of Section 5. The form B is G-invariant, since the action of G on X preserves the orientation. Let $\langle \cdot, \cdot \rangle$ be a positive definite and G-invariant inner product on $IH_m(X)$. An operator A is defined by the equation $B(x,y) = \langle x, Ay \rangle$. This operator commutes with the action of G, and its adjoint is given by $A^* = (-1)^m A$. Since A is G-equivariant, G acts on each eigenspace of A.

Suppose that m is even. Then A is self-adjoint and $IH_m(X)$ decomposes as a direct sum $IH_m(X) = IH_+ \oplus IH_-$, where IH_+ is the direct sum of the eigenspaces of the positive eigenvalues of A and IH_- is the direct sum of the eigenspaces of the negative eigenvalues. These are G-invariant and thus define real G-modules that we will also denote by IH_+, IH_-. Up to isomorphism, they are independent of the choice of inner product $\langle \cdot, \cdot \rangle$, as G has discrete characters, the characters of IH_+ and IH_- vary continuously with the inner product, and the space of G-invariant positive definite inner products is connected. If V is a finite dimensional real vector space and B is a nondegenerate symmetric bilinear pairing on V, then a subspace $V_+ \subset V$ is called *positive* if $B|_{V_+ \otimes V_+}$ is positive definite, i.e. $B(v,v) > 0$ for all $v \in V_+$, $v \neq 0$. The subspace V_+ is called *maximally positive*, if V_+ is not contained in a larger positive subspace. Negative and maximally negative subspaces are defined analogously.

Lemma 8.1. *Let $V_+ \subset V$ be a positive subspace for B and $V_- \subset V$ a negative subspace. If $V = V_+ \oplus V_-$, then V_+ is maximally positive and V_- is maximally negative.*

This is well-known and straightforward. Sylvester's law of inertia states that any two maximally positive (respectively, negative) subspaces have the same dimension. The signature of B is the difference $\dim V_+ - \dim V_-$, for any maximal positive subspace V_+ and any maximal negative V_-.

Lemma 8.2. *The subspace IH_+ is maximally positive for the intersection form B. The subspace IH_- is maximally negative for B.*

Proof. The spectral theorem applied to the self-adjoint operator A can be used to show that IH_+ is positive for B. Similarly, IH_- is negative. Since $IH_m(X) = IH_+ \oplus IH_-$, Lemma 8.1 implies that IH_+ is in fact maximally positive and IH_- is maximally negative. □

For the following definition see also [9]. Let $RO(G)$ denote the real, and $R(G)$ the complex, representation ring of G.

Definition 8.3. For m even, the *G-signature* of the G-Witt space X^{2m} is the virtual representation

$$\mathrm{Sign}(G,X) := IH_+ - IH_- \in RO(G) \subset R(G).$$

On elements $g \in G$, we will in particular consider the real numbers

$$\mathrm{Sign}(g,X) := \mathrm{tr}(g|_{IH_+}) - \mathrm{tr}(g|_{IH_-}),$$

where tr denotes the trace of a linear endomorphism. This number depends only on the action of g on $IH_*(X)$.

If m is odd, then A is skew-adjoint. Let $(AA^*)^{1/2}$ denote the positive square root of $AA^* = -A^2$. Since the square of the operator $J = A/(AA^*)^{1/2}$ is $J^2 = -1$, J defines a complex structure on $IH_m(X)$. As J commutes with the G-action, we obtain a complex G-module $IH_m(X)$. Again, this module is independent of the choice of inner product.

Definition 8.4. For m odd, the *G-signature* of the G-Witt space X^{2m} is the virtual representation

$$\mathrm{Sign}(G,X) := IH_m(X) - IH_m(X)^* \in R(G).$$

On elements $g \in G$, we will in particular consider the complex numbers

$$\mathrm{Sign}(g,X) := \mathrm{tr}(g|_{IH_m(X)}) - \overline{\mathrm{tr}(g|_{IH_m(X)})} = 2i\ \mathrm{Im}\,\mathrm{tr}(g|_{IH_m(X)}),$$

where one takes the trace of g as an automorphism of a complex vector space. This number is again independent of choices.

Remark 8.5. For $g = 1$, one obtains the ordinary signature $\mathrm{Sign}(X)$. Indeed, if m is even, then

$$\mathrm{Sign}(1,X) = \mathrm{tr}(\mathrm{id}\,|_{IH_+}) - \mathrm{tr}(\mathrm{id}\,|_{IH_-}) = \dim IH_+ - \dim IH_- = \mathrm{Sign}(X),$$

using Lemma 8.2. If m is odd, then

$$\mathrm{Sign}(1,X) = 2i\ \mathrm{Im}\,\mathrm{tr}(\mathrm{id}\,|_{IH_m(X)}) = 2i\ \mathrm{Im}\,\dim_{\mathbb{C}} IH_m(X) = 0 = \mathrm{Sign}(X).$$

If X is odd dimensional, one sets $\mathrm{Sign}(g,X) = 0$. The next lemma is a consequence of the topological invariance of the intersection form B (see e.g. [23, Remark 8.2.7 and Thm. 9.3.16]).

Lemma 8.6. *Let X and Y be oriented closed G-Witt pseudomanifolds. If X and Y are G-equivariantly and orientation preservingly homeomorphic, then* $\mathrm{Sign}(g,X) = \mathrm{Sign}(g,Y)$ *for every $g \in G$.*

Proposition 8.7. *Let X be a closed oriented G-Witt pseudomanifold such that the G-action preserves the orientation. Then* $\mathrm{Sign}(hgh^{-1},X) = \mathrm{Sign}(g,X)$ *for all $g,h \in G$.*

Proof. We consider the automorphism $h : X \to X$ given by $h(x) = h \cdot x$. This is generally not a G-equivariant map. We make it G-equivariant by defining a new G-action $\bullet$ on X: For $x \in X$, we set $g \bullet x := hgh^{-1} \cdot x$. Then $h : (X, \cdot) \to (X, \bullet) =: Y$ is indeed G-equivariant, as

$$h(g \cdot x) = h \cdot (g \cdot x) = hgh^{-1} \cdot (h \cdot x) = g \bullet (h(x)).$$

The action $\bullet$ is again simplicial and preserves the orientation of X, since $x \mapsto g \bullet x$ is the composition of the homeomorphisms $x \mapsto h^{-1} \cdot x$, $x \mapsto g \cdot x$, and $x \mapsto h \cdot x$, each of which is simplicial and preserves the orientation of X by assumption. By Lemma 8.6, $\mathrm{Sign}(g, (X, \cdot)) = \mathrm{Sign}(g, (X, \bullet)) = \mathrm{Sign}(hgh^{-1}, (X, \cdot))$. □

The proof of Proposition 8.12 on the signature of the orbit space requires the following standard fact from the representation theory of finite groups; see e.g. Fulton-Harris [24, p. 15f], or Hirzebruch-Zagier [31, Thm. p. 21f].

Lemma 8.8. *Let G be a finite group acting linearly on a finite dimensional real or complex vector space V. Then the dimension of the linear subspace $V^G = \{v \in V \mid gv = v \; \forall g \in G\}$ of invariant vectors is given by*

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(g|_V).$$

Let $\pi : X \to X/G$ be the orbit projection, a simplicial map. By Corollary 7.3, X/G is an oriented Witt pseudomanifold of dimension $2m$, and thus has a well-defined signature $\mathrm{Sign}(X/G)$, which can be described as follows: Since X/G is Witt, the canonical map from lower to upper middle perversity intersection homology is an isomorphism, and the bilinear intersection form $\overline{B} : IH_m(X/G) \times IH_m(X/G) \to \mathbb{R}$ is nondegenerate. This form is symmetric when m is even and skew-symmetric when m is odd.

Suppose that m is even. Using the map $\pi_* : IH_m(X) \to IH_m(X/G)$, we define linear subspaces of $IH_m(X/G)$ by $\overline{IH}_+ := \pi_*(IH_+)$, $\overline{IH}_- := \pi_*(IH_-)$.

Lemma 8.9. *The image $\pi_!(\overline{IH}_\pm)$ consists precisely of the G-invariant elements in $IH_\pm$, that is, $\pi_!(\overline{IH}_+) = IH_+ \cap IH_m(X)^G$, $\pi_!(\overline{IH}_-) = IH_- \cap IH_m(X)^G$.*

Proof. For $w \in IH_+$, one has $\pi_!\pi_*(w) = \sum_g g_*(w)$. The elements $g_*(w)$ lie in IH_+ for every $g \in G$, as IH_+ is G-invariant. Thus their sum, and hence $\pi_!\pi_*(w)$ is in IH_+. Since every element of $\overline{IH}_+$ is of the form $\pi_*(w)$, with $w \in IH_+$, this shows that $\pi_!(\overline{IH}_+) \subset IH_+$. By Proposition 5.10, $\pi_!(\overline{IH}_+) \subset \pi_!(IH_m(X/G)) \subset IH_m(X)^G$. We have proved that $\pi_!(\overline{IH}_+) \subset IH_+ \cap IH_m(X)^G$. Conversely, let $w \in IH_+ \cap IH_m(X)^G$ be any element. Set $d := |G|$ and $v := \frac{1}{d}\pi_*(w)$. Then $v \in \overline{IH}_+$ and

$$\pi_!(v) = \tfrac{1}{d}\pi_!\pi_*(w) = \tfrac{1}{d} \sum_{g \in G} g_*(w) = \tfrac{1}{d} dw = w,$$

since $w \in IH_m(X)^G$ is G-invariant so that $g_*(w) = w$ for all g. We conclude that $w \in \pi_!(\overline{IH}_+)$, which establishes the converse inclusion. □

Lemma 8.10. *The subspace $\overline{IH}_+$ is positive for the intersection form $\overline{B}$. The subspace $\overline{IH}_-$ is negative for $\overline{B}$.*

Proof. Given a nonzero vector $v \in \overline{IH}_+$, we must prove that $\overline{B}(v, v) > 0$. Using the up-down Proposition 5.8, we can write v as $v = \frac{1}{d}\pi_*\pi_!(v)$. Using the adjointness relation $B(\pi_!(-), -) = \overline{B}(-, \pi_*(-))$ ([28, p. 390, 3.]) and up-down,

$$\begin{aligned}\overline{B}(v, v) &= \tfrac{1}{d^2}\overline{B}(\pi_*\pi_! v, \pi_*\pi_! v) = \tfrac{1}{d^2} B(\pi_!\pi_*\pi_! v, \pi_! v) \\ &= \tfrac{1}{d^2} B(\pi_!(dv), \pi_! v) = \tfrac{1}{d} B(\pi_! v, \pi_! v).\end{aligned}$$

According to Lemma 8.9, the vector $\pi_! v$ is in IH_+, and it is nonzero since $\pi_!$ is injective. By Lemma 8.2, IH_+ is positive for B. Therefore, $B(\pi_! v, \pi_! v) > 0$. □

Lemma 8.11. *The vector space $IH_m(X/G)$ decomposes as a direct sum $IH_m(X/G) = \overline{IH}_+ \oplus \overline{IH}_-$.*

Proof. Given any $v \in IH_m(X/G)$, we put $w := \pi_!(v) \in IH_m(X)^G \subset IH_m(X)$ (Proposition 5.10). Using the decomposition $IH_m(X) = IH_+ \oplus IH_-$, there are $w_+ \in IH_+$, $w_- \in IH_-$ with $w = w_+ + w_-$. It follows that $v = \frac{1}{d}\pi_*\pi_!(v) = \frac{1}{d}\pi_*(w) = \frac{1}{d}\pi_*(w_+) + \frac{1}{d}\pi_*(w_-)$, where the first summand is in $\overline{IH}_+$ and the second one in $\overline{IH}_-$. It remains to prove that $\overline{IH}_+ \cap \overline{IH}_- = \{0\}$. Suppose that v is a vector in this intersection. If v were nonzero, then by Lemma 8.10, $\overline{B}(v,v) > 0$ since $v \in \overline{IH}_+$ and $\overline{B}(v,v) < 0$ since $v \in \overline{IH}_-$. This contradiction shows that $v = 0$. □

Using Lemma 8.1, we deduce from Lemmas 8.10 and 8.11 that $\overline{IH}_+$ is in fact maximal positive for $\overline{B}$, and $\overline{IH}_-$ is maximal negative. In particular, $\mathrm{Sign}(X/G) = \dim \overline{IH}_+ - \dim \overline{IH}_-$. If m is odd, then $\overline{B}$ is skew-symmetric and hence $\mathrm{Sign}(X/G) = 0$.

Proposition 8.12. *The signature of the orbit space is given in terms of equivariant signatures by*

$$\mathrm{Sign}(X/G) = \frac{1}{|G|}\sum_{g\in G} \mathrm{Sign}(g,X).$$

Proof. Suppose first that m is even. The transfer isomorphism of Proposition 5.10)

$$\pi_! : IH_m(X/G) \xrightarrow{\simeq} IH_m(X)^G$$

restricts to a monomorphism $\pi_! : \overline{IH}_+ \hookrightarrow IH_m(X)^G$, whose image is $IH_+ \cap IH_m(X)^G = (IH_+)^G$ by Lemma 8.9. Hence the restriction is an isomorphism $\pi_! : \overline{IH}_+ \xrightarrow{\simeq} (IH_+)^G$. The restriction to the negative subspace similarly yields an isomorphism $\pi_! : \overline{IH}_- \xrightarrow{\simeq} (IH_-)^G$. Thus, using Lemma 8.8,

$$\begin{aligned}\mathrm{Sign}(X/G) &= \dim \overline{IH}_+ - \dim \overline{IH}_- = \dim (IH_+)^G - \dim (IH_-)^G \\ &= \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}(g_*|_{IH_+}) - \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}(g_*|_{IH_-}) \\ &= \frac{1}{|G|}\sum_{g\in G} \big(\mathrm{tr}(g_*|_{IH_+}) - \mathrm{tr}(g_*|_{IH_-})\big) = \frac{1}{|G|}\sum_{g\in G} \mathrm{Sign}(g,X).\end{aligned}$$

Now assume that m is odd. In this case, $\mathrm{Sign}(X/G) = 0$. Again using Lemma 8.8, this agrees with

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G} \mathrm{Sign}(g,X) &= \frac{1}{|G|}\sum_{g\in G} 2i\ \mathrm{Im}\,\mathrm{tr}(g_*|_{IH_m(X)}) = \frac{2i}{|G|}\mathrm{Im}\sum_{g\in G} \mathrm{tr}(g_*|_{IH_m(X)}) \\ &= 2i\,\mathrm{Im}\dim(IH_m(X)^G) = 0.\end{aligned}$$

□

9. Equivariant Zagier-L-Classes

Let X be an oriented closed PL pseudomanifold of dimension n that satisfies the Witt condition. Suppose that a finite group G acts on X. The action is to be simplicial with respect to some triangulation and preserves the orientation. For any $g \in G$, we shall define equivariant L-classes

$$L_*(g,X) \in H_*(X;\mathbb{C}).$$

In the special case where X is a rational homology manifold, these classes have been defined by Zagier in [45]. In the smooth case, Zagier's classes, and hence the classes defined in this paper, agree under a Gysin homomorphism with the Poincaré duals of the equivariant Atiyah-Singer classes introduced in [5].

We triangulate the oriented i-sphere S^i by the complex C given as the boundary of a standard $(i+1)$-simplex. We endow S^i with the trivial G-action. Let $\overline{f}: X/G \to S^i$ be a continuous map. We approximate it by a simplicial map as follows: Let K be the (finite) simplicial complex with polyhedron $|K| = X$ such that G acts simplicially on K. As pointed out in Section 7, we may assume that this action is regular so that the complex $L := K/G$ triangulates X/G and the orbit projection $\pi : K \to L$ is simplicial. By the finite simplicial approximation theorem, there exists an N such that $\overline{f}$ has a simplicial approximation $\mathrm{bsd}^N(L) \to C$. (Here, bsd^N denotes the N-th abstract barycentric subdivision. Note that the complex C does not have to be subdivided.) We denote this simplicial approximation again by $\overline{f}$. The simplicial map $\pi : K \to L$ induces a simplicial map $\pi : \mathrm{bsd}^N(K) \to \mathrm{bsd}^N(L)$. (Here, *abstract* barycentric subdivision is relevant, since a simplicial map will not generally map *geometric* barycenters to *geometric* barycenters.) Any simplicial action induces a simplicial action on barycentric subdivision. Hence, G still acts simplicially on $\mathrm{bsd}^N(K)$. Composing, we obtain a G-invariant map f:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X/G \\ & \searrow^{f} & \downarrow \overline{f} \\ & & S^i. \end{array}$$

Let $\Delta^\circ \in C$ be the interior of an i-simplex and let $p \in \Delta^\circ$ be any point, viewed as the (abstract) barycenter of Δ. Then the preimage $f^{-1}(p) \subset X$ is a G-invariant subspace. We give $f^{-1}(p)$ the structure of a simplicial subcomplex as follows: The point p is a vertex of $\mathrm{bsd}(C)$. Then the diagram of simplicial maps

$$\begin{array}{ccc} \mathrm{bsd}^{N+1}(K) & \xrightarrow{\pi} & \mathrm{bsd}^{N+1}(L) \\ & \searrow^{f} & \downarrow \overline{f} \\ & & \mathrm{bsd}(C) \end{array}$$

shows that $f^{-1}(p)$ is a subcomplex of $\mathrm{bsd}^{N+1}(K)$. Since G acts simplicially on $\mathrm{bsd}^{N+1}(K)$ and $f^{-1}(p)$ is a G-invariant subcomplex, the restricted action of G on $f^{-1}(p)$ is still simplicial. The preimage is compact since it is closed in the compact space X. It is contained in the open neighborhood $f^{-1}(\Delta^\circ)$. Let $\phi : f^{-1}(\Delta^\circ) \to \Delta^\circ \times f^{-1}(p)$ be the PL homeomorphism provided by Milnor-Stasheff [37, p. 236, Lemma 20.5]. (See also Curran [21, p. 121f], as well as, in the equivariant case, Zagier [45, p. 22].) This homeomorphism is such that $f \circ \phi^{-1}$ is the first factor projection to Δ°. Furthermore, ϕ restricts to the identity map on $f^{-1}(p) \subset f^{-1}(\Delta^\circ)$. This shows that $f^{-1}(p)$ is normally nonsingular in X, with trivial normal bundle. As $f^{-1}(\Delta^\circ) = \pi^{-1}(\overline{f}^{-1}(\Delta^\circ))$, the tube is G-invariant. We endow $\Delta^\circ \times f^{-1}(p)$ with the G-action that makes ϕ G-equivariant. Let $(q,x) \in \Delta^\circ \times f^{-1}(p)$ be a point and $g \in G$. The translate $g(q,x)$ has the form (q',x'). As f is G-invariant,

$$q' = f \circ \phi^{-1}(q',x') = f \circ \phi^{-1}(g(q,x)) = f(g \cdot \phi^{-1}(q,x)) = f \circ \phi^{-1}(q,x) = q.$$

Thus $g(q,x) = (q,x')$, i.e. g acts trivially on the first component. Now, the PL pseudomanifold property desuspends, i.e. if $A \times \mathbb{R}^i$ is a PL pseudomanifold, then A is a pseudomanifold. (The reason is that the pseudomanifold property can be checked with respect to any PL stratification. Now use the PL intrinsic stratification. For this stratification, the intrinsic strata of $A \times \mathbb{R}^i$ are the products with $\mathbb{R}^i$ of the intrinsic strata of A, see [8, Prop. A.1].) Similarly, the Witt condition desuspends, as is shown in [10, p. 46, Lemma 14.1]. Therefore, $f^{-1}(p)$ is a pseudomanifold satisfying the Witt condition; its dimension is $n-i$. The orientation of S^i induces an orientation of Δ°. Together with the orientation of X, it induces an orientation of $f^{-1}(p)$. Let $\pi^i(X)$ denote the Borsuk-Spanier cohomotopy set of X, consisting of pointed homotopy classes of continuous maps $X \to S^i$. Suppose now that $n \leq 2(i-1)$. Then $\pi^i(X)$ is an abelian group. There is a natural action of G on $\pi^i(X)$ and the subgroup of invariant elements will be denoted by $\pi^i(X)^G$. For $g \in G$, we define a map

$$S_p(g) : \pi^i(X)^G \otimes \mathbb{C} \to \mathbb{C}$$

by

$$S_p(g)([f] \otimes \lambda) := \lambda \cdot \mathrm{Sign}(g, f^{-1}(p)),$$

using the G-signature of G-Witt pseudomanifolds as described in Section 8 and in [9]. If p' is any other point in Δ°, then ϕ restricts to an orientation preserving G-homeomorphism $\phi| : f^{-1}(p') \cong \{p'\} \times f^{-1}(p)$ and thus $S_p(g)([f] \otimes \lambda) = S_{p'}(g)([f] \otimes \lambda)$ by Lemma 8.6. Suppose that $F : X \times I \to S^i$ is a G-equivariant homotopy from f to f'. Since G acts trivially on S^i, F factors through a homotopy $\overline{F} : (X/G) \times I \to S^i$, which we may take to be simplicial, as described above. Composing with π, we receive a G-invariant simplicial version of the homotopy F such that the preimage $F^{-1}(p)$ is a G-invariant subcomplex of a simplicial subdivision of $K \times I$ and a simplicial G-space. Since $F^{-1}(p)$ is also a Witt bordism from $f^{-1}(p)$ to $f'^{-1}(p)$, we obtain a simplicial Witt G-bordism from $f^{-1}(p)$ to $f'^{-1}(p)$. We now use the invariance of G-signatures under Witt G-bordisms, a result due to Eric Leichtnam, Paolo Piazza and the author.

Proposition 9.1. *(Banagl-Leichtnam-Piazza.) The equivariant signature* $\mathrm{Sign}(g, X)$ *of closed oriented Witt pseudomanifolds X is a simplicial Witt G-bordism invariant.*

Proof. In the nonsingular case, this was shown by Ossa in [39]. In the singular Witt situation, this follows from the equivariant signature operator methods of [9]. In more detail, one argues as follows. Let W^{n+1} be a simplicial Witt G-bordism with $\partial W = X \sqcup -X'$, n even. Then W can in particular be smoothly stratified by taking the open simplices as strata, see also [3, p. 37, Prop. 4.5]. Since G acts linearly on simplices, it acts in particular by stratified diffeomorphisms. There exists a G-invariant wedge metric on W, and it can be taken to be a product near the boundary. Then the signature operator D_W (see Albin, Leichtnam, Mazzeo, Piazza [2]) with respect to such a metric is G-invariant and thus defines a class $[D_W] \in K^G_{n+1}(W, \partial W)$ in G-equivariant analytic relative K-homology such that $\partial_*[D_W] = [D_X] - [D_{X'}]$ (up to a multiple of 2). The constant maps $C : W \to \mathrm{pt}$, $c : X \to \mathrm{pt}$, $c' : X' \to \mathrm{pt}$ (and indeed more general G-maps $W \to Y$) fit into a commutative diagram

$$\begin{array}{ccc} K^G_{n+1}(W, \partial W) & \xrightarrow{C_*} & K^G_{n+1}(\mathrm{pt}, \mathrm{pt}) = 0 \\ \downarrow{\scriptstyle \partial_*} & & \downarrow{\scriptstyle \partial_*} \\ K^G_n(X) \oplus K^G_n(X') & \xrightarrow[c_* + c'_*]{} & K^G_n(\mathrm{pt}) \end{array}$$

It follows that $c_*[D_X] = c'_*[D_{X'}]$. The pushforward $c_*[D_X] \in K^G_n(\mathrm{pt}) = R(G)$ under the constant map $c : X \to \mathrm{pt}$ to a point is the equivariant index of D_X. At an element $g \in G$, this index is nothing but $\mathrm{Sign}(g, X)$, see [9] for more information. □

By the proposition, $S_p(g)[f] = S_p(g)[f']$. The map f is equivariantly homotopic to $s \circ f$ for any simplicial automorphism s of C. Thus we may move Δ° to any other i-simplex of C and $S_p(g)[f]$ does not change. So altogether then, $S_p(g)[f] = S_{p'}(g)[f']$ for G-homotopic maps f, f' and almost all $p, p' \in S^i$ (not necessarily in the interior of the same i-simplex). This shows that $S_p(g)$ is well-defined and independent of p for almost all p. We may thus simply write $S(g)$ for $S_p(g)$. The map $S(g)$ is linear. Hurewicz maps provide vertical homomorphisms such that the diagram

$$\begin{array}{ccc} H^i(X/G;\mathbb{C}) & \xrightarrow[\pi^*]{\simeq} & H^i(X;\mathbb{C})^G \\ \simeq\uparrow & & \uparrow\simeq \\ \pi^i(X/G)\otimes\mathbb{C} & \xrightarrow[\pi^*]{\simeq} & \pi^i(X)^G\otimes\mathbb{C} \end{array}$$

commutes. In the range $n \le 2(i-1)$, a well-known theorem of Serre ([41]) asserts that the Hurewicz maps are isomorphisms. Under these isomorphisms, $S(g)$ defines an element

$$S(g) \in \mathrm{Hom}(H^i(X/G;\mathbb{C}),\mathbb{C}) = H_i(X/G;\mathbb{C}).$$

Its transfer is an invariant element $\pi_! S(g) \in H_i(X;\mathbb{C})^G$. To compensate for the difference between $\pi_*[X]$ and $[X/G]$ in $H_*(X/G)$, we divide by $\deg\pi$. (If G acts effectively, then $\deg\pi = |G|$, the order of G.)

Definition 9.2. The *Atiyah-Singer-Zagier equivariant L-class* of the closed oriented G-Witt pseudomanifold X is on an element $g \in G$ defined to be

$$L_*(g,X) := \frac{1}{\deg\pi}\pi_! S(g) \in H_*(X;\mathbb{C})^G.$$

The restriction $n \le 2(i-1)$ can be removed by taking the product of X with a G-invariant sphere of large dimension.

Proposition 9.3. *Let $u \in H^i(S^i;\mathbb{Z})$ be the generator compatible with the orientation and let $\langle -,- \rangle$ denote evaluation of a cohomology class on a homology class. The equivariant class $L_*(g,X)$ satisfies*

$$\langle f^*(u),\ L_*(g,X)\rangle = \mathrm{Sign}(g, f^{-1}(p)). \tag{12}$$

For $\dim X \le 2(i-1)$, *it is the unique G-invariant homology class satisfying this equation for all G-invariant homotopy classes $f: X \to S^i$.*

Proof. The Hurewicz isomorphism sends $[\overline{f}]$ to $\overline{f}^*(u)$. Thus, by the construction of $S(g)$, we have $\langle \overline{f}^*(u),\ S(g)\rangle = \mathrm{Sign}(g, f^{-1}(p))$. Using the up-down formula for the transfer $\pi_!$, we calculate

$$\begin{aligned} \langle \overline{f}^*(u),\ S(g)\rangle &= \langle \overline{f}^*(u),\ \frac{1}{\deg\pi}\pi_*\pi_! S(g)\rangle = \langle \overline{f}^*(u),\ \pi_* L_*(g,X)\rangle \\ &= \langle \pi^*\overline{f}^*(u),\ L_*(g,X)\rangle = \langle f^*(u),\ L_*(g,X)\rangle. \end{aligned}$$

□

For $g = 1$, Remark 8.5 shows that $L_*(1,X) = L_*(X)$, the Goresky-MacPherson-Siegel class.

Proposition 9.4. *For any $g,h \in G$, $h_* L_*(g,X) = L_*(hgh^{-1},X)$.*

Proof. We use notation as in the construction of $L_*(g,X)$. Thus $f: X \to S^i$ is a simplicial G-invariant map, $f^{-1}(p) \subset X$ is a G-invariant simplicial subcomplex, $p \in \Delta^\circ \subset S^i$, and there is a G-equivariant PL homeomorphism $f^{-1}(\Delta^\circ) \cong \Delta^\circ \times f^{-1}(p)$ under which f looks like the factor projection to Δ°. It suffices to verify that the equality of evaluations

$$\langle f^*(u),\, h_* L_*(g,X)\rangle = \langle f^*(u),\, L_*(hgh^{-1},X)\rangle$$

holds. Using Proposition 9.3, the left hand side is the equivariant signature

$$\begin{aligned}\langle f^*(u),\, h_* L_*(g,X)\rangle &= \langle h^* f^*(u),\, L_*(g,X)\rangle = \langle (f\circ h)^*(u),\, L_*(g,X)\rangle \\ &= \operatorname{Sign}(g, h^{-1}(f^{-1}(p))),\end{aligned}$$

while the right hand side is the equivariant signature

$$\langle f^*(u),\, L_*(hgh^{-1},X)\rangle = \operatorname{Sign}(hgh^{-1}, f^{-1}(p)),$$

The space $Y = f^{-1}(p) \subset X$ is G-invariant and thus $h^{-1}(Y) = Y$. Hence by Proposition 8.7, $\operatorname{Sign}(g,h^{-1}(Y)) = \operatorname{Sign}(g,Y) = \operatorname{Sign}(hgh^{-1},Y)$. We note that h does indeed preserve the orientation of Y, since h preserves the orientation of the open G-tube $f^{-1}(\Delta^\circ) \cong \Delta^\circ \times f^{-1}(p)$ and it acts trivially in the normal direction Δ°, as pointed out earlier. □

Suppose that X is a rational homology manifold, assumed to be oriented, closed and triangulated. In that case, Zagier constructed a cohomology class $L^*(g,X) \in H^*(X;\mathbb{C})$, which is the invariant class uniquely determined by

$$\langle L^*(g,X) \cup f^*(u),\ [X]\rangle = \operatorname{Sign}(g, f^{-1}(p))$$

for all simplicial G-invariant maps $f : X \to S^i$ with p a sufficiently general point as above, see [45, p. 21, Thm. 1]. Comparing this equation with (12) above, it follows that the equivariant L-class constructed in this paper is the Poincaré dual of Zagier's cohomological class when X is a rational homology manifold. In that case, the cohomological expression of Proposition 9.4 is $h^* L^*(g,X) = L^*(h^{-1}gh, X)$, see [45, p. 18 (16), p. 21 (3)].

Suppose that X is a smooth closed oriented manifold such that G acts smoothly on X, preserving the orientation. For an element $g \in G$, let X^g be the submanifold of points fixed by g. Atiyah and Singer constructed an equivariant class in $H^*(X^g; \mathrm{or}_{X^g} \otimes \mathbb{C})$, explicitly given in terms of characteristic classes of X^g and of the equivariant normal bundle of the inclusion $j : X^g \subset X$, see [5, p. 582]. We use the formulation $L'(g,X) \in H^*(X^g; \mathrm{or}_{X^g} \otimes \mathbb{C})$ of these classes as given by Zagier in [45, p. 12, (27)]. The G-Signature Theorem of Atiyah and Singer is the relation

$$\operatorname{Sign}(g,X) = \langle L'(g,X), [X^g]\rangle.$$

The inclusion has an associated Gysin homomorphism $j_! : H^*(X^g) \to H^*(X)$ and the image of the AS-class under $j_!$ is Zagier's class, i.e.

$$j_! L'(g,X) = L^*(g,X),$$

[45, p. 4, (3); p. 21, Thm. 1]. As the Gysin map corresponds under Poincaré duality to covariant homological pushforward $j_* : H_*(X^g) \to H_*(X)$, the AS-class is related to our homological class by

$$j_*(L'(g,X) \cap [X^g]) = L_*(g,X)$$

in the differentiable case.

10. APPLICATION: THE GORESKY-MACPHERSON L-CLASS OF THE ORBIT SPACE

If a finite group G acts orientation preservingly on a Witt pseudomanifold X, then we have seen earlier (Corollary 7.3) that the orbit space X/G is again a Witt pseudomanifold. Thus the L-class $L_*(X/G) \in H_*(X/G;\mathbb{Q})$ is well-defined ([29], [42], [21]). We now apply the equivariant L-classes $L_*(g,X)$ constructed in Section 9 to compute $L_*(X/G)$, considered as an element of $H_*(X/G;\mathbb{C})$.

Theorem 10.1. *Let G be a finite group and X an oriented closed Witt pseudomanifold upon which G acts simplicially, preserving the orientation. Then X/G is a Witt pseudomanifold and its Goresky-MacPherson-Siegel L-class $L_*(X/G)$ is related to the equivariant L-classes by*

$$L_*(X/G) = \frac{1}{|G|}\sum_{g\in G} \pi_* L_*(g,X), \tag{13}$$

where $\pi : X \to X/G$ is the orbit projection.

Proof. The orbit space X/G is an oriented compact Witt pseudomanifold by Corollary 7.3. Therefore, it has a well-defined L-class $L_*(X/G)$. By the universal coefficient theorem for complex coefficients, it suffices to show that

$$\langle v,\, L_*(X/G)\rangle = \langle v,\, \pi_*\left(\frac{1}{|G|}\sum_{g\in G} L_*(g,X)\right)\rangle \tag{14}$$

for every $v \in H^i(X/G;\mathbb{C})$. Up to taking the product with a G-invariant large dimensional sphere, it also suffices to work in the range $n \leq 2(i-1)$, where $n = \dim X$. Some multiple kv of v can then be written as $kv = \overline{f}^*(u)$ for some simplicial map $\overline{f} : X/G \to S^i$, $u \in H^i(S^i)$ the generator. Let $f : X \to S^i$ be the G-invariant simplicial map $f = \overline{f}\circ\pi : X \to S^i$. Then $\pi^*(kv) = f^*(u)$ and the right hand side of (14) is

$$\langle v,\, \pi_*\left(\frac{1}{|G|}\sum_{g\in G} L_*(g,X)\right)\rangle = \langle \pi^* v,\, \frac{1}{|G|}\sum_{g\in G} L_*(g,X)\rangle = \frac{1}{k|G|}\sum_{g\in G}\langle f^*u,\, L_*(g,X)\rangle.$$

By Equation (12), $\sum_g \langle f^*u,\, L_*(g,X)\rangle = \sum_g \operatorname{Sign}(g, f^{-1}(p))$, and by Proposition 8.12,

$$\frac{1}{k|G|}\sum_{g\in G}\operatorname{Sign}(g, f^{-1}(p)) = \frac{1}{k}\operatorname{Sign}(f^{-1}(p)/G).$$

Now, the orbit space of the preimage is $f^{-1}(p)/G = (\pi^{-1}(\overline{f}^{-1}(p)))/G = \overline{f}^{-1}(p)$. Since $\overline{f}$ is transverse regular to p (as $\overline{f}$ is simplicial and p is in the interior of a top dimensional simplex), it follows from the construction of the (nonequivariant) Goresky-MacPherson-Siegel L-class (see also Curran [21, p. 121f]) that $\langle \overline{f}^*(u),\, L_*(X/G)\rangle = \operatorname{Sign}(\overline{f}^{-1}(p))$. Thus

$$\begin{aligned}\langle v,\, \pi_*\left(\frac{1}{|G|}\sum_{g\in G} L_*(g,X)\right)\rangle &= \frac{1}{k}\operatorname{Sign}(f^{-1}(p)/G) = \frac{1}{k}\operatorname{Sign}(\overline{f}^{-1}(p)) \\ &= \frac{1}{k}\langle \overline{f}^*(u),\, L_*(X/G)\rangle = \langle v,\, L_*(X/G)\rangle,\end{aligned}$$

as required. ∎

The left hand side of (13) comes from rational homology, while the individual summands $L_*(g,X)$ on the right hand side are only known to be elements in homology with complex coefficients. Thus Theorem 10.1 is in particular an integrality result on the equivariant classes

$L_*(g,X)$. Formula (13) corresponds to the formulae of Cappell-Shaneson-Maxim-Schürmann [15, p. 1726f, (1.11), (1.12)] for the equivariant Hirzebruch class in the algebraic setting.

11. Free Actions

If G acts freely, then the only contribution to $L_*(X/G)$ can arise from $L_*(1,X)$ in light of the following result.

Theorem 11.1. *(Free action.) Suppose that G acts freely on X. If $g \neq 1$, then $L_*(g,X) = 0$.*

Proof. By Proposition 9.3, it suffices to show that $\mathrm{Sign}(g,X) = 0$ for free $2m$-dimensional G-Witt spaces X, $g \in G - \{1\}$. Since $\mathrm{Sign}(g,X)$ depends only on the action of the subgroup $\langle g \rangle \cong \mathbb{Z}/_k$ generated by $g \neq 1$, we may as well assume $G = \langle g \rangle$. Let BG be the corresponding classifying space.

Suppose first that X has the form $X = Y \times \mathbb{Z}/_k$, where $G = \mathbb{Z}/_k$ acts trivially on the Witt space Y and by translation on the second factor. Then $IH_m(X) = \bigoplus_{j=0}^{k-1} IH_m(Y \times g^j)$ and the intersection form B on $IH_m(X)$ is given by an orthogonal sum of k copies of the intersection form B_Y on $IH_m(Y)$. Choose a positive definite inner product $\langle \cdot,\cdot \rangle$ on $IH_m(Y) = IH_m(Y \times g^0)$ and extend it to $IH_m(X)$ by $\langle (v,g^j),(w,g^j) \rangle = \langle (v,g^0),(w,g^0) \rangle$ and $\langle (v,g^i),(w,g^j) \rangle = 0$ for $g^i \neq g^j$. This is then a positive definite and G-invariant inner product on $IH_m(X)$. Suppose that m is even. Let $IH_+(Y)$ and $IH_-(Y)$ be the positive and negative eigenspaces of the operator A_Y defined on $IH_m(Y)$ by $B_Y(v,w) = \langle v, A_Y w \rangle$, yielding a decomposition $IH_m(Y) = IH_+(Y) \oplus IH_-(Y)$. The operator A on $IH_m(X)$ defined by $B(v,w) = \langle v, Aw \rangle$ is a block diagonal sum of k copies of A_Y. The positive and negative eigenspaces of $A = k \cdot A_Y$ are given by $IH_\pm = \bigoplus_j IH_\pm(Y) \times g^j$. Now let $\{(e_1^+,g^0),\ldots,(e_s^+,g^0)\}$ be a basis for $IH_+(Y)$. Then a basis for IH_+ is given by $\{(e_1^+,g^j),\ldots,(e_s^+,g^j) \mid j = 0,\ldots,k-1\}$. If $g_*|_{IH_+}$ is written as a matrix with respect to this basis, then this matrix has zero blocks, corresponding to g^j, along the diagonal since g_* acts by sending a basis vector (e_i^+,g^j) to the basis vector (e_i^+,g^{j+1}). This implies that the trace $\mathrm{tr}(g_*|_{IH_+})$ is zero. Similarly, $\mathrm{tr}(g_*|_{IH_-}) = 0$. Thus the difference $\mathrm{Sign}(g,X)$ of these two traces is zero. The case of odd m is treated similarly: The operators A_Y and A are skew-adjoint and the complex structure $J = A(AA^*)^{-1/2}$ on $IH_m(X)$ is the block diagonal sum of k copies of the complex structure $J_Y = A_Y(A_Y A_Y^*)^{-1/2}$ on $IH_m(Y)$. Thus there is a direct sum decomposition of complex vector spaces $(IH_m(X),J) = \bigoplus_{j=0}^{k-1} (IH_m(Y \times g^j), J_Y)$. Let $\{(e_1,g^0),\ldots,(e_s,g^0)\}$ be a basis for the complex vector space $(IH_+(Y),J_Y)$. Then a basis for the complex vector space $(IH_m(X),J)$ is given by $\{(e_1,g^j),\ldots,(e_s,g^j) \mid j = 0,\ldots,k-1\}$. If the complex linear automorphism $g_*|_{(IH_m(X),J)}$ is written as a matrix with respect to this basis, then this matrix again has zero blocks along the diagonal since $g_*|_{(IH_m(X),J)}$ acts by sending a basis vector (e_i,g^j) to the basis vector (e_i,g^{j+1}). This implies that the trace $\mathrm{tr}(g_*|_{(IH_m(X),J)})$ is zero. Thus $\mathrm{Sign}(g,X) = 0$ also when m is odd.

We show next that some multiple of a free simplicial $G = \mathbb{Z}/_k$-Witt pseudomanifold X is simplicially Witt G-bordant to a G-Witt pseudomanifold of the above form $Y \times \mathbb{Z}/_k$. This will imply the desired statement by the Witt G-bordism invariance of the g-signature, Proposition 9.1. If $G = \langle g \rangle$ acts freely on X, then the orbit map $\pi : X \to X/G$ is a regular covering space classified by a map $f : Y := X/G \to BG$. The orbit space X/G is an oriented Witt pseudomanifold by Corollary 7.3. Hence, the map f defines an element $[f] \in \Omega_*^{\mathrm{Witt}}(B\mathbb{Z}/_k)$, where $\Omega_*^{\mathrm{Witt}}(-)$ denotes Witt bordism theory. We recall that a connected space S is called (homotopy theoretically) *simple* if $\pi_1(S)$ is abelian and the action of $\pi_1(S)$ on $\pi_n(S)$ is trivial for every $n \geq 2$. A simple space has a rational localization $S_{\mathbb{Q}}$, which is again a simple space. Standard localization methods in stable homotopy theory show that $\widetilde{E}_n(S_{\mathbb{Q}}) = \widetilde{E}_n(S) \otimes \mathbb{Q}$

for every spectrum E. The classifying space $BG = B\mathbb{Z}/_k = K(\mathbb{Z}/_k, 1)$ is a simple space, since $\mathbb{Z}/_k$ is abelian and $\pi_n(K(\mathbb{Z}/_k, 1)) = 0$ for all $n \geq 2$. (Explicitly, $B\mathbb{Z}/_k$ is an infinite dimensional lens space $L = L_k^\infty$.) The localization $L_\mathbb{Q}$ is homotopy equivalent to a point, since $\pi_1(L) \otimes \mathbb{Q} = \mathbb{Z}/_k \otimes \mathbb{Q} = 0$, and for $n \geq 2$, $\pi_n(L) \otimes \mathbb{Q} = 0$. Given any spectrum E, we deduce that $\widetilde{E}_n(L) \otimes \mathbb{Q} = \widetilde{E}_n(L_\mathbb{Q}) = \widetilde{E}_n(\mathrm{pt}) \otimes \mathbb{Q} = 0$. The long exact sequence of the pair $(L, \{l_0\})$, where $l_0 \in L$ is a base point, shows that the inclusion $j : \{l_0\} \hookrightarrow L$ induces an isomorphism $E_n(\{l_0\}) \otimes \mathbb{Q} \cong E_n(L) \otimes \mathbb{Q}$. Applying this to the spectrum $E = \mathrm{MWITT}$ representing Witt bordism theory, we receive an isomorphism $\Omega_n^{\mathrm{Witt}}(\mathrm{pt}) \otimes \mathbb{Q} \cong \Omega_n^{\mathrm{Witt}}(B\mathbb{Z}/_k) \otimes \mathbb{Q}$. We return to the element $[f] \in \Omega_*^{\mathrm{Witt}}(BG)$ represented by the classifying map $f : X/G \to BG$. The above isomorphism shows that some multiple of $[f]$ is Witt bordant over BG to a map which factors as

$$Y \xrightarrow{c} \mathrm{pt} \overset{j}{\hookrightarrow} BG,$$

where Y is some closed oriented Witt pseudomanifold. Let $F : W \to BG$ be such a bordism. Let $EG \to BG$ be the universal principal G-bundle. (Explicitly, $EG = S^\infty$.) This is the universal cover of $BG = K(\mathbb{Z}/_k, 1)$. Since f classifies the $\mathbb{Z}/_k$-space X, we have $X = f^*EG$. Let $V := F^*EG$ and $X' := c^* j^* EG$. Then V is a covering space of W and thus also a Witt pseudomanifold. The group $G = \mathbb{Z}/_k$ is the deck transformation group and in particular operates freely on V. Let C be any simplicial complex triangulating W which is fine enough so that every simplex is contained in a connected open set which is evenly covered by $V \to W$. Then C can be lifted to a triangulation of V by a simplicial complex K such that $V \to W$ becomes a simplicial map $K \to C$ under the triangulation homeomorphisms. Since the restriction of this map to any slice over an evenly covered subset is invertible, the group of deck transformations (i.e. G) acts simplicially on K. Since W is oriented, its covering space V is oriented as well. The boundary of V is $(F|_{\partial W})^* EG = f^*EG \sqcup c^* j^* EG = X \sqcup X'$. Thus V is a simplicial Witt G-bordism between a multiple rX of X and X'. Now $j^*EG = \mathbb{Z}/_k$, the principal G-bundle over a point. Thus $X' = c^*(\mathbb{Z}/_k) = Y \times \mathbb{Z}/_k$, where $\mathbb{Z}/_k$ acts trivially on Y and by translation on the second factor. Hence by Proposition 9.1, $r\mathrm{Sign}(g, X) = \mathrm{Sign}(g, rX) = \mathrm{Sign}(g, X') = \mathrm{Sign}(g, Y \times \mathbb{Z}/_k) = 0 \in \mathbb{C}$. This implies that $\mathrm{Sign}(g, X) = 0$. □

When π is a regular covering projection, Theorem 10.1 together with Theorem 11.1 asserts that

$$L_*(X/G) = \frac{1}{|G|} \pi_* L_*(X).$$

This was already established in [7].

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