Vector bundles on the Fargues-Fontaine curve III

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1 Statement of the Theorem

Fix a prime p. Let $\mathbb{Q}_p$ be the p-adic numbers and let $E/\mathbb{Q}_p$ be a finite extension. For any $n \geq 0$, denote by E_n the composite of E with the unique unramified extension of $\mathbb{Q}_p$ of degree n. As before, fix a uniformizer π_E of E.

We recall that the complexes

$$B_e \oplus t^{-d}\mathbf{B}_{\mathrm{dR}}^{+} \longrightarrow \mathbf{B}_{\mathrm{dR}}^{+}[1/t] = \mathbf{B}_{\mathrm{dR}} \tag{1}$$

compute (in a functorial way) the cohomology of $\mathcal{O}_{X_E}(d)$ on X_E for integers d, where the map is given by $(x, y) \to x - y$ via the obvious inclusions. Hence:

Proposition 1.1. a) $H^0(X_E, \mathcal{O}_{X_E}(d,h)) = \begin{cases} 0, & \text{if } d < 0, \\ (B_E^{+})^{\varphi_E^h = \pi_E^d}, & \text{if } d \geq 0. \end{cases}$

b) Let $t \in P_{E,\pi,1} \setminus \{0\}$, $\{\infty\} = V^{+}(t)$ and $\mathbf{B}_{\mathrm{dR}}^{+} = \widehat{\mathcal{O}}_{X_E,\infty}$. Then

$$H^1(X_E, \mathcal{O}_{X_E}(d,h)) = \begin{cases} 0, & \text{if } d \geq 0, \\ \mathbf{B}_{\mathrm{dR}}^{+}/(t^{-d}\mathbf{B}_{\mathrm{dR}}^{+} + E_h), & \text{if } d < 0. \end{cases}$$

The purpose of this talk is to show the following theorem:

Theorem 1.2. ([2] Theorem 12.9) Let

$$0 \to \mathcal{O}_{X_E}\left(-\frac{1}{n}\right) \to \mathcal{E} \to \mathcal{O}_{X_E}(1) \to 0$$

be an exact sequence of vector bundles on X_E. Then $H^0(X_E, \mathcal{E}) \neq 0$.

In order to prove this theorem, we introduce the theory of Banach-Colmez Spaces.

2 Banach-Colmez Spaces

Let C/E be a algebraically closed field, complete for a valuation v which extends the valuation on E. Let $\mathrm{Spec}(\Lambda)$ denote the set of continuous C-algebra maps $\Lambda \to C$. We equip $\mathrm{Spec}(\Lambda)$ with the coarsest topology such that all the evaluation maps $\mathrm{Spec}(\Lambda) \to C$, $s \mapsto s(\lambda)$ for $\lambda \in \Lambda$ are continuous.

Definition 2.1. Let Λ denote a topological commutative C-algebra.

a) The **spectral norm** $\|\cdot\|$ on Λ is defined by $\|\lambda\| := \sup_{s\in\mathrm{Spec}(\Lambda)} |s(\lambda)| \in \mathbb{R}\cup\{\infty\}$ for $\lambda\in\Lambda$. One can show that the spectral norm, if it is a norm, is power-multiplicative.

b) Λ is a **sympathetic algebra** if it is a C-Banach algebra equipped with the spectral norm, such that it is connected (i.e. there exists no non-trivial idempotent) and such that any $\lambda\in\Lambda$ with $\|\lambda-1\|<1$, that is, $\lambda\in 1+\Lambda^{\circ\circ}$ admits a p-th root (i.e. is p-closed; this implies that the map $\Lambda^\circ/p \xrightarrow{p} \Lambda^\circ/p$ is surjective). These form a category with the obvious continuous morphisms.

Sympathetic algebras occur "naturally", since any spectral connected C-Banach algebra possesses a sympathetic closure $\widetilde{\Lambda}$: one takes the inductive family of elementary extensions, where one adjoins a p-th root of an element in $1+\Lambda^{\circ\circ}$, then repeats this process for this extension, etc. On the limit there is the natural spectral norm (since we take this norm on every finite étale elementary extension). Completing this gives a sympathetic algebra. One can show that the sympathetic closure is unique up to (possibly non-unique) isomorphism.

Example 2.2. Consider $C[X]$, equipped with the Gauß norm $\|\cdot\|_G$ given by

$$\|\sum a_i X^i\|_G = \sup|a_i|$$

and let $C\{X\}$ be the completion. Then $C\{X\}$ is spectral. Let $\widehat{\overline{C\{X\}}}$ be the completion of the integral closure of $C\{X\}$ in the the algebraic closure of the fraction field of $C\{X\}$. This is then the basic example of a sympathetic algebra.

The sympathetic closure $\widetilde{C\{X\}}$ injects into $\widehat{\overline{C\{X\}}}$, but is not equal to it.

Definition 2.3. A **Banach Space** is a covariant functor X from the category of sympathetic algebras to the category of $\mathbb{Q}_p$-Banach algebras, satisfying the following two conditions:

a) $X(\Lambda)\times\mathrm{Spec}(\Lambda)\to X(C)$ is continuous,

b) The induced map $X(\Lambda)\to\mathrm{Maps}(\mathrm{Spec}(\Lambda), X(C))$ is injective.

Maps between Banach Spaces are natural transformations. A sequence of maps of Banach Spaces is **exact** if it is exact at each Λ-point.

Example 2.4. a) Let V be a finite dimensional $\mathbb{Q}_p$-vector space. The functor $\Lambda\mapsto V$ with the identity maps everywhere is a Banach Space, denoted by $V^{\text{ét}}$.

b) Let W be a finite dimensional C-vector space. The functor $\Lambda\mapsto W\otimes\Lambda$ with the obvious morphisms is a Banach Space, denoted by W^{an}.

Definition 2.5. An **effective Banach-Colmez Space** Y is a Banach Space that is an extension of some W^{an} by some $V^{\text{ét}}$, i.e. there is an exact sequence $0\to V^{\text{ét}}\to Y\to W^{\text{an}}\to 0$. A **Banach-Colmez Space** X is a Banach Space such that there exists an effective Banach-Colmez Space Y and some $V'^{\text{ét}}$ such that $0\to V'^{\text{ét}}\to Y\to X\to 0$ is exact. We call these two exact sequences a **presentation** of E. We denote by $\mathcal{BC}$ the category of Banach-Colmez Spaces.

If $X\in\mathcal{BC}$ and we have a representation of X as above, we define the tupel $(\dim X, \mathrm{ht}X) := (\dim_C W, \dim_{\mathbb{Q}_p} V - \dim_{\mathbb{Q}_p} V') \in \mathbb{N}\times\mathbb{Z}$ for this presentation of X.

Theorem 2.6. ([1], Corollary 6.17) Let X be a Banach-Colmez Space. Then $(\dim X, \mathrm{ht}X)$ does not depend on the presentation of X.

It is clear by definition that

$$\{X \in \mathcal{BC} |\ \dim X = 0\} = \mathrm{Vect}_{\mathbb{Q}_p} = \{V^{\text{ét}} |\ V \text{ f. d. } \mathbb{Q}_p \text{ vector space}\}.$$

Remark 2.7. a) Consider the pro-étale topos of the category of locally noetherian adic spaces over C, or more precisely, the subcategory $\mathcal{T}$ of all $\widehat{\mathbb{Q}}_p$-sheaves that are extensions of sheaves W^{an} by $V^{\text{ét}}$, which are defined as $V^{\text{ét}} = V \otimes_{\mathbb{Q}_p} \widehat{\mathbb{Q}_p}$ and $W^{\text{an}} = W \otimes_C \mathcal{O}_Y$ for Y a locally noetherian adic space. If Λ is a sympathetic algebra then Λ can be written as the p-adic completion of a direct limit of strongly noetherian Tate C-algebras Λ_i. Evaluating a sheaf $\mathcal{F} \in \mathcal{T}$ at $\Lambda = \varinjlim \Lambda_i$ via $\mathcal{F}(\varinjlim_i \Lambda_i)$ defines a fully faithful functor $\mathcal{T} \to \mathcal{BC}^{\text{eff}}$ that can be shown to be an equivalence of categories. The reason being that the sympathetic algebras form a basis of open neighborhoods for the pro-étale topology for locally noetherian adic spaces over C.

b) Another way to view the whole of $\mathcal{BC}$ was established by Le Bras. He gives a description of $\mathcal{BC}$ as a certain core of a derived category of coherent sheaves on the Fargues-Fontaine curve (cf. [3]).

Theorem 2.8. ([1], §6) The catogory of Banach-Colmez Spaces $\mathcal{BC}$ is abelian. The functor "evaluation at C"

$$\mathcal{BC} \longrightarrow \mathbb{Q}_p\text{-Banach Spaces},\ X \longmapsto X(C)$$

is exact and faithful. The functions $\dim : \mathcal{BC} \to \mathbb{N}$ and $\mathrm{ht} : \mathcal{BC} \to \mathbb{Z}$ are additive on exact sequences.

Kernels of morphisms of Banach Spaces are again Banach Spaces, but it is not true in general that images are again Banach Spaces, since they need need not be closed for each Λ-point. But one can check that this holds for Banach-Colmez Spaces. More or less by definition, the other properties of an abelian category are satisfied.

The exactness of the evaluation functor is clear by definition. The faithfulness is checked first on effective spaces, and from this the general case follows.

Proposition 2.9. ([1], Proposition 6.11) Let $\psi : C^{\text{an}} \to C^{\text{an}}$ be a map of Banach-Colmez Spaces with $\psi \neq 0$. Then ψ is surjective.

Proposition 2.10. Let $\varphi : C^{\text{an}} \to \mathbb{Q}_p^{\text{ét}}$ be a map of Banach-Colmez Spaces. Then $\varphi = 0$.

Proof. We have a canonical (non-zero) map $f : \mathbb{Q}_p^{\text{ét}} \to C^{\text{an}}$, such that $\psi = f \circ \varphi$ is a map on C^{an}. If φ is non-zero, then ψ surjective by the above proposition. But then $\mathrm{Im}(\psi(C))$ is a non-zero C-vector space which contradicts the fact that it is a finite-dimensional $\mathbb{Q}_p$-vector space. $\square$

3 Period (pre-)sheaves

The Topological Ring $\mathbb{A}$ is defined as $\mathbb{A}(\Lambda) = \Lambda^\circ$. The Topological Ring $\mathbb{R}$ is defined as $\mathbb{R} = \varprojlim_n \mathbb{A}/p\mathbb{A}$, where the transition maps are the Frobenius, with norm $\|\cdot\|_\Lambda$ on $\mathbb{R}(\Lambda)$ given by $\|(x_n)\|_\Lambda := \|x^{(0)}\|_\Lambda$ via the usual identification

$$\mathbb{R}(\Lambda) \cong \{(x^{(n)})_{n \in \mathbb{N}} |\ x^{(n)} \in \Lambda^\circ, (x^{(n)})^p = x^{(n-1)}\}.$$

Similarly, one defines $\mathbb{A}_{\text{inf}} = W_{\mathcal{O}_E}(\mathbb{R})$, the surjective map $\theta : \mathbb{A}_{\text{inf}} \to \mathbb{A}$, $\mathbb{J} = \ker \theta$, $\mathbb{I} = \theta^{-1}(p\mathbb{A}) \supset \mathbb{J}$, $\mathbb{B}_{\text{dR}}^+$ as the $\mathbb{J}$-adic completion of $\mathbb{A}_{\text{inf}}[1/p]$, $\mathbb{A}_{\text{max}}$ as the p-adic completion of the

$\mathbb{A}_{\text{inf}}$-subalgebra of $\mathbb{A}_{\text{inf}}[1/p]$ generated by $\pi_E^{-1}\mathbb{I}$, $\mathbb{B}^+_{\max} = \mathbb{A}_{\max}[1/p]$, $\mathbb{B}^+ = \bigcap_n \varphi_E^n(\mathbb{B}^+_{\max})$, since we have a Frobenius action φ_E on $\mathbb{B}^+_{\max}$.

For example, $\mathbb{B}^+_{\max}(\Lambda)$ is a $\mathbb{Q}_p$-Banach algebra by demanding that $\|x\| = 1$ if and only if $x \in \mathbb{A}_{\max} \setminus p\mathbb{A}_{\max}$, and similarly for $\mathbb{B}^+_{\text{dR}}$. Since $(\mathbb{B}^+)^{\varphi_E=\pi_E} = (\mathbb{B}^+_{\max})^{\varphi_E=\pi_E}$ we have that the first space is also a Banach Space.

If we specialize one of these Banach algebras at C we obtain all the algebras we defined before. For instance, $\mathbb{B}^+(C) = B^+_E, \mathbb{B}^+_{\text{dR}}(C) = \mathbf{B}^+_{\text{dR}}$, etc.

Theorem 3.1. ([1], Proposition 8.19) One has the exact sequence

$$0 \to t \cdot E^{\text{ét}} \to (\mathbb{B}^+)^{\varphi_E=\pi_E} \to C^{\text{an}} \to 0$$

of Banach Spaces, where $t \cdot E^{\text{ét}} \subset B^+_E$ as before, that is, $(\mathbb{B}^+)^{\varphi_E=\pi_E}$ is an effective Banach-Colmez Space of dimension $(1, \dim_{\mathbb{Q}_p} E)$.

4 Finishing the proof of Theorem 1.2

Proof. (of Theorem 1.2) By the exact sequence

$$H^0(X_E, \mathcal{E}) \longrightarrow H^0(X_E, \mathcal{O}_{X_E}(1)) \xrightarrow{\delta} H^1(X_E, \mathcal{O}_{X_E}(-1/n))$$

it suffices to show that $\text{Ker}(\delta) \neq 0$. Let

$$x \in H^1(X_E, \mathcal{O}_{X_E}(-1 - \frac{1}{n})) = H^1(X_{E_n}, \mathcal{O}_{X_{E_n}}(-1-n))$$

correspond to the extension

$$0 \to \mathcal{O}_{X_E}\left(-\frac{1}{n}\right) = \pi_{n,*}\mathcal{O}_{X_{E_n}}(-1) \to \mathcal{E} \to \mathcal{O}_{X_E}(1) \to 0,$$

on X_E, so that we may also assume that this correpsonds to an extension

$$0 \to \mathcal{O}_{X_{E_n}}(-1) \to \mathcal{E}' \to \mathcal{O}_{X_{E_n}}(n) = \pi_n^*\mathcal{O}_{X_E}(1) \to 0$$

on X_{E_n}. By writing down the resolutions for these sequences, we can check that the map δ factors as

$$H^0(X_E, \mathcal{O}_{X_E}(1)) \xrightarrow{\pi_n^*} H^0(X_{E_n}, \mathcal{O}_{X_{E_n}}(n)) \xrightarrow{\delta} H^1(X_{E_n}, \mathcal{O}_{X_{E_n}}(-1)).$$

In general, if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles on X, we can describe the connecting homomorphism via the diagram

$$\begin{tikzcd}
H^0(X, E'') \otimes H^0(E''^{\vee} \otimes E'') \arrow[r] \arrow[d, "\text{id}\otimes\delta'"] & H^0(X, E'') \arrow[d, "\delta"] \\
H^0(X, E'') \otimes H^1(E''^{\vee} \otimes E') \arrow[r] & H^1(X, E').
\end{tikzcd}$$

By the identification $H^i(E''^{\vee} \otimes E') = \text{Ext}^i(E'', E')$, one checks that the connecting homomorphism δ' has the property that it sends the identity in $\text{Ext}^i(E'', E') = \text{Hom}(E'', E')$ to x.

Hence, we can describe the map δ as the map induced by the cup product

$$\cup : H^0(X_{E_n}, \mathcal{O}_{X_{E_n}}(n)) \times H^1(X_{E_n}, \mathcal{O}_{X_E}(-1-n)) \to H^1(X_{E_n}, \mathcal{O}_{X_{E_n}}(-1)),$$

i.e. $\delta(-) = -\cup x$. Via the identifications from Proposition 1.1 and the description of the complex (1) that computes the cohomology of the $\mathcal{O}_X(d)$ we can describe δ as

$$(B_E^+)^{\varphi_E=\pi_E} \longrightarrow \mathbf{B}_{\mathrm{dR}}^+/(t^{n+1}\mathbf{B}_{\mathrm{dR}}^+) \xrightarrow{\times x} \mathbf{B}_{\mathrm{dR}}^+/(t^{n+1}\mathbf{B}_{\mathrm{dR}}^+) \longrightarrow \mathbf{B}_{\mathrm{dR}}^+/(t\mathbf{B}_{\mathrm{dR}}^+ + E_n) \cong C/E_n.$$

where the first and the last map are the canonical inclusion resp. projection (the x here corresponds to cupping with the class x). By this description it is clear that it is induced by a map of Banach-Colmez Spaces $u : (\mathbb{B}^+)^{\varphi_E=\pi_E} \to C^{\mathrm{an}}/E_n^{\text{ét}}$. If u were injective, then would obtain an exact sequence

$$0 \to (\mathbb{B}^+)^{\varphi_E=\pi_E} \to C^{\mathrm{an}}/E_n^{\text{ét}} \to \mathrm{coker}(u) \to 0.$$

By the additivity of the dimension, since $\dim(\mathbb{B}^+)^{\varphi_E=\pi_E} = \dim C^{\mathrm{an}}/E_n^{\text{ét}} = 1$, $\dim \mathrm{coker}(u) = 0$, so that $\mathrm{coker}(u) = U^{\text{ét}}$ for some finite dimensional $\mathbb{Q}_p$-vector space U. But then we obtain a map $C^{\mathrm{an}} \to C^{\mathrm{an}}/E_n^{\text{ét}} \to \mathrm{coker}(u)$ which must be zero by Proposition 2.10. Hence $\mathrm{coker}(u) = 0$, and u is an isomorphism. But this contradicts the fact that $\mathrm{ht}(\mathbb{B}^+)^{\varphi_E=\pi_E} = \dim_{\mathbb{Q}_p} E > 0$, but $\mathrm{ht}(C^{\mathrm{an}}/E_n^{\text{ét}}) = -\dim_{\mathbb{Q}_p} E_n < 0$. $\square$

References

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