TALK II : VECTOR BUNDLES ON THE ALGEBRAIC FARGUES-FONTAINE CURVE

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These notes are an exposition of my talk at the workshop "Gal($\overline{\mathbb{Q}}_p/\mathbb{Q}_p$) as a geometric fundamental group, May 4-8, 2015, Neckarbischofseim, Germany." We closely follow ([Durham, Section 6]). We shall use results from the previous talks in the workshop. In the first section we recall briefly the notation from the previous talks. In the second section we recall the definition of algebraic Fargues-Fontaine curve X and compute certain cohomology groups associated to these curves. In the final section we state the classification theorem of vector bundles on X and prove an important result which will be used to prove this classification theorem in the next talk.

0.1. **Notation and definition.**

Throughout the talk we fix the following notation.
Let p be a fix prime,
$E/\mathbb{Q}_p$ be a local field extension, $\mathcal{O}_E$ denote the ring of integer of $E/\mathbb{Q}_p$.
Put $m_E = \pi\mathcal{O}$, $\mathbb{F}_q = \mathcal{O}/m_E$, $q = p^f$ where f denote the degree of $\mathbb{F}$ over $\mathbb{Z}/p$
$F/\mathbb{F}_q$ be a perfect, complete, non-archimedean algebraic closed field for a non-trivial valuation $v : F \longrightarrow \mathbb{R} \cup \{+\infty\}$. Put $|-| := q^{-v(-)}$.
Let $\mathcal{E}/E$ be the unique complete unramified extension of inducing the residue field extension $F/\mathbb{F}_q$.
Let $|.| : F \longrightarrow \mathcal{O}_{\mathcal{E}}$ denote the Teichmuller lift.
Then we have, $\mathcal{E} = \{\Sigma_{n\geq -\infty}|x_n|\pi^n \text{ (unique writing) } \mid x_n \in F\}$. Let $\varphi_E : \mathcal{E} \longrightarrow \mathcal{E}$ be the Frobenius morphism defined as $\varphi_E(\Sigma_{n\geq-\infty}|x_n|\pi^n) = \Sigma_{n\geq-\infty}|x_n^q|\pi^n$.
Put, $B^b = \{\Sigma_{n\geq-\infty}|x_n|\pi^n \in \mathcal{E}|\exists\ C, \forall|x_n| \leq C\}$ and
$B^{b,+} = \{\Sigma_{n\geq-\infty}|x_n|\pi^n \in \mathcal{E}|x_n \in \mathcal{O}_F\}$.
For $x = \Sigma_{n\geq-\infty}|x_n|\pi^n \in B^b$ and $r \geq 0$ put $v_r(x) = inf_{n\in\mathbb{Z}}\{v(x_n) + nr\}$ and if $\rho \in (0, 1]$ then put $|x|_\rho = q^{-v_r(n)}$.
Let B (resp B^+) denote the completion of B^b (resp. $B^{b,+}$) with respect to the family of multiplicative norms $(|-|_\rho)_{\rho\in(0,1]}$ (see [Durham, definition 1.5]). The Frobenius morphism φ_E extends to B (resp. B^+) by continuity.

For an interval $I \subset (0,1)$ let B_I denote the completion of B^b with respect to I. Set,

$$B := \varprojlim_{I \subset (0,1)} B_I.$$

Let $|Y|$ denote the set of closed points of B and for a maximal ideal $m \in |Y|$ let $B^+_{dR,m}$ denote the m-adic completion of B (see [Durham, corollary 3.11, definition 3.1]).

0.2. **The graded algebra P and the curve X.**
For an integer r, put $P_r := \{\mathfrak{b} \in B | \varphi_E(\mathfrak{b}) = \pi^r \mathfrak{b}\} = B^{\varphi_E = \pi^r}$. From [Durham, 1.15] we get that $P_r = 0$ when $r < 0$, $P_r = E$ when $r = 0$ and $P_r = (B^+)^{\varphi = \pi^r}$ if $r \geq 0$. Consider the graded E algebra

$$P_{F,E,\pi} = P_E := \bigoplus_{r \geq 0} P_r.$$

Let $X_{F,E,\pi} = X_E := Proj P_E$ denote the projective spectrum of P_E with the structure sheaf $\mathcal{O}_{X_E}$.

Proposition 0.1. *(*[FF, Theorem 10.2]*)* .
(a) The scheme X_E is noetherian, integral and regular of dimension one. We have the following bijection

$$div : (P_1 \backslash \{0\})/E^\times \longrightarrow |X_E| := \{x \in X_E \ closed\}$$

$$t \longmapsto \infty_t = V^+(t) = Proj(P_E/tP_E)$$

(b) $\Gamma(X_E, \mathcal{O}_{X_E}) = P_0 = E$.
(c) If E'/E is a finite extension of fields then there is a canonical isomorphism $X_{E'} \cong X_E \times_{SpecE} SpecE'$ and $X_{E'} \longrightarrow X_E$ is a finite étale Galois cover of degree $[E' : E]$.

For an integer $h \geq 0$, let E_h denote the unique unramified Galois extension of E of degree h. Put $X_h := X_{E_h}$ and $X := X_E$. We have that $\pi_h : X_h \longrightarrow X$ is a finite étale Galois extension of curves with Galois group $\mathrm{Gal}(E_h/E) \cong \mathbb{Z}/h\mathbb{Z}$.

Definition 0.2. *Let $d, h \in \mathbb{Z}$ such that $(d, h) = 1$.*
Put $P_{E_h}[d] := \bigoplus_{r \in \mathbb{Z}} P_{E_h}[d]_r$ where $P_{E_h}[d]_r := P_{E_h, d+r}$.
Let $\mathcal{O}_{E_h}(d) := \widetilde{P_{E_h}[d]}$ be the sheaf associated to the graded module $P_{E_h}[d]$ and put $\mathcal{O}_X(d/h) := \pi_{h*}(\mathcal{O}_{X_h}(d))$.

Let $\infty \in |X|$ and choose an element $t \in P_1$ such that $V^+(t) = \infty$ (see Proposition 0.1). Put $B_e := \Gamma(X\backslash\{\infty\})$. From Theorem 5.5 there is a bijection between $|Y|/\varphi^{\mathbb{Z}}$ and $|X|$. Let $m \in |Y|$ be a be the maximal ideal corresponding to $\{\infty\}$ under this bijection. Then we have that $B^+_{dR,m} = B^+_{dR} \cong \widehat{\mathcal{O}_{X,\infty}}$ where $\widehat{\ }$ denote the t-adic completion.(see [Durham, Theorem 5.3]). Put, $B_{dR} = B^+_{dR}[1/t]$. The ring B^+_{dR} is a discrete valuation ring with maximal ideal m generated by t (see [Durham, Theorem 5.3]) . Let $VB_{X_E}/\sim$ denote the isomorphism classes of vector bundles on X and $\mathcal{C}$ be the category of pairs (M, W) where W is a free B^+_{dR} module of finite type and $M \subset W[1/t]$ is a sub B_e-module of finite type with an isomorphism $M \otimes_{B_e} B_{dR} \xrightarrow{\sim} W[1/t]$.

Proposition 0.3. *(See* [FF, Section 2]*) There is an equivalence of categories*

$$VB_{X_E} \xrightarrow{\sim} \mathcal{C}$$

$$\mathcal{E} \longmapsto (\Gamma(X\backslash\{\infty\}, \mathcal{E}), \widehat{\mathcal{E}_\infty}).$$

Furthermore, if $\mathcal{E}$ corresponds to the pair (M, W) then,

$$0 \longrightarrow H^0(X, \mathcal{E}) \longrightarrow M \oplus W \xrightarrow{\delta} W[1/t] \longrightarrow H^1(X, \mathcal{E}) \longrightarrow 0.$$

is an exact sequence. In particular,

$$H^0(X_E,) \cong M \cap W \quad \textit{and} \quad H^1(X_E,) \cong W[1/t]/(W + M).$$

Theorem 0.4. *Let $d \in \mathbb{Z}$. We have, $H^1(X_E, \mathcal{O}_{X_E}) = 0$ if $d \geq 0$ and $H^1(X_E, \mathcal{O}_{X_E}) = B^+_{dR}/(t^{-d}B^+_{dR} + E)$ if $d < 0$.*

Proof. Following the notation of Proposition 0.3, the line bundle $\mathcal{O}_{X_E}$ corresponds to the pair (B_e, B^+_{dR}). Therefore, $H^1(X_E, \mathcal{O}_{X_E}) = B^+_{dR}[1/t]/(B^+_{dR} + B_e)$. For a positive integer i, put $Fil_iB_e := \{x/t^i | x \in P_i\}$. From the Fundamental Exact sequence (see [Durham, Theorem 5.1] we have that the natural map $Fil_iB_e/Fil_{i-1}B_e \xrightarrow{\sim} t^{-i}B^+_{dR}/t^{-i+1}B^+_{dR}$ is an isomorphism for every positive integer i. This implies that $B^+_{dR}[1/t] = B^+_{dR} + B_e$ and therefore $H^1(X_E, \mathcal{O}_E) = 0$. Now consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{t^d} \mathcal{O}_X(d) \longrightarrow i_*\widehat{\mathcal{O}}_{X,\infty}/t^d\widehat{\mathcal{O}}_{X,\infty} \longrightarrow 0.$$

where $i : \{\infty\} \hookrightarrow X$. Since $i_*\widehat{\mathcal{O}}_{X,\infty}/t^d\widehat{\mathcal{O}}_{X,\infty}$ is a torsion sheaf with finite support on the reduced curve X, we get that $H^1(X, i_*\widehat{\mathcal{O}}_{X,\infty}/t^d\widehat{\mathcal{O}}_{X,\infty}) = 0$. From the associated long exact cohomology sequence and using the vanishing of $H^1(X, \mathcal{O}_E)$, it follows that $H^1(X, \mathcal{O}_E(d)) = 0$ for every positive integer d. Next, we suppose that $d < 0$. From the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(d) \xrightarrow{t^{-d}} \mathcal{O}_X \longrightarrow i_* \widehat{\mathcal{O}}_{X,\infty}/t^{-d}\widehat{\mathcal{O}}_{X,\infty} \longrightarrow 0.$$

we get the following long exact cohomology sequence

$$H^0(X_E, \mathcal{O}_X(d)) \xrightarrow{t^{-d}} H^0(X, \mathcal{O}_X) \to i_* H^0(X, \widehat{\mathcal{O}}_{X,\infty}/t^{-d}\widehat{\mathcal{O}}_{X,\infty}) \to H^1(X, \mathcal{O}_X(d)) \to 0.$$

Now, the theorem follows by using the facts $H^0(X, \mathcal{O}_X(d)) = 0$, $H^0(X, \mathcal{O}_X) = P_0 = E$ and $H^0(X, \widehat{\mathcal{O}}_{X,\infty}/t^{-d}\widehat{\mathcal{O}}_{X,\infty}) = B^+_{dR}/t^{-d}B^+_{dR}$. $\square$

Corollary 0.5. *Let $\lambda, \mu \in \mathbb{Q}$.*
(a) $Ext^0(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu)) = 0$ if and only if $\lambda > \mu$.
(b) $Ext^1(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu)) \neq 0$ if and only if $\lambda > \mu$.

Proof. From [FF, Proposition 4.23] we have that $Ext^i(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu)) \cong H^i(X, \mathcal{O}_X(\mu - \lambda))$. Let $\mu - \lambda = d/h$, for some $d, h \in \mathbb{Z}$ such that $(d, h) = 1$ and $h \geq 1$. From Shapiro's Lemma we get that $H^i(X, \mathcal{O}_X(\mu - \lambda)) = H^i(X, \pi_{h*}\mathcal{O}_X(d)) = H^i(X_h, \mathcal{O}_{X_h}(d))$. Now the corollary follows from Theorem 0.4. $\square$

Proposition 0.6. *Let $\mathcal{E}$ be a vector bundle on X and $h \geq 1$ be a positive integer.*
(a)The vector bundle $\mathcal{E}$ is semi-stable of slope $\mu(\mathcal{E}) = \lambda$ if and only if $\pi_h^*\mathcal{E}$ is semi-stable of slope $\mu(\pi_h^*\mathcal{E}) = h\lambda$.
(b) $\mathcal{E} \cong \mathcal{O}_X^r(\lambda)$ for some integer r if and only if $\pi_h^*\mathcal{E} \cong \mathcal{O}_{X_h}^{r'}(h\lambda)$ for some integer r'.

Proof. We have $\mu(\pi_h^*\mathcal{E}) = deg(\pi_h^*\mathcal{E})/rk(\pi_h^*\mathcal{E}) = hdeg(\mathcal{E})/rk(\mathcal{E}) = h\mu(\mathcal{E})$. Here, $deg(\mathcal{E})$ (resp. $rk(\mathcal{E})$) denote the degree (resp. rank) of $\mathcal{E}$.

Recall that the map $\pi_h : X_h \longrightarrow X$ is Galois with Galois group $\mathrm{Gal}(E_h/E)$. Therefore we have an equivalence of categories

$$\pi_h^* : \mathrm{VB}_X/\sim \quad \longrightarrow \mathrm{Gal}(E_h/E) - \text{ equivariant vector bundle on } X_h$$

$$\mathcal{E} \longmapsto \pi_h^*\mathcal{E}.$$

Now suppose that $\mathcal{E}$ is semi-stable and consider the Harder-Narasimhan filtration (HNF)

$$0 = Z_0 \subsetneq Z_1 \subsetneq \cdots Z_n = \pi_h^*\mathcal{E}$$

of $\mathcal{E}$. By uniqueness of (HNF) and the fact that $\mu(\tau^* Z_1) = \mu(Z_1)$ for every $\tau \in \mathrm{Gal}(E_h/E)$ it follows that Z_1 is a Galois equivariant sub-bundle of $\pi_h^*\mathcal{E}$. Let $Z_1' \hookrightarrow \mathcal{E}$ be a vector bundle such that $\pi_h^* Z_1' = Z_1$. Since $\mathcal{E}$ is semi-stable we get that

$\mu(Z_1') \leq \mu(\mathcal{E})$. Therefore,$\mu(Z_1) \leq \pi_h^*\mathcal{E}$. This implies that $n = 1$ and therefore $\pi_h^*\mathcal{E}$ is semi-stable.

Now suppose that $\pi_h^*\mathcal{E}$ is semi-stable and let $Z \hookrightarrow \mathcal{E}$ be a vector bundle. Since $\pi_h^*\mathcal{E}$ is semi-stable and $\pi_h^*\mathcal{Z} \subset \pi_h^*\mathcal{E}$ we get that $\mu(\pi_h^*\mathcal{Z}) \leq \mu(\pi_h^*\mathcal{E})$. Therefore $\mu(Z) \leq \mu(\mathcal{E})$. This proves that $\mathcal{E}$ is semi-stable.

The statement (b) is an immediate consequence of Hilbert Theorem 90. □

0.3. **Classification of vector bundles on X.**

Next we state the main theorem about vector bundles.

Theorem 0.7. *(i) The semi-stable vector bundle of slope λ on X are the direct sums of $\mathcal{O}_X(\lambda)$.*
(ii) The Harder-Narsimhan filtration) (HNF) of a vector bundle on X is split.
(iii) There is a bijection

$$\{\lambda_1 \geq \cdots \geq \lambda_n | n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} \xrightarrow{\sim} VB_X/\sim$$

$$(\lambda_1, \cdots, \lambda_n) \mapsto [\bigoplus_{i=1}^n \mathcal{O}_X(\lambda_i)]$$

Remark

(a) (i)+(ii) $\Longleftrightarrow$ (iii) (easy).
(b) Using the fact that $Ext^1(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu)) = 0$ for $\lambda \geq \mu$ it can be easily shown that (i) $\implies$ (ii).
The aim of this talk is to prove the following theorem

Theorem 0.8. *Theorem 0.7 is equivalent to the following statement: for any $n \geq 1$ and any vector bundle $\mathcal{E}$ that is an extension*

$$0 \longrightarrow \mathcal{O}_X(-1/n) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

one has $H^0(X, \mathcal{E}) \neq 0$.

Proof. Let $\mathcal{E}$ be a vector bundle that is an extension as in the statement of the theorem. Suppose that Theorem 0.7 is true. Then, we have that $\mathcal{E} \cong \oplus_i \mathcal{O}_X(\lambda_i)$. Since $deg(\mathcal{O}_X(-1/n) = deg(\pi_n^*\mathcal{O}_{X_h}(-1)) = deg(\mathcal{O}_{X_h}(-1)) = -1$ Therefore $\deg(\mathcal{E}) = 0$. This implies that $\lambda_i \geq 0$ for some i. From Corollary we get that 0.2 $H^0(X, \mathcal{E}) \neq 0$.

To prove the other direction let $\mathcal{E}$ be a semi-stable bundle on X. For simplicity we shall assume that $rk(\mathcal{E}) = 2$. The general case can be proved similarly but the complete argument is more technical (see [FF, Theorem 4.26]. Without loss of generality using Proposition 0.6 we can assume that $\mu(\mathcal{E}) \in \mathbb{Z}$. We can also replace $\mathcal{E}$ by a twist $\mathcal{E} \otimes \mathcal{O}(d)$ and assume that $\mu(\mathcal{E}) = 0$.

Since $\mathcal{E}$ is semi-stable, $\mu(\mathcal{L}) \leq 0$ for every sub-line bundle $\mu(\mathcal{L})$ of $\mathcal{E}$ we have that $\mu(\mathcal{L}) \leq \mu(\mathcal{E})$. We mention that here by a sub-line bundle $\mathcal{L}$ of $\mathcal{E}$ we mean that $\mathcal{L}$ is a sub-sheaf of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{L}$ is a vector bundle. From [Durham, Proposition 6.3]) we have an isomorphism $\mathbb{Z} \xrightarrow{\sim} Pic(X)$ given by $d \mapsto \mathcal{O}_X(d)$. Let $\mathcal{L}$ be a sub-line bundle with maximal degree. Writing $\mathcal{L} \cong \mathcal{O}_X(-d)$ for some integer $d \geq 0$ and using the assumption that $rk\mathcal{E} = 2$, we get the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

If $d = 0$ then using the fact that $H^0(X, \mathcal{O}_X) = 0$, we have $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X$. Now suppose that $d > 0$. Since $-d + 2 \leq d$, from Corollary 0.2 we have $Hom(\mathcal{O}_X(-d + 2), \mathcal{O}_X(d)) \neq 0$. Let

$$u : \mathcal{O}_X(-d+2) \xrightarrow{\neq 0} \mathcal{O}_X(d)$$

be a non-trivial morphism. We consider the pull-back exact sequence

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}_X(-d+2) \longrightarrow 0$$

with a morphism $\mathcal{E}' \longrightarrow \mathcal{E}$ which is a generic isomorphism. By twisting the above exact sequence by $\mathcal{O}_X(d-1)$, we get the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{E}'(d-1) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0.$$

From the assumption of the theorem we get that $H^0(X, \mathcal{E}'(d-1)) \neq 0$ and therefore there is a non-trivial morphism $\mathcal{O}_X(1-d) \longrightarrow \mathcal{E}'$. Composing it with the generic isomorphism $\mathcal{E}' \longrightarrow \mathcal{E}$, we get a non-trivial morphism $u' : \mathcal{O}_X(1-d) \longrightarrow \mathcal{E}$. Since the image of u' has degree $\geq -d$ we get a contradiction. Therefore $d = 0$. This proves the theorem. $\square$

References

[Durham] Laurant Fargues and Jean-Marc Fontaine, Vector bundles on curves and p-adic Hodge theory, http://webusers.imj-prg.fr/ laurent.fargues/Durham.pdf .

[FF] Laurant Fargues and Jean-Marc Fontaine, Courbes et fibres vectoriels en theorie de hodge p-adique, http://math.bu.edu/people/jsweinst/AWS/Files/FarguesFontaineCourbes.pdf .