

Vector bundles on X_E , I

①

As before, let

• E/\mathbb{Q}_p finite, $m_E = \pi_E \cdot \mathcal{O}_E$, $\mathbb{F} = k_E$

• \mathbb{F}/\mathbb{F} perfect, complete field, w.r.t. non-triv., non-arch. v_F
 $q^{-v_F} = 1 \cdot 1: \mathbb{F}^* \rightarrow \mathbb{R}_{>0}$; we will always assume $\boxed{\mathbb{F} = \mathbb{F}^{\text{alg}}}$

• $E = W_{\mathcal{O}_E}(\mathbb{F})[\frac{1}{\pi_E}] \curvearrowright \varphi^{\text{-Frobenius}}$
 $[x_i] \mapsto [x_i^q]$

$$\begin{array}{ccc} U & & \\ B^b & \xrightarrow[\text{cplth}]{\text{Fréchet}} & B \supseteq \\ U & & U \\ B^{b,+} & \xrightarrow{\text{minus}} & B^+ \supseteq \end{array}$$

• $P := \bigoplus_{d \geq 0} B^{d, \pi_E^d} \quad \text{graded } E\text{-algebra} \quad (\text{better: } P_{E, \mathbb{F}, \pi_E})$

• $X_E := X_{E, \mathbb{F}, \pi_E} := \text{Proj}(P_{E, \mathbb{F}, \pi_E})$ is Spec E -scheme

Rem: Choosing a different uniformizer π'_E gives a canonically isomorphism $X_{E, \mathbb{F}, \pi_E} \cong X_{E, \mathbb{F}, \pi'_E}$; [F-F] 8.7.2

Proposition:

i) X_E is integral, noetherian, one-dim., regular E -scheme

ii) $H^0(X_E, \mathcal{O}_{X_E}) = P_0 = E$

iii) For E'/E finite extⁿ, there is a canon. iso of schemes

$$X_{E'} \xrightarrow{\sim} X_E \otimes_E E'$$

② i) + ii) ✓

For iii): We only consider (and need) $E' = E_h$, the unramified ext^u of deg $h \geq 1$ over E . In this case

$$P_{E'} = \bigoplus_{d \geq 0} (B_{E'}^+)^{\varphi' = \pi'^{ad}}$$

with $\varphi' = \varphi^h$, $\pi' = \pi$ (by unramifiedness),

$$\text{and } B_{E'}^+ = \left(\lim_{\leftarrow} (W_{O_{E'}}(\mathbb{F})[\frac{1}{\pi}]) \right)_I = \left(\lim_{\leftarrow} (W_{O_E}(\mathbb{F})[\frac{1}{\pi}]) \right)_I \cong B_E^+.$$

$$\text{Also } (B_E^+)^{\varphi^h = \pi^{hd}} \cong (B_E^+)^{\varphi = \pi^a} \otimes_E E'$$

argument (basically $\varphi \cdot \pi^{ad}$ gives an $\text{Gal}(E_h/E) \cong \mathbb{Z}/h$ -action on the left side, which by Hilbert 90 must descend to E).

Hence $\bigoplus_d P_{E', dh} \cong \bigoplus_d P_{E, d} \otimes_E E'$ and $\text{Proj}(\bigoplus_d P_{E', dh}) \cong \text{Proj}(\bigoplus_d P_{E, d})$

□

We get a tower of firs. Galois covers

$$\dots \rightarrow X_{E_h} \xrightarrow{\pi_{h,h}} X_{E_h} \rightarrow \dots \xrightarrow{\pi_h} X_E, \text{ for } h|h'$$

$$\text{where } \text{Gal}(X_h/X_h) \cong h\mathbb{Z}/h'\mathbb{Z}.$$

This is called 'generalized Riemannian sphere' in [F-F]
 ↓
 i.e. complete curve + \mathbb{Z} -cover

Def: A vector bundle on a scheme (Y, \mathcal{O}_Y) is a loc. free \mathcal{O}_Y -module of finite constant rank

in category $V\mathcal{G}_Y$

(3)

Def: Let $d \in \mathbb{Z}$, $h \in \mathbb{Z}_{\geq 1}$

i) Set $P_{E_h}[d] := \bigoplus_{r \in \mathbb{Z}} P_{E_h, d+r}$ (graded P_{E_h} -mod.)

$\mathcal{O}_{X_h}(d) := P_{E_h}[d] \sim$ (assoc. \mathcal{O}_{X_h} -module)

ii) If $\gcd(d, h) = 1$, set $\mathcal{O}_X(\frac{d}{h}) := \mathcal{O}_X(d, h) := \pi_{h*}(\mathcal{O}_{X_h}(d)) \in VB_{X_E}$

Expl: $\mathcal{O}_{X_h}(0) = \mathcal{O}_{X_h}$, $h \geq 1$

$\mathcal{O}_X(\frac{d}{1}) = \mathcal{O}_X(d)$, $d \in \mathbb{Z}$

Q: How much of the theory of vec. bun. on X can we get with these?

A: If $F = F^{alg}$, all of it!

For line bundles, i.e. vec. bun. of rank 1, there is the usual exact seq.

$$0 \rightarrow E^* \rightarrow E(X_\bullet)^* \xrightarrow{\text{div}} \text{Div}(X_\bullet) \rightarrow \text{Pic}(X_\bullet) \rightarrow 0$$

$\downarrow \deg$
 \mathbb{Z}

where $E(X)^* = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \text{Frac}(P)^* \mid x, y \text{ homogenous, of same degree} \right\}$,

$\text{Div}(X_\bullet) \ni D \mapsto [\mathcal{O}_X(D)] \in \text{Pic}(X)$ with

$$\Gamma(U, \mathcal{O}_X(D)) := \left\{ f \in E(X) / \text{div}(f)|_U + D|_U \geq 0 \right\}.$$

Since $\deg(\text{div}(f)) = 0$, \deg factors over $\text{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$, which is isom. with inverse $d \mapsto [\mathcal{O}_X(d)]$.

For general vector bundles $E \in VB_X$, $E \in VB_{X_h}$

$$\begin{aligned} \deg(\pi_h^* E) &= h \cdot \deg(E) & \deg(\pi_{h*} E') &= \deg(E') \\ \text{rk}(\pi_h^* E) &= \text{rk}(E) & \text{rk}(\pi_{h*} E') &= h \cdot \text{rk}(E') \end{aligned}$$

④ where degree of a rk or $\text{vector bundle } E$ is defined as the degree of the line bundle $\Lambda^r E$.

The above formulas are a consequence of X_n/X_* being finite stalks of degree n .

Interlude: Hander-Narasimhan formalism

Let \mathcal{C} be an exact category (in the sense of Quillen)

↑ additive + class of 's.e.s.', that is closed under iso's, composit., contains split ext", pullb., pushouts, ...

+ two functions on iso'classes

$$\begin{aligned} \deg : \text{Ob } \mathcal{C} &\rightarrow \mathbb{R} \\ \text{rk} : \text{Ob } \mathcal{C} &\rightarrow \mathbb{N} \end{aligned} \quad \left. \begin{array}{l} \text{additive on s.e.s.} \\ \text{additive on s.e.s.} \end{array} \right\}$$

+ a functor $\mathcal{C} \xrightarrow{F} \text{ab}$ to an abelian cat., s.t.

- F is exact + faithful 'generic fibers'
- F induces bij. $\left\{ \begin{array}{l} \text{strict subobj.} \\ \text{of } X \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{l} \text{subobj. of } \\ F(X) \end{array} \right\}$
- rk is induced from function $\text{rk} : \text{Ob } \mathcal{C} \rightarrow \mathbb{N}$ that satisfies $\text{rk}(X) = 0 \Leftrightarrow X = 0$

(!) • for $u : X \rightarrow Y \in \mathcal{C}$ with $F(u) : F(X) \xrightarrow{\sim} F(Y)$ iso

we have $\deg(X) \leq \deg(Y)$
with equality iff u is iso.

Def: $\mu(X) := \frac{\deg(X)}{\text{rk}(X)} \in \mathbb{R}$ 'slope'

Call $X \in \mathcal{C}$ semi-stable, if every strict subobject X' has $\mu(X') \leq \mu(X)$.

Thm: $X \in \mathcal{C}$ has a unique finite filtration, s.t. the successive quotients are semi-stable of decreasing slope. ⑤
 strictly

Now, apply this to

$$\mathcal{C} = VB_X, \text{ rk } \deg$$

$$\mathcal{A} = E(X)\text{-Mod}$$

$$F: \mathcal{C} \rightarrow \mathcal{A}, E \mapsto E_{\mathbb{Q}}$$

Proposition:

Let $\alpha, \alpha' \in \mathbb{Z}$, $h, h' \in \mathbb{Z}_{\geq 1}$, s.t. $\gcd(\alpha, h) = \gcd(\alpha', h') \geq 1$.

Then i) $\mathcal{O}_X(\frac{\alpha}{h})$ has rank h , slope $\frac{\alpha}{h}$ on X

$$\text{ii)} \Gamma(X, \mathcal{O}_X(\frac{\alpha}{h})) = \begin{cases} \mathbb{C}, & \alpha < 0 \\ P_{E_{\mathbb{Q}}, \alpha} = \mathbb{B}^{q^h = \pi^{\alpha}}, & \alpha \geq 0 \end{cases}$$

$$\text{iii)} \text{Hom}_X(\mathcal{O}_X(\frac{\alpha}{h}), \mathcal{O}_X) \cong \mathcal{O}_X(-\frac{\alpha}{h})$$

$$\text{iv)} \mathcal{O}_X(\frac{\alpha}{h}) \otimes \mathcal{O}_X(\frac{\alpha'}{h'}) \cong \bigoplus_{\text{fin.}} \mathcal{O}_X(\frac{\alpha''}{h''})$$

$$\frac{\alpha}{h} + \frac{\alpha'}{h'} \text{ (reduced)}$$

To compute the cohomology of $E \in VB_X$:

Choose $\infty \in |X|$, let $\mathcal{B}_e := \Gamma(X \setminus \{\infty\}, \mathcal{O}_X)$, $\mathcal{B}_{eR}^+ := \mathbb{B}_{X, \infty}^+$, $\mathcal{B}_{eR} := \mathcal{B}_{eR}^+[\frac{1}{t}]$

Def: The cat. of pairs (M, W) with

$$W \text{ fin. free } / \mathcal{B}_{eR}^+$$

$M \subset W[\frac{1}{t}]$ finite type \mathcal{B}_e -submodule, s.t.

$$M \otimes \mathcal{B}_{eR} \cong W[\frac{1}{t}]$$

⑥

Proposition: $E \mapsto (\Gamma(X, \{e\}, E), \widehat{E}_e)$ gives equiv. of cat.

$$VB_X \simeq C$$

identifying $OSVB_{\text{an}}_X^{\text{rk} = r}$ with $GL_r(\mathbb{B}_{\text{dR}}) \backslash GL_r(\mathbb{B}_{\text{dR}})/GL_r(\mathbb{B}_{\text{dR}}^+)$

Proposition: For $E \in VB_X$: $R\Gamma(X, E) \simeq [M \otimes W \xrightarrow{W \otimes W \xrightarrow{(x,y) \mapsto x-y} M + W} W[\frac{1}{t}]]$

$$\text{i.e. } H^0(X, E) \simeq M \otimes W$$

$$H^1(X, E) \simeq W[\frac{1}{t}] \bigg/ M + W$$

Remark: The proof of the first proposition is an application of the 'descent' theorem of Beauville-Laszlo:

you can glue * modules over $R[\frac{1}{t}]$

and \widehat{R}^\pm along an iso over $\widehat{R}^\pm[\frac{1}{t}]$

* The second prop. can be understood as computing
Čech-cohomology on the covering by an affine
open and a formal open subspace.