

# TILTING EQUIVALENCE FOR THE ÉTALE SITES

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Fix  $K$  a perfectoid base field with tilt  $K^\flat$ , and write  $\varpi \in K$  resp.  $\varpi^\flat \in K^\flat$  for pseudo-uniformizers.

We already have

- tilting equivalence for the sites of analytic topology, identifying rational subsets;
- partial results for the étale sites, including
  - tilting equivalence for perfectoid fields, identifying finite étale extensions;
  - for perfectoid algebras, we have the untilting functor  $R_{\text{fet}}^\flat \rightarrow R_{\text{fet}}$  passing from characteristic  $p > 0$  to characteristic zero, which is fully faithful with essential image being strongly fet perfectoid algebras over  $R$ .

To complete the picture, including a global version for the étale sites of perfectoid spaces, Scholze starts with the strongly étale version and prove, essentially involving the almost purity of Faltings, that the strengthened version coincides with the usual one defined in terms of fet and étale morphisms of adic spaces. The positive characteristic case is known, and the general case is obtained by tilting.

We write fet for finite étale.

The notion of pro-étale topology is NOT discussed in this talk.

## 1. ÉTALE MORPHISMS STRENGTHENED

First recall the classical version of Huber:

**Definition 1.1** (étale morphisms for adic spaces). Fix for simplicity  $k$  a non-archimedean base field.

- (1) A map of affinoid  $k$ -algebras  $(R, R^+) \rightarrow (S, S^+)$  is fet :=
  - $R \rightarrow S$  is fet as a map of  $k$ -algebras
  - $S^+$  equals the integral closure of  $R^+ \rightarrow S$ .
- (2) A map of adic spaces  $f : X \rightarrow Y$  over  $k$  is fet := for some affinoid open cover  $Y = \bigcup_i V_i$ 
  - each pull-back  $U_i = V_i \times_Y X$  is affinoid, and
  - $f : U_i \rightarrow V_i$  is given by a fet map  $(O_X U_i, O_X^+ U_i) \leftarrow (O_Y V_i, O_Y^+ V_i)$ .
- (3) A map of adic spaces  $f : X \rightarrow Y$  is étale := étale everywhere, i.e. for any  $x \in X$  there exists open neighborhoods  $x \in U \subset X$  and  $f(U) \subset V \subset Y$  such that  $f : U \rightarrow V$  factors through

$$U \xrightarrow{j} W \xrightarrow{p} V$$

with  $p$  fet and  $j$  open embedding.

A few useful facts in the (locally noetherian) adic case:

**Facts 1.2.** (1) Both fet morphisms and étale morphisms are open, stable under composition and arbitrary base change. If  $g$  and  $g \circ f$  are both étale then so it is with  $f$ .

The (small) étale site  $X_{\text{et}}$  of  $X$  is the category of étale morphisms toward  $X$ , with covers being jointly surjective families of morphisms.

(2) If  $X \rightarrow Y$  is fet with  $Y$  affinoid and locally noetherian, then  $X$  is also affinoid (actually valid for finite morphisms).

(3) The functor  $O_X : U \in X_{\text{et}} \mapsto O_U U$  is an étale sheaf. For  $X$  affinoid with a finite fetaffinoid cover  $(U_i)$ , the associated Čech complex for the presheaf  $O_X^+$  is exact up to  $\pi$ -power torsions,  $\pi$  being some pseudo-uniformizer of  $k$ .

For perfectoid spaces over  $K$  (the perfectoid base field), one starts with a stronger version:

**Definition 1.3** (strengthened étale morphisms for perfectoid spaces). (1) A map of affinoid perfectoid  $K$ -algebras  $f : (R, R^+) \rightarrow (S, S^+)$  is strongly fet :=

- $f$  is fet as a map of affinoid  $K$ -algebras;
- $f : R^{\circ a} \rightarrow S^{\circ a}$  is fet.

(2) A map of perfectoid spaces  $f : X \rightarrow Y$  over  $K$  is strongly fet if there exists an open affinoid cover  $Y = \bigcup_i V_i$  such that

- each pull-back  $U_i = V_i \times_Y X$  is an affinoid perfectoid space over  $K$ ;
- the restriction  $f : U_i \rightarrow V_i$  gives  $(O_X U_i, O_X^+ U_i) \leftarrow (O_Y V_i, O_Y^+ V_i)$  strongly fet for each  $i$ .

(3) A map of perfectoid spaces  $f : X \rightarrow Y$  over  $K$  is strongly étale if it is strongly étale everywhere on  $X$ , i.e. for any  $x$  there exist open neighborhoods  $x \in U \subset X$  and  $f(U) \subset V \subset Y$  such that  $f : U \rightarrow V$  is a strongly fet morphism composed with an open embedding.

It is already clear that this strengthened version is stable under tilting (and untilting).

**Remark 1.4** (integral structure). Given a fet morphism of affinoid algebras  $(R, R^+) \rightarrow (S, S^+)$ , it is clear that  $S^+$  is uniquely determined by  $R^+$  and the fet map  $R \rightarrow S$ . In this way we see that  $(R, R^+)_{\text{fet}} \cong R_{\text{fet}}$  as categories once a choice of  $R^+$  is specified.

For  $R$  a perfectoid  $K$ -algebra, there could be different choices of  $R^+$ . By definition,  $R^\circ$  is the maximal one; on the other hand, the intersection  $R_{\min}^\circ = \bigcap \{R^+ \subset R\}$  of all possible integral structures of  $R$  is open and integrally closed: it is open as it is a subring containing the open ideal  $R^\circ$ , and it is integrally closed because so it is with each  $R^+$  in the intersection. In general the difference between  $R_{\min}^\circ$  and  $R^\circ$  could be non-trivial.

There are of course cases when  $R^\circ = R_{\min}^\circ$ . For example, in the completed perfection  $R = \mathbb{F}_p((t^{1/p^\infty}))$  of  $\mathbb{F}_p((t))$  we have  $R^\circ = \mathbb{F}_p[[t^{1/p^\infty}]] = R_{\min}^\circ$ , and by tilting arguments we see that the same holds for  $(\mathbb{Q}_p(\mu_{p^\infty}))^\wedge$  and  $(\mathbb{Q}_p(p^{1/p^\infty}))^\wedge$  (the  $p$ -adic completions).

## 2. STABILITY UNDER BASE CHANGE AND COMPOSITION

**Lemma 2.1** (base change). (1) Given affinoid perfectoid spaces  $X = \text{Spa}(A, A^+)$ ,  $Y = \text{Spa}(B, B^+)$ ,  $Z = \text{Spa}(C, C^+)$  and morphisms

$$(A, A^+) \xleftarrow{f} (B, B^+) \rightarrow (C, C^+)$$

with  $f$  strongly fet, then  $X \times_Y Z = \text{Spa}(D, D^+)$  with  $D = A \otimes_B C$  and  $D^+$  the integral closure of  $C^+$  in  $D$ . Moreover  $(C, C^+) \rightarrow (D, D^+)$  is strongly fet.

Globally, for  $X \rightarrow Y$  a strongly fet resp. strongly étale morphism between perfectoid spaces, its base change along an arbitrary map  $Z \rightarrow Y$  of perfectoid spaces remains of the same type, and the map of topological spaces  $|X \times_Y Z| \rightarrow |X| \times_{|Y|} |Z|$  is onto.

(2) In characteristic  $p$ , the classical notion of fet resp. étale morphisms (in the adic sense) coincides with the strengthened one.

*Proof.* (1) Note that  $D = A \otimes_B C$  is already complete and finite projective as a  $C$ -module as  $B \rightarrow A$  is fet, which implies that  $D$  is complete and perfectoid. It is also clear that the base change  $D^{\circ a} = A^{\circ a} \otimes_{B^{\circ a}} C^{\circ a}$  is fet over  $C^{\circ a}$ .

The global statements are reduced to the local affinoid case and the surjectivity does involve some delicate construction of valuative spectra, which we refer to the arguments in [2] 3.9(i) for details.

(2) For a map of affinoid algebras  $(R, R^+) \rightarrow (S, S^+)$  in characteristic  $p > 0$ , fet in the sense of Huber,  $R$  being perfectoid implies the same with  $S$ , and  $R^{\circ a} \rightarrow S^{\circ a}$  is also fet due to the positive characteristic.  $\square$

Hence strongly fet morphisms and strongly étale morphisms are stable under base change.

**Lemma 2.2** (completion and decompletion). (1) Let  $A$  be a flat  $K^\circ$ -algebra henselian along  $(\varpi)$  with  $\varpi$ -adic completion  $\hat{A}$ , the  $\varpi$ -adic completion functor is an equivalence:

$$A[\varpi^{-1}]_{\text{fet}} \rightarrow \hat{A}[\varpi^{-1}]_{\text{fet}}, B \mapsto \hat{B}.$$

(2) Let  $(A_i)_i$  be a filtered inductive system of complete flat  $K^\circ$ -algebras, and  $A = (\varinjlim_i A_i)^\wedge$  the completed inductive limit, which is clearly flat over  $K^\circ$ . Then we have a category equivalence:

$$A[\varpi^{-1}]_{\text{fet}} \cong 2 - \varinjlim (A_i[\varpi^{-1}]_{\text{fet}}).$$

(3) In particular, for  $(R_i)$  a filtered system of perfectoid  $K$ -algebras and  $R = (\varinjlim R_i)^\wedge$  the completed inductive limit, we have  $R_{\text{fet}} \cong 2 - \varinjlim (R_{i, \text{fet}})$ .

If moreover  $K$  is of characteristic  $p$  and  $(R, R^+) = ((S, S^+)^{1/p^\infty})^\wedge$  is  $p$ -finite, i.e. completed perfection of some affinoid algebra  $(S, S^+)$  reduced and topologically of finite type. Then  $R_{\text{fet}} \cong S_{\text{fet}}$ .

*Proof.* (1) This is done in Gabber-Ramero (5.4.53).

(2) Use standard reductions of filtered inductive limits in combination with the completion functor in (1):

$$2 - \varinjlim (A_i[\varpi^{-1}]_{\text{fet}}) \cong (\varinjlim A_i[\varpi^{-1}])_{\text{fet}} \cong A[\varpi^{-1}]_{\text{fet}}.$$

(3) This is the special case of (2) for  $R = (\varinjlim S^{1/p^n})^\wedge$  and  $S_{\text{fet}}^{1/p^n} \cong S_{\text{fet}}$  for all  $n$ .  $\square$

The lemma above allows us to descend, after tilting if necessary, to (locally) noetherian adic spaces:

**Proposition 2.3.** (1) Let  $f : X \rightarrow Y$  be a strongly fet morphism of perfectoid spaces over  $K$ , and  $V \subset Y$  an open affinoid perfectoid subspace. Then  $U = V \times_Y X$  is also affinoid perfectoid, and the map of global sections  $(O_Y V, O_Y^+ V) \rightarrow (O_X U, O_X^+ U)$  is strongly fet.

(2) In characteristic  $p > 0$ , if  $f : X \rightarrow Y$  is étale, then for any  $x \in X$ , there exist open affinoid perfectoid neighborhoods  $x \in U \subset X$  and  $f(U) \subset V \subset Y$  such that  $f : U \rightarrow V$  is pulled back from some étale morphism  $U_0 \rightarrow V_0$  of affinoid noetherian adic spaces.

In fact one may take  $V \rightarrow V_0$  as the completed perfection of  $V_0$ .

*Proof.* (1) Tilting reduces the proof to positive characteristic, and up to shrinking  $K$  to a perfectoid subfield we may assume that  $K^\circ = K_{\min}^\circ$  in the sense of Remark 1.4 and  $Y = V = \text{Spa}(R, R^+)$  is affinoid with  $R^+$  a  $K^\circ$ -algebra.

We proceed to show that  $X$  is affinoid itself. Since  $(R, R^+)$  is the completion of the filtered inductive limit  $\varinjlim (R_i, R_i^+)$  of its  $p$ -finite affinoid perfectoid subalgebras, the morphism  $X \rightarrow Y$  descends to the case where  $Y = \text{Spa}(R, R^+)$  is  $p$ -finite. In this case a strongly fet morphism between perfectoid spaces is the same as a fet morphism defined in terms of adic spaces, and  $X \rightarrow Y$  descends further to some fet morphism  $X_0 \rightarrow Y_0 = \text{Spa}(R_0, R_0^+)$  with  $(R_0, R_0^+)$

reduced and topologically of finite type, and  $(R, R^+) = ((R_0, R_0^+)^{1/p^\infty})^\wedge$ . Then Huber's result in the adic case implies that  $X_0$  is affinoid with  $(O_{Y_0} Y_0, O_{Y_0}^+ Y_0) \rightarrow (O_{X_0} X_0, O_{X_0}^+ X_0)$  fet.

(2) Similar to the arguments in (1), we are reduced to the case where  $X$  is a rational subset of some fet cover of  $Y$  and further to the  $p$ -finite case which is descendable by completed perfection.  $\square$

**Corollary 2.4** (composition). *Strongly fet morphisms and strongly étale morphisms are both open and stable under composition.*

*Proof.* Again reduce to positive characteristic by tilting, and reduce further to Huber's classical case of étale morphisms for locally noetherian adic spaces upon  $p$ -finite devissage.  $\square$

### 3. ALMOST PURITY AND ÉTALE TOPOLOGY

So far the strengthened étale morphisms already define a topology, which will be indicated by the subscript sfet. This section discusses its relation to the classical one.

**Theorem 3.1** (almost purity). *Let  $(R, R^+)$  be an affinoid perfectoid  $K$ -algebra, with  $X = \text{Spa}(R, R^+)$  and  $X^\flat = \text{Spa}(R^\flat, R^{\flat+})$  the tilt. Then:*

(1) *For any open affinoid perfectoid subspace  $U \subset X$  the functor*

$$U_{\text{sfet}} \rightarrow (O_X U)_{\text{fet}}, (V \rightarrow U) \mapsto (O_U U \rightarrow O_V V),$$

*where  $U_{\text{sfet}}$  is the category of strongly fet morphisms toward  $U$ , is a category equivalence respecting covers (reduced to faithfully flat objects)*

(2) *If  $R \rightarrow S$  is a fet cover of  $K$ -algebras, then  $S$  is perfectoid and  $R^{\circ a} \rightarrow S^{\circ a}$  is fet, and  $S^{\circ a}$  is uniformly almost finitely generated as an  $R^{\circ a}$ -module.*

*In particular, the strengthened version of fet resp. étale morphisms between perfectoid spaces is the same as the classical one computed in adic spaces.*

*Proof.* We may assume that  $U = X = \text{Spa}(R, R^+)$ .

(1) The full faithfulness is clear: if  $V \rightarrow X$  is strongly fet, then  $V$  is affinoid and is determined by the fet  $(O_X X)^{\circ a}$ -algebra  $(O_V V)^{\circ a}$ .

It remains to prove the essential surjectivity: start with an arbitrary  $S \in R_{\text{fet}}$ , find  $Y \rightarrow X$  strongly fet so that  $O_Y Y = S$ .

We first do this locally: for any  $x \in X$ , there exists an affinoid perfectoid neighborhood  $x \in U \subset X$  (not the  $U$  specified in the statement) and  $V \rightarrow U$  strongly fet such that  $O_V V \cong O_X U \otimes_{O_X U} S$  as  $O_X U$ -algebras. Note that  $U$  could be taken arbitrarily small. In fact using Hensel lifting and tilting we have

$$2 - \varinjlim_{x \in U} (O_X U)_{\text{fet}} \cong \hat{k}(x)_{\text{fet}} \cong \hat{k}(x^\flat)_{\text{fet}} \cong 2 - \varinjlim_{x^\flat \in U^\flat} (O_{X^\flat} U^\flat)_{\text{fet}}.$$

The object in  $2 - \varinjlim_{x \in U} (O_X U)_{\text{fet}}$  represented by  $R \rightarrow S$  passes to some  $V^\flat \rightarrow U^\flat$  fet for some  $U \subset X$  small affinoid neighborhood of  $x$  and untilting gives  $V \rightarrow U$  strongly fet. Thus  $X$  admits a finite cover  $\bigcup_i U_i$  by open affinoid perfectoid subspaces, and each  $U_i$  is equipped with a strongly fet cover  $V_i \rightarrow U_i$  defined by the same  $R$ -algebra  $S$  via  $O_{V_i} V_i = S_i = O_{U_i} U_i \otimes_R S$  such that  $V_i = \text{Spa}(S_i, S_i^+)$ , where  $S_i^+$  is the integral closure of  $O_X^+ U_i$  in  $S_i$ .

Note that these  $V_i$ 's do glue to an perfectoid space  $Y$ , where the gluing condition is lifted from the one for the open affinoid cover  $X = \bigcup_i U_i$  as the base change along  $R \rightarrow S$  pulls back the gluing condition for the cover (one may write  $U_i \cap U_j = \bigcup_\alpha W_\alpha$  with open affinoid  $W_\alpha = \text{Spa}(R'_\alpha, R'^+_\alpha)$  and then  $V_i \cap V_j = \bigcup_\alpha W'_\alpha$  with  $W'_\alpha = \text{Spa}(S'_\alpha, S'^+_\alpha)$  pulled back using  $S'_\alpha = R'_\alpha$  and verify the cocycle condition similarly).

Thus we get  $Y \rightarrow X$  a strongly fet morphism between perfectoid spaces, which implies that  $Y = \mathrm{Spa}(A, A^+)$  is affinoid itself. Finally the maps of  $S$ -algebras

$$\prod_i O_X U_i \otimes_R S \rightarrow \prod_{i,j} O_X(U_i \cap U_j) \otimes_R S$$

and

$$\prod_i O_Y V_i \rightarrow \prod_{i,j} O_Y(V_i \cap V_j)$$

are identical, hence their kernels coincide, i.e.  $A = S$ .

(2) This is immediate from (1) by taking global sections and tilting.  $\square$

**Corollary 3.2** (fet base change). *Both fet and étale morphisms between perfectoid spaces (computed in adic spaces) are stable under arbitrary base change along morphisms between perfectoid spaces, and the evident map  $|X \times_Y Z| \rightarrow |X| \times_{|Y|} |Z|$  similar to the one used in 2.1 is onto.*

*Proof.* This is reduced to the positive characteristic case in 2.1(2) by tilting.  $\square$

**Definition 3.3** (étale site). For  $X$  a perfectoid space, the (small) étale site  $X_{\mathrm{et}}$  is the category of étale morphisms toward  $X$ , with jointly surjective families of morphisms as covers. Denote by  $X_{\mathrm{et}}^{\sim}$  the associated topos of set-valued sheaves.

Given  $f : X \rightarrow Y$  a morphism between perfectoid spaces, we have  $f : X_{\mathrm{et}} \rightarrow Y_{\mathrm{et}}$  and  $f : X_{\mathrm{et}}^{\sim} \rightarrow Y_{\mathrm{et}}^{\sim}$  the associated morphisms between sites and topoi.

**Theorem 3.4** (tilting equivalence for étale sites). *For  $X$  a perfectoid space over  $K$  with tilt  $X^b$  over  $K^b$ , the tilting operation induces an isomorphism of sites  $X_{\mathrm{et}} \cong X_{\mathrm{et}}^b$  and it is functorial in  $X$ .*

*Proof.* The proof is reduced to the following commutative diagram for affinoid perfectoid algebras

$$(R, R^+)_{\mathrm{fet}} \cong R_{\mathrm{fet}}^{\circ a} \cong R_{\mathrm{fet}}^{b \circ a} \cong (R^b, R^{b+})_{\mathrm{fet}}.$$

$\square$

Similar to the classical case, the structure sheaf functor is a étale sheaf:

**Proposition 3.5** (structure sheaf). *For  $X$  a perfectoid space over  $K$ , the functor*

$$X_{\mathrm{et}}^{\mathrm{op}} \rightarrow K - \mathrm{Perf}, U \mapsto O_U U$$

*is a sheaf, which is denoted as  $O_X$ . If moreover  $X$  is affinoid, then  $H^i(X_{\mathrm{et}}, O_X^{\circ a})$  vanishes for any  $i > 0$ .*

*Proof.* The exactness of

$$0 \rightarrow (O_X X)^{\circ a} \rightarrow \prod_i (O_{U_i} U_i)^{\circ a} \rightarrow \prod_{i,j} (O_{U_{ij}} U_{ij})^{\circ a}$$

given by an étale cover  $(U_i)$  of  $X$  with  $U_{i,j} = U_i \times_X U_j$  is reduced first to the characteristic  $p$  case by tilting, then to the  $p$ -finite case, and finally descend finally to the classical noetherian case treated by Huber, which proves that the Čech complex of the presheaf  $O_X^+$  associated to an étale cover is acyclic up to  $\pi$ -torsions,  $\pi$  being suitable topological nilpotent element in  $K$ .  $\square$

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