Perfectoid Spaces

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We closely follow Chapter 6 in [Sch12]. Let K be a perfectoid field with maximal ideal $\mathfrak{m}$ and $\varpi \in \mathfrak{m}$ such that $|p| \leq |\varpi| < 1$.

Definition 1. *A perfectoid affinoid K-algebra is an affinoid K-algebra (R, R^+) such that R is a perfectoid K-algebra.*

Remark 2. - *If (R, R^+) is a perfectoid affinoid K-algebra, we have $\mathfrak{m}R^\circ \subseteq R^+ \subseteq R^\circ$. In particular, if R^+ is a K°-algebra, R^+ is almost equal to R°.*

Proof. Let $x \in \mathfrak{m}R^\circ$. Then x is topologically nilpotent, so some power of x lies in R^+, i.e. x is integral over R^+. As R^+ is integrally closed in R by definition, the claim follows. $\square$

Lemma 3. *The categories of perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras are equivalent.*

Proof. First of all, R° is integrally closed in R: If $x \in R$ is integral over R°, $R^\circ[x]$ is a finitely generated R°-module and hence $R^\circ[x] \subseteq aR^\circ$ for some $a \in K$. Hence, x is power bounded, i.e. $x \in R^\circ$. This implies that a subring $R^+ \subseteq R^\circ$ is integrally closed in R° if and only if it is integrally closed in R.

We claim that we have a bijective correspondence

$$\{\text{open integrally closed subrings of } R^\circ\} \overset{\cong}{\leftrightarrow} \{\text{integrally closed subrings of } R^\circ/\varpi R^\circ\}$$

mapping R^+ in the left hand side to $R^+/\varpi R^\circ$ and S in the right hand side to its preimage under the canonical projection $pr : R^\circ \to R^\circ/\varpi R^\circ$. This is well-defined by Remark 2.

Firstly, let $S \subseteq R^\circ/\varpi R^\circ$ be an integrally closed subring. Then $pr^{-1}(S)$ contains ϖR° and hence is open. If $x \in R^\circ$ is integral over $pr^{-1}(S)$, $pr(x)$ is integral over S which implies $pr(x) \in S$. Hence $x \in pr^{-1}(S)$. On the other hand, if $R^+ \subseteq R^\circ$ is open and integrally closed and $\overline{x}$ is integral over $R^+/\varpi R^\circ$, choose a lift $x \in R^\circ$ of $\overline{x}$. Then there exists a monic polynomial $f \in R^+[X]$ such that $f(x) \in \varpi R^\circ \subseteq R^+$. Hence, x is integral over R^+, i.e. lies in R^+ and $\overline{x} \in R^+/\varpi R^+$. This implies the claim.

Now we have $R^\circ/\varpi R^\circ \cong {R^\flat}^\circ/\varpi^\flat {R^\flat}^\circ$ from the construction of the tilting equivalence for perfectoid algebras. Applying the claim to R and $R^\flat$ instead of R yields a correspondence between open integrally closed subrings of R° and open integrally closed subrings of $R^{\flat\circ}$. $\square$

Theorem 4. *Let (R, R^+) be a perfectoid affinoid K-algebra and $X = \mathrm{Spa}(R, R^+)$ with tilt $(R^\flat, R^{\flat +})$ and $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$.*

(i) We have a homeomorphism $X \cong X^\flat$ mapping $x \in X$ to the valuation $x^\flat \in X^\flat$ defined by $|f(x^\flat)| = |f^\sharp(x)|$. It identifies rational subsets.

(ii) For any rational subset $U \subseteq X$ mapping to $U^\flat \subseteq X^\flat$ the complete affinoid K-algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is perfectoid with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}^+_{X^\flat}(U^\flat))$.

These are the first two parts of Theorem 6.3 in [Sch12]. The goal of these notes is to outline the proof of the theorem above. The fact that $x^\flat$ is a valuation can be seen analogously to the case when R is a field (see the proof of [Sch12][Proposition 3.6]).

To see the continuity of the map $X \to X^\flat$ let $U(\frac{f_1,\ldots,f_n}{g}) \subseteq X^\flat$ be a rational subset. We may assume that f_n is a power of $\varpi^\flat$ without changing the rational subset $U(\frac{f_1,\ldots,f_n}{g})$ (cf. [Sch12][Remark 2.8]). Then the preimage of $U(\frac{f_1,\ldots,f_n}{g})$ is given by $U(\frac{f_1^\sharp,\ldots,f_n^\sharp}{g^\sharp})$ and this is rational because $f_n^\sharp \in K^\times$, i.e. the ideal generated by $f_1^\sharp, \ldots, f_n^\sharp$ is equal to R. Hence, $X \to X^\flat$ is continuous.

Lemma 5. *Let $U = U(\frac{f_1,\ldots,f_n}{g}) \subseteq X^\flat$ be rational with preimage $U^\sharp \subseteq X$. Without changing U we may assume that all $f_i, g \in R^{\flat\circ}$ and that $f_n = \varpi^{\flat N}$ for some N.*

(i) Let $A := R^\circ\langle(\frac{f_i}{g})^{1/p^\infty\sharp}\rangle$ be the ϖ-adic completion of $R^\circ[(\frac{f_i}{g})^{1/p^\infty\sharp}] \subseteq R[\frac{1}{g^\sharp}]$. Then A^a is a perfectoid $K^{\circ a}$-algebra.

(ii) $\mathcal{O}_X(U^\sharp)$ is a perfectoid K-algebra and $\mathcal{O}_X(U^\sharp)^{\circ a} = A^a$.

(iii) $\mathcal{O}_{X^\flat}(U)$ is the tilt of $\mathcal{O}_X(U^\sharp)$.

Proof. (i) in characteristic p: If K is of characteristic p we may identify K with $K^\flat$ and R with $R^\flat$. Thus, we may leave out the superscripts $\sharp$ and $\flat$. By definition, A is ϖ-adically complete and ϖ-torsion free.

We have a surjective homomorphism $\varphi : R^\circ[T_i^{1/p^\infty}] \to R^\circ[(\frac{f_i}{g})^{1/p^\infty}]$. Its kernel contains the ideal I generated by the elements $T_i^{1/p^m} g^{1/p^m} - f_i^{1/p^m}$. Hence we obtain a surjective homomorphism $\overline{\varphi} : R^\circ[T_i^{1/p^\infty}]/I \to R^\circ[(\frac{f_i}{g})^{1/p^\infty}]$. We claim that this is an almost isomorphism. The exact sequence

$$0 \to \ker(\overline{\varphi}) \to R^\circ[T_i^{1/p^\infty}]/I \to R^\circ[(\frac{f_i}{g})^{1/p^\infty}] \to 0$$

induces the exact sequence

$$0 \to \ker(\overline{\varphi}) \otimes_{K^\circ} K \to R[T_i^{1/p^\infty}]/\langle I\rangle \to R[(\frac{f_i}{g})^{1/p^\infty}] \to 0.$$

But in $R[T_i^{1/p^\infty}]/\langle I\rangle$ we have $gT_n = \varpi^N$. Hence, g is invertible, $T_i = \frac{f_i}{g}$ and the right map is an isomorphism. This implies that $\ker(\overline{\varphi}) \otimes_{K^\circ} K = 0$ which means that for each $f \in \ker(\overline{\varphi})$ there exists some k such that $\varpi^k f = 0$. Then $(\varpi^{k/p^m} f)^{p^m} = 0$ for each $m \geq 1$ and because I is closed under taking p-power roots, we have $\varpi^{k/p^m} f = 0$. This shows that $\mathfrak{m} f = 0$ and hence $\overline{\varphi}$ is almost injective and consequently an almost isomorphism. This implies that the horizontal maps in the commutative diagram

$$\begin{array}{ccc} R^\circ[T_i^{1/p^\infty}]/\langle I, \varpi \rangle & \longrightarrow & R^\circ[(\frac{f_i}{g})^{1/p^\infty}]/\varpi \\ \uparrow & & \uparrow \\ R^\circ[T_i^{1/p^\infty}]/\langle I, \varpi^{1/p} \rangle & \longrightarrow & R^\circ[(\frac{f_i}{g})^{1/p^\infty}]/\varpi^{1/p} \end{array}$$

are almost isomorphisms. The left Frobenius map is an isomorphism by definition of I and hence the right Frobenius map is an almost isomorphism. This concludes the proof of (i) in characteristic p.

We will now show that (i) implies (ii) for K of arbitrary characteristic. We have inclusions

$$R^\circ[(\frac{f_i}{g})^\sharp] \subseteq R^\circ[(\frac{f_i}{g})^{1/p^\infty \sharp}] \subseteq \varpi^{-nN} R^\circ[(\frac{f_i}{g})^\sharp].$$

The first inclusion is obvious. To see the second one note that $\frac{1}{g^\sharp} = \varpi^{-N} \frac{f_n^\sharp}{g^\sharp} \in \varpi^{-N} R^\circ[(\frac{f_i}{g})^\sharp]$. But every element of $R^\circ[(\frac{f_i}{g})^{1/p^\infty \sharp}]$ is a linear combination with coefficients in $(\frac{1}{g^\sharp})^n R^\circ$ by terms contained in $R^\circ[(\frac{f_i}{g})^\sharp]$.

Because the quotient of $R^\circ[(\frac{f_i}{g})^{1/p^\infty \sharp}]$ by $R^\circ[(\frac{f_i}{g})^\sharp]$ is killed by a finite power of ϖ, we still have inclusions after ϖ-adic completion and we get

$$R^\circ\langle(\frac{f_i}{g})^\sharp\rangle \subseteq R^\circ\langle(\frac{f_i}{g})^{1/p^\infty \sharp}\rangle \subseteq R\langle(\frac{f_i}{g})^\sharp\rangle = \mathcal{O}_X(U^\sharp).$$

Thus, $\mathcal{O}_x(U^\sharp) = (A^a)_*[\varpi^{-1}]$ is perfectoid by part (i) and [Sch12][Theorem 5.2].

Finally, we will show (i) and (iii) for general K. As before A is flat and ϖ-adically complete. We still have a surjective ring homomorphism $R^\circ[T_i^{1/p^\infty}] \to A$ and its kernel contains the ideal I generated by all $T_i^{1/p^m} g^{1/p^m \sharp} - f_i^{1/p^m \sharp}$. Now we see from part (ii) in characteristic p that $(\mathcal{O}_{x^\flat}(U), \mathcal{O}^+_{X^\flat}(U))$ is a perfectoid affinoid $K^\flat$-algebra. Let (S, S^+) be its untilt. The map $\mathrm{Spa}(S, S^+) \to X$ induced by the untilt of $(R^\flat, R^{\flat +}) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ factors over $U^\sharp$ and the universal property of the structure presheaf yields a homomorphism $(\mathcal{O}_X(U^\sharp), \mathcal{O}_X^+(U^\sharp)) \to (S, S^+)$. The composite

$$R^\circ\langle T_i^{1/p^\infty}\rangle^a \to R^\circ\langle(\frac{f_i}{g})^{1/p^\infty \sharp}\rangle^a \to \mathcal{O}_X(U^\sharp)^{\circ a} \to S^{\circ a}$$

is the untilt of the morphism

$$R^{\flat\circ}\langle T_i^{1/p^\infty}\rangle^a \to \mathcal{O}_{X^\flat}(U)^{\circ a}.$$

If $I^\flat$ is the ideal of $R^{\flat\circ}[T_i^{1/p^\infty}]$ defined analogously to I this implies that

$$R^\circ\langle T_i^{1/p^\infty}\rangle/\langle I, \varpi\rangle^a \to A^a/\varpi \to S^{\circ a}/\varpi \tag{1}$$

is the untilt of

$$R^{\flat\circ}\langle T_i^{1/p^\infty}\rangle/\langle I^\flat, \varpi^\flat\rangle^a \xrightarrow{\cong} \mathcal{O}^{\circ a}_{X^\flat}(U)/\varpi^\flat$$

which is an isomorphism by the proof of (i) for characteristic p. Hence the composite (1) is an isomorphism and as the first morphism is induced by a surjective map, both individual morphisms are isomorphisms. In particular $A^a/\varpi \cong S^{\circ a}/\varpi$ which implies that A^a is perfectoid and $S \cong (A^a)_*[\varpi^{-1}] = \mathcal{O}_X(U^\sharp)$ is a perfectoid K-algebra. $\square$

For the surjectivity of $X \to X^\flat$ we need a technical approximation lemma which makes up for the fact that $\sharp$ is not surjective.

Lemma 6. *Let (R, R^+) be a perfectoid affinoid K-algebra with tilt $(R^\flat, R^{\flat+})$, $X = \mathrm{Spa}(R, R^+)$ and $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$. Then for any $f \in R$, $c \geq 0$ and $\epsilon > 0$ there exists $g_{c,\epsilon} \in R^\flat$ such that for all $x \in X$*

$$|f(x) - g^\sharp_{c,\epsilon}(x)| \leq |\varpi|^{1-\epsilon} \max(|f(x)|, |\varpi|^c).$$

If $\epsilon < 1$ we have

$$\max(|f(x)|, |\varpi|^c) = \max(|g^\sharp_{c,\epsilon}(x)|, |\varpi|^c).$$

Proof. See Lemma 6.5 and Corollary 6.6 (i) in [Sch12]. $\square$

Corollary 7. *Let (R, R^+) be a perfectoid affinoid K-algebra with tilt $(R^\flat, R^{\flat+})$, $X = \mathrm{Spa}(R, R^+)$ and $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$.*

(i) For any $x \in X$, the completed residue field $\widehat{K(x)}$ is a perfectoid field.

(ii) The morphism $X \to X^\flat$ is a homeomorphism identifying rational subsets.

Proof. (i) for $\mathrm{char}(K) = p$: From Lemma 5 (ii) we know that $\mathcal{O}_X(U)^{\circ a}$ is perfectoid for any rational subset U. This implies that the ϖ-adic completion of $\mathcal{O}^{\circ a}_{X,x} = \varinjlim \mathcal{O}_X(U)^{\circ a}$ is a perfectoid $K^{\circ a}$-algebra. By [Sch12][Prop. 2.25] we have

$$\widehat{K(x)} = \widehat{K(x)^\circ}[\varpi^{-1}] = \widehat{\mathcal{O}^\circ_{X,x}}[\varpi^{-1}] = (\widehat{\mathcal{O}^\circ_{X,x}}^a)_*[\varpi^{-1}] = (\widehat{\mathcal{O}^{\circ a}_{X,x}})_*[\varpi^{-1}]$$

and thus $\widehat{K(x)}$ is perfectoid. As it is also a nonarchimedean field, it is a perfectoid field.
(ii): Let $U = U(\frac{f_1,\ldots,f_n}{g}) \subseteq X$ be a rational subset with $f_n = \varpi^N$ and let $\epsilon < 1$. By Lemma 6 we can find $f_i^\flat, g^\flat \in R^\flat$ such that

$$|f_i(x) - f_i^{\flat\sharp}(x)| \leq |\varpi|^{1-\epsilon} \max(|f_i(x)|, |\varpi|^{N+1})$$

and similarly for g. A calculation shows that $U = U(\frac{f_1^{\flat\sharp},\ldots,f_n^{\flat\sharp}}{g^{\flat\sharp}})$. Hence, every rational subset is the preimage of a rational subset and we have shown that $X \to X^\flat$ identifies rational subsets.

As X is T_0, this implies injectivity: If two points in X would be mapped to the same point of $x \in X^\flat$, the preimage of each rational subset containing x would contain all points mapped to x so the two points which are mapped to x could not be separated by a rational subset.

Now any $x \in X^\flat$ factors as $R^\flat \to \widehat{K(x)} \to \Gamma \cup \{0\}$ and we can untilt $R^\flat \to \widehat{K(x)}$ and each valuation of the perfectoid field $\widehat{K(x)}$ by [Sch12][Prop. 3.6]. This shows that $X \mapsto X^\flat$ is surjective. Using this we can copy the proof of part (i) for general K. $\square$

The previous corollary shows part (i) of Theorem 4. Let us now prove part (ii): By Corollary 7 (ii) and Lemma 5 (ii) $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is perfectoid affinoid. It is characterized among the perfectoid affinoid K-algebras by its universal property: It is the unique perfectoid affinoid K-algebra such that for any $\varphi : (R, R^+) \to (S, S^+)$ into a perfectoid affinoid K-algebra (S, S^+) for which the image of $\mathrm{Spa}(\varphi)$ is contained in U there exists a unique morphism of perfectoid affinoid K-algebras $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (S, S^+)$ which makes the obvious diagram commutative. Tilting this diagram shows that its tilt has the universal property characterizing $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$ in the category of perfectoid affinoid $K^\flat$-algebras.

References

[Sch12] Peter Scholze. Perfectoid spaces. *Publications mathématiques de l'IHÉS*, 116(1):245–313, 2012.