

K perfectoid field $\left[\Leftrightarrow \text{complete n.a. field of res. char. } p > 0 \text{ whose rank-1-valuation is nondiscrete, Frobenius surjective on } K^\circ/\pi \right]$

$\pi = \varpi \in K^\times \text{ with } |\varpi| \leq |\pi| < 1$

$m = \text{max. ideal in } K^\circ$

Definition: (i) A perfectoid K -algebra is a Banach K -algebra R such that R° $\left[:= (\text{the set of power bounded elts.}) := \{x \in R \mid \{x^n \mid n \geq 0\} \text{ is bounded}\} \right]$

is open and bounded, and Frobenius $\Phi: R^\circ/\pi \rightarrow R^\circ/\pi$ is surjective.

new Category K -Perf, with morphisms = continuous K -algebra morphisms.

(ii) A perfectoid K^{0a} -algebra is a π -adically complete flat K^{0a} -algebra A on which Frobenius induces isomorphism $\Phi: A/\pi^{1/p} \xrightarrow{\sim} A/\pi$

new Category K^{0a} -Perf, with morphisms = K^{0a} -algebra morphisms

Examples: (i) $\left(\varinjlim_n K^\circ[\pi^{1/p^n}] \right)^a \left[\frac{1}{\pi} \right] \in K\text{-Perf}$

$$\left(\quad - \quad \right)^a \in K^{0a}\text{-Perf}$$

(ii) Let $\text{char}(K) = p$ and R a Banach K -algebra with $R^\circ \subseteq R$ bounded open.

Then: R perfectoid $\stackrel{(1)}{\Leftrightarrow} R^\circ$ perfect $\left[\text{perfect} \Leftrightarrow \text{Frobenius bijective} \right]$

$\stackrel{(2)}{\Leftrightarrow} R$ perfect

Proof: " \Leftarrow " : use $(R \setminus R^\circ)^p \cap R^\circ = \emptyset$ " \Rightarrow " : use that K is perfect [Sch, Lem. 3.2]

"(1)": By induction on $n \geq 0$ we prove $\Phi: R^\circ/\pi^{n/p} \xrightarrow{\sim} R^\circ/\pi^n$ (all n)

The case $n=1$ is Proposition 3 below. Next consider

$$\begin{array}{ccc} (\pi^{1/p})/(R^\circ) & \xrightarrow[\pi^{-1/p}]{} & R^\circ/\pi^{\frac{n-1}{p}} \\ \Phi \downarrow & & \Phi \downarrow \quad \leftarrow \simeq \text{ by induction hypothesis} \\ (R^\circ)/(\pi^n) & \xrightarrow[\pi^{-1}]{} & R^\circ/\pi^{n-1} \end{array}$$

Therefore also \downarrow on LHS is \simeq

$$\begin{array}{ccccc} 0 \rightarrow (\pi^{1/p})/(R^\circ) & \rightarrow & R^\circ/(\pi^{1/p}) & \rightarrow & R^\circ/\pi^{1/p} \rightarrow 0 \\ \Phi \downarrow & & \Phi \downarrow & & \Phi \downarrow \\ 0 \rightarrow (R^\circ)/(\pi^n) & \rightarrow & R^\circ/\pi^n & \rightarrow & R^\circ/\pi \rightarrow 0 \end{array}$$

□

Definition = (iii) A perfectoid K^{ur}/π -algebra is a flat K^{ur}/π -algebra \bar{A} on which Frobenius induces an isomorphism $\Phi: \bar{A}/\pi^{1/p} \xrightarrow{\sim} \bar{A}$.

now Category $(K^{\text{ur}}/\pi)\text{-Perf}$, with morphisms = (K^{ur}/π) -algebra morphisms.

Theorem 1 (Tilting equivalence) Equivalences of categories

$$K\text{-Perf} \xleftarrow[\sim]{(a)} K^{\text{ur}}\text{-Perf} \xrightarrow[\sim]{(b)} (K^{\text{ur}}/\pi)\text{-Perf}$$

$$K^b\text{-Perf} \xleftarrow[(d)]{\sim} K^{b\text{ur}}\text{-Perf} \xrightarrow[(c)]{\sim} (K^{b\text{ur}}/\pi^b)\text{-Perf}$$

PF: (a)+(d) via almost math's

(b)+(c) via deformation theory (cotangent cplx.)

Lemma 2: Let M be a K^{oa} -module. Then \mathbb{N}^o -adic complete.

- (i) M flat over K^{oa} $\stackrel{(1)}{\Leftrightarrow} M_{\mathfrak{A}}$ flat over $K^o \stackrel{(2)}{\Leftrightarrow} M_{\mathfrak{A}}$ has no \mathbb{N} -torsion
- (ii) If N is a flat K^o -module, then N^a is flat over K^{oa} , and

$$(N^a)_{\mathfrak{A}} = \left\{ x \in N \left[\frac{1}{\pi} \right] : \varepsilon x \in N \text{ for all } \varepsilon \in \mathfrak{m} \right\}.$$

(iii) Let M be flat over K^{oa} . Then M is \mathbb{N}^o -adically complete.

M is \mathbb{N} -adically complete $\Leftrightarrow M_{\mathfrak{A}}$ is \mathbb{N} -adically complete.

Proof: (i) M flat $\stackrel{[Sch \text{ 4.7}][GR]}{\Leftrightarrow} \text{Tor}_i^{K^o}(M_{\mathfrak{A}}, N) \stackrel{\text{almost}}{=} 0$ for all $i > 0$, all K^o -modules N

This shows " $\stackrel{(1)}{\Leftarrow}$ ".

" $\stackrel{(1)+(2)}{\Rightarrow}$ ": Consider $\text{Tor}_1^{K^o}(M_{\mathfrak{A}}, K^o/\mathbb{N}) \rightarrow M_{\mathfrak{A}} \xrightarrow{\pi} M_{\mathfrak{A}} \rightarrow M_{\mathfrak{A}/\mathbb{N}} \rightarrow 0$

$M_{\mathfrak{A}} \stackrel{df}{=} \text{Hom}_{K^{oa}}(K^{oa}, M) = \text{Hom}_{K^o}(M, M_{\mathfrak{A}})$ \leftarrow
 $\text{Hom}_{K^{oa}}(K^{oa}, M) \stackrel{\text{almost}}{=} \text{Hom}_{K^o}(M, M_{\mathfrak{A}})$ without non trivial almost-zero-els. f
 $(M_{\mathfrak{A}})^a$ [Sch, Prop. 4.4] [because $\mathfrak{m} f(\mathfrak{m}) = f(\mathfrak{m}^2) = f(\mathfrak{m})$],
hence without \mathbb{N} -torsion

" $\stackrel{(2)}{\Leftarrow}$ ": May assume $N := M_{\mathfrak{A}}$ finitely generated. Choose maximal free submodule N' .
Then find $n \in N$ with $\pi^n N \subset N'$. By assumption π^n is injective, i.e.
 N is submodule of a free module.

(ii) $(N^a)_{\mathfrak{A}} \stackrel{df}{=} \text{Hom}_{K^{oa}}(K^{oa}, N^a) = \text{Hom}_{K^o}(M, N)$
[Sch, Prop. 4.4]

N is flat,
hence \mathbb{N} -torsion free $\rightarrow = \left\{ x \in \text{Hom}_K(K, N \left[\frac{1}{\pi} \right]) : \varepsilon x \in N \text{ for all } \varepsilon \in \mathfrak{m} \right\}$

Flatness of N^a over K^{oa} is trivial [like " \Leftarrow " in (ii)]

(iii) Let $x \in K^o$. Read $(xM)_* = x(M_*)$. Then $xM_* = x(M_{\frac{1}{\varepsilon}})$, hence

$$(xM)_* = \text{Hom}_{K^{oa}}(K^{oa}, xM) = \text{Hom}_{K^{oa}}(K^{oa}, (xM_*)^a)$$

$$= \text{Hom}_K(M, xM_*)$$

[Sch. Prop. 4.4]

$$= \{y \in M_*[\frac{1}{\varepsilon}]; \varepsilon y \in xM_* \text{ for all } \varepsilon \in M\}$$

xM_* is flat over K^o

by (ii), so the last step in the proof of (ii) applies.

$$= xM_*$$

$$\uparrow \exists \varepsilon \in M : |\varepsilon| < \varepsilon < 1$$

(.)_{*} left exact

$$\Rightarrow M_* / xM_* \simeq M_* / (xM)_* \quad \Downarrow \Rightarrow M_* / xM_* \rightarrow (M / xM)_* \text{ injective}$$

Now let $\varepsilon \in M$ and consider

$$\begin{array}{ccccc}
 & M_* & & M_* & \\
 & \downarrow \varepsilon & & \downarrow \varepsilon & \\
 M_* / xM_* & \leftarrow \quad \quad \quad \rightarrow & M_* & \leftarrow \tilde{n} \uparrow & \\
 \text{can} & & \text{can} & & \\
 \text{im(can)} \subseteq \frac{M_*}{xM_*} & & \text{eval at } \varepsilon & & \\
 & & & & \\
 & & \leftarrow \cdot \varepsilon \rightarrow & & \\
 & & \left(\frac{M}{xM} \right)_* & \simeq & \left(\frac{M_*}{x\varepsilon M_*} \right) \\
 & & \text{can} & & \text{eval at } \varepsilon \\
 & & & & \Rightarrow \tilde{m} \\
 & & & & \\
 & & & & \left(\frac{M}{x\varepsilon M} \right)_* = \text{Hom}(M, M_* / x\varepsilon M_*) \\
 & & & & \Rightarrow \tilde{m}
 \end{array}$$

Claim:

$$\text{im(can)} \subseteq \frac{M_*}{xM_*}$$

[and hence " $=$ "]

Pf. of claim: Let $\tilde{m} \in \left(\frac{M}{xM} \right)_*$, let $n = \tilde{m}(\varepsilon) \in \frac{M_*}{x\varepsilon M_*}$, let $\tilde{n} \in M_*$ lift n .

For $s \in M$ we have $s_n = s \tilde{m}(\varepsilon) = \varepsilon \tilde{m}(s) \in \frac{\varepsilon M_*}{x\varepsilon M_*}$, hence $s \tilde{n} \in \varepsilon M_*$

$$\Rightarrow \tilde{n} \in \varepsilon M_* / \varepsilon \Rightarrow m_1 := \frac{\tilde{n}}{\varepsilon} \in M_* \text{ lifts } m = \text{can}(\tilde{m}) \in \left(\frac{M}{xM} \right)_*$$

$$\begin{matrix} \uparrow \\ \tilde{n} \in M_* \\ (\text{S}) \Rightarrow \lambda \end{matrix}$$

because \tilde{n} maps to $\varepsilon M = \tilde{m}(\varepsilon) = n$ in $\left(\frac{M}{x\varepsilon M} \right)_*$

and $\left(\frac{M}{xM} \right)_* \xrightarrow{\cdot \varepsilon} \left(\frac{M}{x\varepsilon M} \right)_*$ is injective [as (.)_{*} is left exact [right adjoint to (.)^a]]

(5)

Now: $(\cdot)_*$ has left adjoint $(\cdot)^a$

$$(\cdot)^a \dashv \dashv (\cdot)_! = M \otimes (\cdot)_*$$

 $\Rightarrow (\cdot)_*$ and $(\cdot)^a$ commute with inverse limits• M π -adically complete \Rightarrow claim above

$$M_* = \left(\lim_{\leftarrow n} M/\pi^n M \right)_* = \lim_{\leftarrow n} (M/\pi^n M)_* = \lim_{\leftarrow n} M_*/\pi^n M_*$$

i.e. M_* π -adically complete• M_* π -adically complete

commutation

$$M = (M_*)^a = \left(\lim_{\leftarrow n} M_*/\pi^n M_* \right)^a = \lim_{\leftarrow n} (M_*/\pi^n M_*)^a = \lim_{\leftarrow n} M/\pi^n M$$

($(\cdot)^a$ right exact
[left adjoint to $(\cdot)_*$])

i.e. M π -adically complete. \square Proposition 3: Let R be a perfectoid K -algebra. Then $\Phi: R^o/\pi^o P \xrightarrow{\sim} R^o/\pi$ and R^{oa} is a perfectoid K^{oa} -algebra.Proof: Φ surjective by assumption. Let $x \in R^o$ such that $x^P \in \pi R^o$, i.e. $x^P = \pi y$ for some $y \in R^o$. Thus $\left(\frac{x}{\pi^{1/p}}\right)^p = y \in R^o$, so $\frac{x}{\pi^{1/p}} \in R^o$, i.e. $x \in \pi^{1/p} R$, so Φ is injective. R^o is π -adically complete and flat over $K^o \Rightarrow R^{oa}$ π -adically complete and flat over K^{oa} .↑
by Lemma 2 \square

Lemma 4: Let A be a perfectil K^{op} -algebra, $R := A[\frac{1}{\pi}]$ with K -Banach algebra structure making A_* open and bounded. Then $A_* \stackrel{!}{=} R^\circ$ and R is perfectil.

Proof: By definition $\Phi: A/\pi^{np} \xrightarrow{\sim} A/\pi$ $\Rightarrow \Phi: A_*/\pi^{np} \xrightarrow{\sim} A_*/\pi$

$\Rightarrow \Phi: A_*/\pi^{np} \rightarrow A_*/\pi$ is an almost isomorphism.

\Rightarrow For $x \in A_*$ with $x^p \in \pi A_*$ we have $\varepsilon \cdot x \in \pi^{np} A_*$ for all $\varepsilon \in M$,

Lemma 2 (ii) $x \in ((\pi^{np} A_*)^a)_* = \pi^{np} A_*$

$\Rightarrow \Phi: A_*/\pi^{np} \rightarrow A_*/\pi$ injective, hence bijective.

[Alternative argument: $\Phi: (A/\pi^{np})_* \xrightarrow{\sim} (A/\pi)_*$ by definition
 $\Rightarrow \Phi: A_*/\pi^{np} \xrightarrow{\sim} A_*/\pi$ by right exactness of Φ .
 \Rightarrow Define since $\Phi((\pi^{np} A_*)^a)_* = \pi^{np} A_*$ by Lemma 2 (ii)]

Claim: $x \in R$ with $x^p \in A_*$ implies $x \in A_*$

PF: Let $k \in \mathbb{N}$ with $y := \pi^{k/p} x \in A_*$. Then $y^p = \pi^{kp/p} x^p \in \pi A_*$.

Injectivity of Φ above $\Rightarrow y \in \pi^{np} A_*$ $\Rightarrow \pi^{\frac{k-1}{p}} x \in A_*$.

Continue to decrease k . The claim is proven.

$A_* \subseteq R^\circ$: clear

$R^\circ \subseteq A_*$: Let $x \in R^\circ$. For $\varepsilon \in M$ we find $N \in \mathbb{N}$ with $(\varepsilon x)^p \in A_*$.

Claim $\Rightarrow \varepsilon x \in A_*$, for any $\varepsilon \in M$.

Lemma 2 (ii) $\Rightarrow x \in A_*$

$\Phi: A_*/\pi^{np} \rightarrow A_*/\pi$ is surjective:

It is almost surjective \Rightarrow it is enough to show that $A_*/\pi^{np} \xrightarrow{\Phi} A_*/\pi \rightarrow A_*/m$ is surjective

Let $x \in A_{\frac{1}{p}}$. By almost surjectivity we find $y \in A_{\frac{1}{p}}$ with [7]
 $\pi^{np} \cdot x \equiv y^p \pmod{\pi A_{\frac{1}{p}}}$. Then $z = \frac{y}{\pi^{\frac{1}{p^2}}}$ satisfies $z^p \in A_{\frac{1}{p}} \subseteq R^{\circ}$, so
 $z \in R^{\circ} \subseteq A_{\frac{1}{p}}$. But $x \equiv z^p \pmod{\pi^{\frac{p-1}{p}} A_{\frac{1}{p}}} \subseteq \pi A_{\frac{1}{p}}$. □

Algebraic argument for b) going from $\mathbb{B} \otimes_R (\mathfrak{m}/\mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2$.

By functoriality we have $\mathbb{B} \otimes_R (\mathfrak{m}/\mathfrak{m}^2) = \mathbb{B} \otimes_R (A/\mathfrak{m})_{\frac{1}{p}}$.

By right exactness of $\mathbb{B} \otimes_R$ we have $\mathbb{B} \otimes_R A_{\frac{1}{p}} \xrightarrow{\sim} \mathbb{B} \otimes_R A_{\frac{1}{p}}$

By Lemma 1 (right exactness of $\mathbb{B} \otimes_R$) we have $\mathbb{B} \otimes_R A_{\frac{1}{p}} = \mathfrak{m} A_{\frac{1}{p}}$

Proposition 5: A perfectoid K -algebra R is reduced.

Proof: If $a \neq x \in R$ is nilpotent, then also ax for each $a \in K$,
so $Kx \subset R^{\circ}$, contradicting boundedness of R° . □

The cotangent complex: For a map of rings $A \rightarrow B$ there is a complex $L_{B/A} \in D^{\leq 0}(B\text{-Mod})$ with the following properties:

- $L_{B/A} = \Omega^1_{B/A}[0]$ if $A \rightarrow B$ is smooth
- For ring morphisms $A \rightarrow B \rightarrow C$ there is a triangle

$$C \underset{B}{\underset{\otimes}{\overset{H}{\longrightarrow}}} L_{B/A} \longrightarrow L_{C/A} \longrightarrow L_{C/B} \quad \text{in } D(C)$$

- Let R be a ring, $I \subset R$ an ideal with $I^2 = 0$, let $R_0 = R/I$. (8)
 - For a flat R_0 -algebra S_0 there is an obstruction class in $\text{Ext}^2(L_{S_0/R_0}, S_0 \otimes_{R_0} I)$, which vanishes precisely when there exists a flat R -algebra S with $S/I = S_0$.
 - If S as in (1) exists, the set of isomorphism classes of such S is a torsor under $\text{Ext}^1(L_{S_0/R_0}, S_0 \otimes_{R_0} I)$.
 - Every S as in (1) has automorphism group $\text{Aut}(L_{S_0/R_0}, S_0 \otimes_{R_0} I)$.
 - For two flat R -algebras S, S' and morphism $f_0: S_0 = S/I \rightarrow S'_0 = S'/I$ there is an obstruction class in $\text{Ext}^1(L_{S_0/R_0}, S'_0 \otimes_{R_0} I)$, which vanishes precisely when f_0 extends to $f: S \rightarrow S'$.
 - If f as in (4) exists, the set of all such f is a torsor under $\text{Aut}(L_{S_0/R_0}, S'_0 \otimes_{R_0} I)$.

Proposition 6 : (i) $L_{R/\mathbb{F}_p} = 0$ for perfect \mathbb{F}_p -algebras R .

(ii) Let $R \rightarrow S$ be a morphism of \mathbb{F}_p -algebras. Let $R_{(\oplus)}$ be R , with \mathbb{Q} -algebra structure $\oplus: R \rightarrow R = R_{(\oplus)}$, and $S_{(\oplus)}$ similarly.

Assume that the relative Frobenius $\mathbb{F}_{S(R)}: R_{(\oplus)} \otimes_R S \rightarrow S_{(\oplus)}$ induces [R-linear!] an isomorphism $\mathbb{F}: R_{(\oplus)} \otimes_R S \rightarrow S_{(\oplus)}$ in $D(R)$.

Then $L_{S/R} \simeq 0$.

[9]

Proof: (i) For $x \in R$ we find $y \in R$ with $x = y^p$, hence $d =$
 $\Rightarrow dx = dy^p = p y^{p-1} dy = 0 \Rightarrow \Omega^1_{R/\mathbb{F}_p} = 0$
 $\Rightarrow (\dots \text{resolution techniques...}) L_{R/\mathbb{F}_p} = 0$

(ii) Similarly. \square

Gabber and Ramero = For a mapism $A \rightarrow B$ of K^{ur} -algebras

there is an object $L_{B/A}^a \in D^{\leq 0}(B\text{-Mod})$ [can be constructed as an "honest complex" in $D^{\leq 0}(B_{\mathfrak{p}}\text{-Mod})$]. It has the previous properties.

Corollary 7 = $L_{\bar{A}/(K^{ur}/\pi)}^a = 0$ for a perfectoid (K^{ur}/π) -algebra \bar{A} .

$$\begin{array}{ccccc} \bar{A} & \xleftarrow{\sim} & \bar{A}/\pi^{1/p} & \xleftarrow{\quad} & \bar{A} \\ \uparrow \Phi & & \uparrow & & \uparrow \\ K^{ur}/\pi & \xleftarrow{\sim} & K^{ur}/\pi^{1/p} & \xleftarrow{\quad} & K^{ur}/\pi \end{array}$$

Use the almost version of Proposition 6 . \square