

GAGA for $X^{\text{ad}} \rightarrow X$ (Omar Venjakob)

We report on the GAGA-principle as developed in
 [KL] K.-S. Kedlaya, R. Liu: "Relative p-adic Hodge Theory: Foundations"
 arXiv: 1301.0792v4

Theorem 1: $I \subset (0, 1)$ closed interval

$\Rightarrow B_I$ is strongly noetherian, i.e. $B_I < T_1, \dots, T_n >$ is

Pf: [Kedlaya Thm 4.10 by Gröbner basis techniques] \square
 Noetherian properties of Fargues-Fontaine curves, arXiv: 1410.5160v2

Consequences: • $\text{Spa}(B_I)$ is sheafy (difficult proof!)

- Y_I^{ad} and $Y^{\text{ad}}, X^{\text{ad}}$ are locally noetherian, in particular
 for $Z \in \{Y_I^{\text{ad}}, Y^{\text{ad}}, X^{\text{ad}}\}$ an O_Z -module \mathcal{F} is
coherent if and only if
 - \mathcal{F} is of finite presentation
 - or • \mathcal{F} is of finite type + quasi-coherent

Since for any loc. ringed space Z and ring A we
 have (EGA1, 2nd ed., Chap. I, Prop. 1.6.3)

$$\text{Mor}_{\text{loc. ringed spaces}}(Z, \text{Spec}(A)) \cong \text{Hom}_{\text{Ring}}(A, O_Z(Z)) \quad (1)$$

we have for any sheafy adic space $X = \text{Spa}(A, A^+)$

a map

$$\text{Spa}(A, A^+) \xrightarrow{\delta_A} \text{Spec}(A)$$

of loc. ringed spaces, which maps $v \in X$

As its support $\mathcal{F}_0 = \mathcal{V}^{-1}(0)$.

Z loc. ringed space: $VB(Z)$ tensor cat. of finite locally free \mathcal{O}_Z -modules

Theorem 2: $\delta = \delta_A$ induces an equivalence of tensor cat.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad} & \delta^* \mathcal{F} \\ VB(\mathrm{Spec}(A)) & \xrightarrow[\text{1:1}]{} & VB(\mathrm{Spec}(A, A^\dagger)) \quad \tilde{M}(U) := M \otimes_{A^\dagger} \mathcal{O}_X(U) \\ \downarrow \text{1:1} & & \uparrow \text{1:1} \\ P\text{-Mod}(A) := \{ \text{f.g. projective } A\text{-modules} \} & M & \uparrow \\ & & \mathcal{F}(\mathrm{Spec}(A)) \end{array}$$

P: see [KL] Thm 8.2.22 / 2.4.23 / 2.7.7 / 5.3.9

+ standard facts in affine algebraic geometry.

More precisely, for every rational subdomain $U \subset \mathrm{Spec}(A, A^\dagger)$

$P\text{-Mod}(\mathcal{O}_X(U)) \xrightarrow[\text{1:1}]{} VB(U)$ Kiehl gluing property
is an equivalence of categories and

$$H^i(U, \mathcal{O}_X) = H^i(U, \mathcal{O}_X, (U_i)_{i \in I}) = \begin{cases} \mathcal{O}_X(U) & i=0 \\ 0 & i>0 \end{cases}$$

for every rational covering $(U_i)_{i \in I}$ of U .

Tate sheaf property

Idea: Glueing projective modules on multiple Laurent.

Coverings: $\{v \in X \mid v(f) \leq 1\}$ $\{v \in X \mid v(f) \geq 1\}$ for $f \in A$.

For $f \in P_1$ (or P_d) the diagram

$$(2) \quad \begin{array}{ccc} P \subset B^+ \subset B & \longrightarrow & B_I \\ \downarrow & \downarrow & \downarrow \\ P_f & \longrightarrow B_f & \longrightarrow (B_I)_f \\ \vee & \vee & \vee \\ P_{(1)} & \xhookrightarrow{\varphi=1} (B_f)^{\varphi=1} & \longrightarrow ((B_I)_f)^{\varphi=1} \end{array} \quad \text{together with (1)}$$

(degree 0 eth in localization)

induces morphisms $i_I: \text{Spec}(B_I)_f \longrightarrow \text{Spec } P_{(1)} \hookrightarrow X$,

which glue to $\text{Spa}(B) \xrightarrow{\delta_B} \text{Spec}(B_I) \xrightarrow{i_I} X$

(since P_1 generates B by [KL] lem. 6.3.7)

\Rightarrow We obtain $y^{\text{ad}} = \varprojlim \text{Spa}(B_I)$ a monr f (using δ_{B_I})

$$(3) \quad \begin{array}{ccccc} y^{\text{ad}} & \xrightarrow{\varphi} & Y^{\text{ad}} & = & \varprojlim \text{Spa}(B_I) \\ & \downarrow & \downarrow P & & \downarrow f \\ & \text{P} & \text{ad} & & \text{ad} \\ & & & & + \text{ (of loc. ringed space)} \\ & & \dots & \dots & \longrightarrow X \end{array}$$

satisfying $f \circ \varphi^n = f \quad \forall n \in \mathbb{Z}$. By the universal property

of the quotient $X^{\text{ad}} = Y^{\text{ad}}/\varphi\mathbb{Z}$ (as loc. ringed space,

c.f. Q. Liu, Algebraic Geometry and Arithmetic Curves, Exer. 2.14)

f induces a unique morphism of loc. ringed spaces

g.

Fact: p is (topologically) a covering map:

Each $y \in Y^{\text{ad}}$ has an open neighbourhood U

(e.g. of the form $\text{Spa}(B_{[g,g]})$ with $g^q < g' < g$; recall

that $B_I \xrightarrow{\varphi} B_{\varphi(I)}$, where $\varphi([g,g]) = [(\varphi)^q, g^q]$)

and that $U \cap \varphi^n(U) = \emptyset \quad \forall n \neq 0$, whence

$p|_U : U \rightarrow p(U)$ is a homeomorphism and $p^{-1}p(U) = \bigcup_{n \in \mathbb{Z}} \varphi^n(U)$ (4)

$\Rightarrow \mathcal{O}_{X^{\text{ad}}} |_{p(U)} \stackrel{(5)}{\cong} \mathcal{O}_{Y^{\text{ad}}} |_U$, i.e. X^{ad} is an adic space!

$$\mathcal{O}_{X^{\text{ad}}}(p(U)) = \mathcal{O}_{Y^{\text{ad}}}(p(p(U)))^{\varphi=1} = (\prod \mathcal{O}_{Y^{\text{ad}}}(\varphi^n(U)))^{\varphi=1} = \mathcal{O}_{Y^{\text{ad}}}(U)$$

(6)
see (8) below

φ -VB(Y^{ad}) ^{Affine} \cong cat. of $(\tilde{F}, \varphi^* \tilde{F} \stackrel{\cong}{\rightarrow} F)$, $F \in \text{VB}(Y^{\text{ad}})$
 \cong iso of $\mathcal{O}_{Y^{\text{ad}}}$ -modules

ψ induces $\tilde{F} \rightarrow \varphi_* \varphi^* \tilde{F} \stackrel{\cong}{\rightarrow} \varphi_* \tilde{F}$, i.e. isom. of rings

$$\tilde{F}(U) \xrightarrow{\varphi_* \tilde{F}|_U} \tilde{F}(\varphi^{-1}(U)) \quad \text{for all } U \subset Y^{\text{ad}} \text{ open.}$$

For $\tilde{F} \in \varphi$ -VB(Y^{ad}) denote by $\tilde{F}^{(n)} \in \varphi$ -VB(Y^{ad}) the same underlying $\tilde{F} \in \text{VB}(Y^{\text{ad}})$ but with twisted φ -action: $\varphi_{\tilde{F}^{(n)}} = \prod^{\varphi=1} \varphi_{\tilde{F}_n}$, $n \in \mathbb{Z}$

We set $\tilde{T}_*(\tilde{F}) := T(Y^{\text{ad}}, \bigoplus_{n \in \mathbb{Z}} \tilde{F}^{(n)})^{\varphi=1} = \bigoplus_{n \in \mathbb{Z}} T(Y^{\text{ad}}, \tilde{F}^{(n)})^{\varphi=1} = \bigoplus_{n \in \mathbb{Z}} \tilde{F}(Y^{\text{ad}})^{\varphi=1}$
 a graded P-algebra.

Theorem 3. The diagram (3) induces equivalences of tensor categories

$$\begin{array}{ccccc}
 (\tilde{p}^* \mathcal{Y})^{(n)} & \varphi\text{-VB}(\mathcal{Y}^{\text{ad}}) & (\mathcal{F}, \gamma) & & (7) \\
 \uparrow & p_*(-)^{\varphi=1} \downarrow \boxed{1} \uparrow p^* & \swarrow \downarrow \perp \searrow & & \\
 & & \mathcal{F}^* & & \\
 \downarrow & \mathcal{V}\mathcal{B}(X^{\text{ad}}) & \xleftarrow{\quad g^* \quad} & \mathcal{V}\mathcal{B}(X) & \\
 \mathcal{G}^{(n)} & \mathcal{G} & \xrightarrow{\quad \quad} & \mathcal{T}(X^{\text{ad}}, \bigoplus_{n \in \mathbb{Z}} \mathcal{G}^{(n)})^{\sim} &
 \end{array}$$

If (equivalence I): $\mathcal{Y} \in \mathcal{V}\mathcal{B}(X^{\text{ad}})$, $(\mathcal{F}, \gamma) \in \varphi\text{-VB}(\mathcal{Y}^{\text{ad}})$

1. $p^* \mathcal{Y}$ belongs to $\varphi\text{-VB}(\mathcal{Y}^{\text{ad}})$ because $p \circ \varphi = p$
2. For $U \subset \mathcal{Y}^{\text{ad}}$ as in (4) we have a commutative diagram with exact rows for all $V \subset U$ open, $W = \tilde{p}^* p(V)$

$$0 \rightarrow \mathcal{F}(W)^{\varphi=1} \rightarrow \mathcal{F}(W) \xrightarrow{\varphi^* - id} \mathcal{F}(W) \quad (8)$$

$\parallel L$ $\parallel R$ $\parallel L$

$$0 \rightarrow \mathcal{F}(V) \rightarrow \prod_{n \in \mathbb{Z}} \mathcal{F}(\varphi^n(V)) \rightarrow \prod_{n \in \mathbb{Z}} \mathcal{F}(\varphi^n(V))$$

$a \mapsto (\varphi^{-n}(a))_{n \in \mathbb{Z}}$

$$(a_n)_{n \in \mathbb{Z}} \mapsto (\varphi(a_{n+1}) - a_n)_n$$

$$\Rightarrow (\tilde{p}_* \mathcal{F})^{\varphi=1} \Big|_{\tilde{p}(V)} \stackrel{\sim}{=} \mathcal{F}|_V \text{ is finite locally free}$$

and the canonical maps

$$p^*(p_+(\bar{f})^{q=1}) \rightarrow p^*p_+ F \rightarrow \bar{F}$$

and $\xrightarrow{p_*p^*g} g$

$$g \dashrightarrow (p_*p^*g)^{q=1}$$

are isomorphisms (as can be deduced on stalks) \blacksquare

For the equivalence II we need some preparation:

The sheaf properties and Thm 2 imply

$$\varphi\text{-VB}(Y^{\text{ad}}) \cong \varprojlim_I \varphi\text{-mod}_{B_I} := \left\{ (M_I)_I \mid M_I \in \varphi\text{-mod}_{B_I} \text{ plus} \begin{array}{l} M_I \otimes_{B_I} B_J \cong M_J \\ \psi_{I,J} \text{ cocycle} \end{array} \right\}$$

$$(\bar{F}, \varphi^*\bar{F} = \bar{F}) \mapsto (\bar{F}(Y^{\text{ad}}))_I$$

$\psi_{I,J}$ induced from ψ

where for $I = [s', s]$ with $0 < s' < s^q$

$$\varphi\text{-mod}_{B_I} := \left\{ M_I \in P\text{-Mod}(B_I), \varphi^*M_I \otimes_{B_{[s',s]}} B_{[s',s]} \cong M_I \otimes_{B_{[s',s]}} B_{[s',s]} \right\}$$

Note: $B_{[s',s]} \hookrightarrow B_{[s',s^q]}$

$$\downarrow \varphi$$

$$B_{[(s')]^q, s^q]} \hookrightarrow B_{[s', s^q]}$$

$$\varphi - VB(y^{ad}) \cong \varprojlim_I \varphi - \text{mod}_{B_I}$$

$\mathcal{F} \hookrightarrow (\mathcal{M}_I)_I$
 $\downarrow \pi$
 $\mathcal{F}(y^{ad}) = M = \bigcup_I \mathcal{M}_I$
 $\varphi - \text{mod}_{B_I}$

(9)

$$P := P(M) = P_{\infty}(\mathcal{F}) = \left(\bigoplus_{n \in \mathbb{Z}} M(n) \right)^{\varphi=1} = \bigoplus_{n \in \mathbb{Z}} M^{\varphi=\pi^n} = \bigoplus_{n \in \mathbb{Z}} M^{\varphi=1} = (M_{\infty})^{\varphi=1}$$

$$\widetilde{P_{\infty}(\mathcal{F})} \Big|_{D_d(t)} \cong \widetilde{P_{(t)}} \Big|_{D_d(t)} = \bigcup_{k \in \mathbb{Z}} t^{-k} \left(M(kd)^{\varphi=1} \right) \in \text{mod}_{P_{(t)}} \Big|_{D_d(t)}, \quad t \in P_d$$

1. $\widetilde{P_{\infty}(\mathcal{F})} \in VB(X)$, if $P_{(t)} \in P\text{-Mod}(P_{(t)})$! (Well-definedness)
2. $f^* \widetilde{P_{\infty}(\mathcal{F})}$ is determined by $(f^* \widetilde{P}) \Big|_{Spa(B_I)} = \delta_{B_I}^* i_I^* \widetilde{P}$,

i.e. by $i_I^* \widetilde{P}$ by Thm. 2

i.e. by $i_{I,t}^* \widetilde{P} = \widetilde{P_{(t)} \otimes_{P_{(t)}} (B_I)_t}$ for $t \in P_d$

Thus $\boxed{f^* \widetilde{P_{\infty}(\mathcal{F})} \cong \mathcal{F}}$ follows from

Theorem 4: $\widetilde{P_{(t)}}$ is a finitely generated projective $P_{(t)}$ -module

and the canonical map

$$\begin{array}{ccc} \widetilde{P_{(t)} \otimes_{P_{(t)}} (B_I)_t} & \xrightarrow{\cong} & M_I \otimes_{B_I} (B_I)_t \\ m \otimes b \mapsto & & \text{pr}_I(m) \otimes \frac{b}{t^n} \end{array} \quad (10)$$

is an iso.

Before we prove this theorem we show

$$3. \quad \mathcal{F} \in VB(X) \Rightarrow \Gamma_*(f^*\mathcal{F})^\sim \cong \mathcal{F} :$$

Let $M_I := (f^*\mathcal{F})(\text{Spec } B_I)$, $n := \sum_I n_I$

$$\Rightarrow 0 \rightarrow M_I \rightarrow \prod_{t_i} (M_I)_{t_i} \rightarrow \prod_{t_i, t_j} (M_I)_{t_i, t_j} \quad t_i, t_j \in P_I$$

$$\mathcal{F}(D_+(t_i)) \otimes_{P_{(t_i)}} (B_I)_{t_i}$$

$$\Rightarrow 0 \rightarrow (M_I)^{\varphi=\pi^n} \rightarrow \prod_{t_i} \left(\mathcal{F}(D_+(t_i)) \otimes_{P_{(t_i)}} (B_I)_{t_i} \right)^{\varphi=\pi^n} \rightarrow \prod_{t_i, t_j} \dots$$

|| ||

$$\mathcal{F}(D_+(t_i)) \otimes_{P_{(t_i)}} P_{(n)}(t_i)$$

$$0 \rightarrow \mathcal{F}^{(n)}(X) \rightarrow \prod_i \mathcal{F}^{(n)}(D_+(t_i)) \rightarrow \prod_{t_i, t_j} \mathcal{F}^{(n)}(D_+(t_i, t_j))$$

$$\Rightarrow M^{\varphi=\pi^n} = \sum_I (M_I)^{\varphi=\pi^n} = \Gamma(X, \mathcal{F}^{(n)})$$

$$\Rightarrow \Gamma_*(f^*\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}^{(n)}) =: \widetilde{\Gamma}_*(\mathcal{F})$$

$$\Rightarrow \widetilde{\Gamma}_*(f^*\mathcal{F}) = \widetilde{\Gamma}_*(\mathcal{F}) \cong \mathcal{F}, \text{ where the last}$$

isomorphism stems from the fact that

X is quasi-compact and separated

(see Görtz/Wedhorn, Algebraic Geometry I, (13.5.3), (13.5.6))

and the beginning of the proof of Thm 13.20)

Equation (11) uses the fact

$$(B_I)^{\varphi = \pi^n} = (B)^{\varphi = \pi^n} \quad \text{for all } n$$

(see [KL], Corollary 5.2.12, where B_I corresponds

to $\tilde{R}_L^{[s, r]}$ with $s' = q^{-\frac{1}{3}}$, $s = q^{-\frac{1}{2}}$ and $I = [s', s]$, $s' \leq s$,

and $B = \tilde{R}_L^{\cdot}$)

$(M_2)_I \in \varprojlim \varphi\text{-mod } B_I$, $M_2 = \bigcup_{I \in I} M_2|_I$, $\beta \in \{\emptyset, 1, 2\}$

Lemma: Assume $0 \rightarrow M_{1,I} \rightarrow M_{2,I} \rightarrow N_I \rightarrow \text{exact}$ for all I

(i)

$\Rightarrow \exists N$ such that for $n \geq N$

$$0 \rightarrow M_{1,(n)} \xrightarrow{\varphi=1} M_{2,(n)} \xrightarrow{\varphi=1} N(n) \xrightarrow{\varphi=1} 0 \text{ is exact}$$

and $M_{2,I} \xrightarrow{\varphi=1} M_{1,I}$ is surjective for all I .

(ii) $(M_I)_I \in \varprojlim \varphi\text{-mod } B_I$. For $n \geq N$ (as in (i))

there exists a finite set $m_1, \dots, m_r \in N(n)$,

$N = \varprojlim_I M_I$, such that their images generate

M_I as B_I -module for all I .

(iii) For $t \in P_d$, $d > 0$,

$$0 \rightarrow P_{1,(t)} \rightarrow P_{2(t)} \rightarrow P_{(t)} \rightarrow 0 \text{ is exact.}$$

Pf: The surjectivity-statement in (i) is

[KL] Prop. 6.2.2, the first statement follows from the snake-lemma and taking inverse limits.

(ii) [KL] Prop 6.2.4

(iii) follows from (i) and the definition of $P(M_2)$ \square

Proof of Theorem 4: $t \in P_d$

Choose $m_1, \dots, m_r \in M(d_n)^{P=1}$ which generate all M_I (by Lem (ii)).

$\Rightarrow \frac{m_i}{t^n} \in \Gamma_{(t)}$ \Rightarrow (10) is surjective. (12)

$$(M_{2,I}) := \left(B_I(-d_n) \right)_I^r \quad \text{is surj} \quad M_{2,I}(d_n) \longrightarrow M_I(d_n)$$

$e_i \longmapsto m_i$

$$0 \rightarrow M_{1,I} \rightarrow M_{2,I} \rightarrow M_I \rightarrow 0 \quad (13) \Rightarrow (M_{1,I})_I \subset \bigcup_{\substack{I' \\ \text{q-and } B_{I'} \\ \text{exact}}} M_{I'}(d_n)$$

proj. proj. proj. exact

Lem (iii)

$$\Rightarrow 0 \rightarrow \Gamma_{1,(t)} \rightarrow \Gamma_{(2),(t)} \rightarrow \Gamma_{(t)} \rightarrow 0 \quad \text{exact, i.e.}$$

$$\begin{array}{c} P(-d_n)_I^r \xrightarrow{\cong} P(t)_I^r \\ \cong \end{array} \boxed{P_{(t)}^r = P_{d+n}}$$

$\Gamma_{(t)}$ is $f \circ g / P_{(t)}$ since $\Gamma_{(2),(t)}$ is.

Consider the commutative diagram $((B_I)_t / B_I \text{ is K(t)})$

$$\begin{array}{ccccccc} \Gamma_{1,(t)} & \xrightarrow{P_{(t)}} & B_{I,t} & \rightarrow & \Gamma_{2,(t)} & \xrightarrow{P_{(t)}} & B_{I,t} \\ \downarrow & & & & \downarrow \text{12} & & \downarrow \\ 0 & \rightarrow & M_{1,I} & \otimes_{B_I} B_{I,t} & \rightarrow & M_{2,I} & \otimes_{B_I} B_{I,t} \end{array} \rightarrow 0$$

with exact rows and surjective vertical maps by (12) above. Note that the middle one is bijective, whence the right one, too, by the 5-lemma. This proves bijectivity in (10).

Since all $M_{2,I}$ are projective, the extension class (13)

corresponds to an elt. in $H_\varphi^1(N_I^* \otimes M_{1,I})$, $N_I^* = \text{Hom}_{B_I}(N_I, B_I)$

where by def. $0 \rightarrow H_\varphi^0(N_I) \rightarrow N_I \xrightarrow{\varphi^{-1}} N_I \rightarrow H_\varphi^1(N_I) \rightarrow 0$ is exact.

By Lem. (i) $H_\varphi^m(N_I^* \otimes M_{1,I}(\text{dm})) = 0$ for $m > 0$

\Rightarrow the push out sequence below

$$0 \rightarrow M_{1,I} \rightarrow M_{2,I} \rightarrow N_I \rightarrow 0$$

$$\downarrow \quad \square \quad \downarrow \quad \parallel$$

$$0 \rightarrow t^{-m}M_{1,I} \xrightarrow{\text{HS}} N_I \rightarrow N_I \rightarrow 0$$

split in $\varphi\text{-mod } B_I$ for all I , i.e., $\bigcup_I \varphi\text{-mod } B_I$
 Lem (ii)

$$\Rightarrow 0 \rightarrow \Gamma(t^{-m}M_{1,I}) \rightarrow \Gamma(N_I)_{(t)} \rightarrow \Gamma(N_I)_{(t)} \rightarrow 0 \quad \text{exact}$$

Note that $\Gamma(N_I)_{(t)}$ is a finitely generated $P_{(t)}$ -module by construction

and that $\Gamma(N_I)_{(t)} \cong \Gamma(M_2)_{(t)}$ because $(M_{2,I})_t \cong (N_I)_t$
 (Adele φ -invariants).

$$\Rightarrow \Gamma(N_I)_{(t)} \in P\text{-Mod}_{P_{(t)}} \quad \blacksquare$$

One can show that the categories

$$\varphi\text{-VB}(Y^{\text{ad}}) \xrightarrow[1:1]{} \text{VB}(X^{\text{ad}}) \xrightarrow[1:1]{} \text{VB}(X)$$

$$\varprojlim_I \varphi\text{-mod}_{B_I}$$

are also equivalent to (induced by (g))

$$\varphi\text{-mod}_B = \{ M \in P\text{-Mod}_B \text{ + semilinear } \varphi\text{-action} \}$$

$$\text{and } 1:1 \downarrow \tilde{B}_{mg}^{st} \otimes_B - \quad \text{a.t. } \varphi^* M \xrightarrow{\cong} M$$

$$\varphi\text{-mod}_{\tilde{B}_{mg}^{st}} = \{ M \in P\text{-Mod}_{\tilde{B}_{mg}^{st}} \text{ + semilinear } \varphi\text{-action} \}$$

$$\text{where } \tilde{B}_{mg}^{st} = \varprojlim_{\uparrow} \varprojlim_{g \leq 1} B_{[g', g]} = \bigcup_{g < 1} \bigcap_{0 < g' \leq g} B_{[g', g]}$$

$$B \quad = \quad \bigcap_{0 < g' \leq g < 1} B_{[g', g]}$$

See [KL], Thm. 6.3.12 (where $\tilde{R}_L = \tilde{B}_{mg}^{st}$, $\tilde{R}_L^\infty = B$).

\tilde{B}_{mg}^{st} is Berger's notation!