# Divisors on the Fargues-Fontaine curve 

Michael Fütterer<br>Talk FF-III at the Workshop<br>The Galois group of $\mathrm{Q}_{p}$ as a geometric fundamental group<br>Neckarbischofsheim, May 4-8, 2015

Setup. We recall the setup of the previous talks. $E$ is a finite extension of $\mathbb{Q}_{p}$ with residue field $\mathbb{F}_{q}$ and uniformizer $\pi . F$ is an algebraically closed extension of $\mathbb{F}_{q}$ which is complete for some nontrivial valuation $v$. We use the rings $B, B^{+}, B^{\mathrm{b}}, B^{\mathrm{b},+}, B_{I}, W_{\mathcal{O}_{E}}\left(\mathcal{O}_{F}\right)$ introduced in the talk FF-I. $Y^{\text {ad }}$ denotes the adic curve introduced in talk FF-II.
Define $|Y|$ to be th set of ideals of $B$ generated by primitive irreducible elements. We view $|Y|$ as a subset of the maximal spectrum $\operatorname{Spm}(B)$. For $\mathfrak{m} \in|Y|$, let

$$
L_{\mathfrak{m}}=B / \mathfrak{m}, \quad \theta_{\mathfrak{m}}: B \longrightarrow L_{\mathfrak{m}} \text { the canonical projection }
$$

$w_{\mathfrak{m}}: L_{\mathfrak{m}} \longrightarrow \mathbb{R} \cup\{\infty\}$ the unique valuation such that $w_{\mathfrak{m}}\left(\theta_{\mathfrak{m}}([a])\right)=v(a)$ for $a \in \mathcal{O}_{F}$,

$$
\|\mathfrak{m}\|=q^{-w_{\mathfrak{m}}(\pi)}, \quad \operatorname{ord}_{\mathfrak{m}}: B \longrightarrow \mathbb{N} \cup\{\infty\} \text { the normalized } \mathfrak{m} \text {-adic valuation. }
$$

Note that every $\mathfrak{m}$ can be written as $\mathfrak{m}=([a]-\pi)$ for some $a \in \mathcal{O}_{F}$ and that for such an $a$ we have $\|\mathfrak{m}\|=q^{-v(a)}$.

## 1 Newton polygons

We include a short reminder about Newton polygons. For details on all this, see [FF13, §5.3].

Definition 1: For

$$
x=\sum_{n \gg-\infty}\left[x_{n}\right] \pi^{n} \in B^{\mathrm{b},+}
$$

let the Newton polygon $\operatorname{Newt}(f)$ be the convex decreasing hull of the points $\left(n, v\left(x_{n}\right)\right)$ for all $n \in \mathbb{Z}$.

Example 2: For $a \in \mathcal{O}_{F}$, the Newton polygon of $[a]-\pi$ looks like this:


Convention: We define the slopes of a piecewise linear map $\mathbb{R} \longrightarrow \mathbb{R} \cup\{0\}$ to be the negative of the slopes as defined usually. This means that the Newton polygon from Example 2 has one non-zero non-infinite slope which is $v(a)$.

Since elements of $B$ can not always be written in the above form, this definition cannot be used to define their Newton polygon. One uses instead the Legendre transform to define the Newton polygon of a $b \in B$ as the inverse Legendre transform of $r \longmapsto w_{r}(b)$.
For two convex decreasing functions $\varphi_{1}, \varphi_{2}: \mathbb{R} \longrightarrow \mathbb{R} \cup\{\infty\}$, put

$$
\varphi_{1} * \varphi_{2}(x)=\inf \left\{\varphi_{1}(a)+\varphi_{2}(b): a+b=x\right\} \quad \text { for } x \in \mathbb{R}
$$

One can check that if $\varphi_{1}$ and $\varphi_{2}$ are polygons, then $\varphi_{1} * \varphi_{2}$ is also a polygon, and its slopes are obtained by "concatenating" (and reordering) the slopes of $\varphi_{1}$ and $\varphi_{2}$.
Lemma 3: For $x, y \in B$, one has

$$
\operatorname{Newt}(x y)=\operatorname{Newt}(x) * \operatorname{Newt}(y)
$$

## 2 Zeros of elements in $B$

Proposition 4: Let $f \in B$ and $\lambda \neq 0, \infty$ a slope of $\operatorname{Newt}(f)$. Then there is an $\mathfrak{m} \in|Y|$ with $f \in \mathfrak{m}$ and $\lambda=w_{\mathfrak{m}}(\pi)$.

Sketch of proof: For $\mathfrak{m}_{1}, \mathfrak{m}_{2} \in|Y|$ and $x_{\in} \mathcal{O}_{L_{\mathfrak{m}_{1}}}$ such that $\theta_{\mathfrak{m}_{1}}\left(\mathfrak{m}_{2}\right)=\mathcal{O}_{L_{\mathfrak{m}_{1}}} x$, put

$$
d\left(\mathfrak{m}_{1}, \mathfrak{m}_{2}\right)=q^{-w_{\mathfrak{m}_{1}}(x)}
$$

One can show that $d$ defines a metric on $|Y|$ and that for each $\rho \in(0,1)$ the set $\{\mathfrak{m} \in|Y|$ : $\|\mathfrak{m}\| \geq \rho\}$ is complete. We sketch the prove only for $f \in W_{\mathcal{O}_{E}}\left(\mathcal{O}_{F}\right)$, the general case follows by an approximation argument. Write

$$
f=\sum_{n \geq 0}\left[x_{n}\right] \pi^{n}
$$

and put for $d \geq 0$

$$
f_{d}=\sum_{n=0}^{d}\left[x_{n}\right] \pi^{n}, \quad X_{d}=\left\{\mathfrak{m} \in|Y|: f_{d} \in \mathfrak{m} \text { and } w_{\mathfrak{m}}(\pi)=\lambda\right\}
$$

For $d \gg 0$, the multiplicity of the slope $\lambda$ in $\operatorname{Newt}(f)$ equals the multiplicity of $\lambda$ in $\operatorname{Newt}\left(f_{d}\right)$. Hence, the cardinality of $X_{d}$ is bounded by this multiplicity for $d \gg 0$. Further we may without loss of generality assume that $x_{0} \neq 0$, since multiplication by $\pi$ just moves the Newton polygon. Hence we can write $f_{d}=\left[a_{d}\right] g_{d}$ with some $a_{d} \in \mathcal{O}_{F}$ and $g_{d} \in W_{\mathcal{O}_{E}}\left(\mathcal{O}_{F}\right)$ primitive. Then $g_{d}$ has an irreducible primitive divisor such that the ideal $m$ generated by this divisor lies in $X_{d}$, since $\operatorname{Newt}\left(f_{d}\right)=\operatorname{Newt}\left(\left[a_{d}\right]\right) * \operatorname{Newt}\left(g_{d}\right)$, and thus $X_{d} \neq \varnothing$ for $d \gg 0$. Now, given an $\mathfrak{m} \in X_{d}$ for some $d \geq 0$, by looking at $w_{\mathfrak{m}}\left(\theta_{\mathfrak{m}}\left(f_{d+1}\right)\right)$ and doing some calculations one can see that there exists an $\mathfrak{m}^{\prime} \in X_{d+1}$ such that

$$
d\left(\mathfrak{m}, \mathfrak{m}^{\prime}\right) \leq q^{-\left((d+1) \lambda-v\left(x_{0}\right)\right) / \# X_{d}} .
$$

This gives a Cauchy sequence in $|Y|$, and the limit of this sequence has the required properties.
For details, see [ $\mathrm{FF}_{13}$, Thm. 6.49] or [FF14, Thm. 3.3].
Corollary 5: The slopes different from $0, \infty$ of the Newton polygon of some $f \in B$ are the $w_{\mathfrak{m}}(\pi)$ where $\mathfrak{m}$ runs through all $\mathfrak{m} \in|Y|$ such that $f \in \mathfrak{m}$, with multiplicity $\operatorname{ord}_{\mathfrak{m}}(f)$.
Proof: All that remains to note is that if $f$ lies in some $\mathfrak{m}$, then its Newton polygon has a slope $w_{\mathfrak{m}}(\pi)$. If we write $\mathfrak{m}=([a]-\pi)$ with some $a \in \mathcal{O}_{F}$ and $f=g([a]-\pi)$, then $\operatorname{Newt}(f)=\operatorname{Newt}(g) * \operatorname{Newt}([a]-\pi)$, hence the claim follows from Example 2.

Corollary 6: For $f \in B \backslash\{0\}$, we have

$$
f \in B^{\times} \Longleftrightarrow \forall \mathfrak{m} \in|Y|: f \notin \mathfrak{m} .
$$

Proof: This follows with the previous results from the characterization of $B^{\times}$via Newton polygons from talk FF-I.

## 3 Factorization of elements of $B$

Given $f \in B$, if $\lambda_{1}, \ldots, \lambda_{r}$ are some non-zero non-infinite slopes of $\operatorname{Newt}(f)$, one can write

$$
\begin{equation*}
f=g \cdot \prod_{i=1}^{r}\left(1-\frac{\left[a_{i}\right]}{\pi}\right) \tag{1}
\end{equation*}
$$

with a $g \in B$ and $a_{i} \in \mathcal{O}_{F}$ such that $v\left(a_{i}\right)=\lambda_{i}$, as we see from the previous results.
It is easy to see that if $\left(a_{i}\right)_{i \in \mathbb{N}}$ is a sequence in $\mathfrak{m}_{F}$ tending to 0 , then the infinite product

$$
\prod_{i=1}^{\infty}\left(1-\frac{\left[a_{i}\right]}{\pi}\right)
$$

converges in $B^{+}$.

Theorem 7: For each $f \in B$ there exists a sequence $\left(a_{i}\right)_{i \in \mathbb{N}}$ in $\mathfrak{m}_{F}$ tending to 0 and $a g \in B$ such that

$$
f=g \cdot \prod_{i=1}^{\infty}\left(1-\frac{\left[a_{i}\right]}{\pi}\right)
$$

One can choose $g$ such that $\left.\operatorname{Newt}(g)\right|_{(-\infty, A]}=\infty$ for some $A \in \mathbb{R}$. Iff $\in B^{+}$then one can choose $g \in B^{b,+}$.

Idea of proof: Use the above identity for infinitely many slopes and write

$$
f=g_{n} \prod_{i=1}^{n}\left(1-\frac{\left[a_{i}\right]}{\pi}\right)
$$

for each $n$. Then do some calculations to see that the sequence of the $g_{n}$ converges and that the limit has the claimed properties (one uses the characterization of the various subrings of $B$ by Newton polygons from the talk FF-I). For details, see [FF13, Thm. 6.50].

## 4 Divisors on $|Y|$

We define

$$
\operatorname{Div}^{+}(Y)=\left\{\begin{array}{ll}
\sum_{\mathfrak{m} \in|Y|} a_{\mathfrak{m}}[\mathfrak{m}]: & a_{\mathfrak{m}} \in \mathbb{N}_{0}, \text { for each compact interval } I \subseteq(0,1) \\
\text { the set }\left\{\mathfrak{m} \in|Y|: a_{\mathfrak{m}} \neq 0 \text { and }\|\mathfrak{m}\| \in I\right\} \text { is } \\
\text { finite }
\end{array}\right\}
$$

and further

$$
\begin{gathered}
\operatorname{div}(f)=\sum_{\mathfrak{m} \in|Y|} \operatorname{ord}_{\mathfrak{m}}(f)[\mathfrak{m}], \quad \text { for } f \in B, \\
\mathfrak{a}_{-D}=\{f \in B: \operatorname{div}(f) \geq D\}, \quad \text { for } D \in \operatorname{Div}^{+}(Y)
\end{gathered}
$$

One can check that the function ord $_{\mathfrak{m}}$ is upper semi-continuous and therefore the ideals $\mathfrak{a}_{-D}$ are closed.

Theorem 8: Let $I \subseteq(0,1)$ be a nonempty compact subinterval.
(a) If $I=\{\rho\}$ for some $\rho \notin\left|F^{\times}\right|$then $B_{I}$ is a field.
(b) If not, $B_{I}$ is a principal ideal domain with maximal ideals the $B_{I} \mathrm{~m}$ where $m$ runs through the $\mathfrak{m} \in|Y|$ such that $\|\mathfrak{m}\| \in I$.

Sketch of proof: One can define a "restricted Newton polygon" Newt ${ }_{I}(f)$ for $f \in B_{I}$ : for $f \in B^{\mathrm{b}}$, one defines $\operatorname{Newt}_{I}(f)$ just as $\operatorname{Newt}(f)$ but without the slopes $\lambda$ such that $q^{-\lambda} \notin I$, and for a general $f \in B_{I}$ by approximation. Analogously to Corollary 5, one can show that the slopes different from $0, \infty$ of $\operatorname{Newt}_{I}(f)$ are the $w_{\mathfrak{m}}(\pi)$ for the $\mathfrak{m} \in|Y|$ such that $f \in \mathfrak{m}$ and $\|\mathfrak{m}\| \in I$, with multiplicity $\operatorname{ord}_{\mathfrak{m}}(f)$. Similarly to (1), we obtain then a factorization

$$
\begin{equation*}
f=g \cdot \prod_{i=1}^{r} \xi_{i} \tag{2}
\end{equation*}
$$

with the $\xi_{i}$ irreducible primitive and such that $\left\|\left(\xi_{i}\right)\right\| \in I$, and with $g \in B_{I}$ such that $\operatorname{Newt}_{I}(g)$ is the empty polygon. Finally, it is not difficult to see that if one has $\operatorname{Newt}_{I}(g)=\varnothing$ for some $g \in B_{I}$ then $g \in B_{I}^{\times}$.
Now (a) follows since $\rho \notin\left|F^{\times}\right|$, because then all Newton polygons are empty. For (b), it suffices to see that prime ideals are principal. If $\mathfrak{p}$ is a prime ideal and $f \in \mathfrak{p}$, we factor $f$ as in (2) and see that $\xi_{i} \in \mathfrak{p}$ for an $i$. But then $\mathfrak{p}=\left(\xi_{i}\right)$ because the ideal generated by $\xi_{i}$ is maximal.
For details, see [ $\mathrm{FF}_{14}$, Thm. 3.8].
We draw consequences of this theorem. They all can be seen similarly by using

$$
B=\lim _{\overleftarrow{ }} B_{I}, \quad \mathfrak{a}=\lim _{\overleftarrow{ }} B_{I} \mathfrak{a} \text { for closed ideals } \mathfrak{a} \subseteq B
$$

and the above theorem.

- $\operatorname{div}(f)$ is well-defined,
- if $D=\operatorname{div}(f)$ then $\mathfrak{a}_{-D}=(f)$,
- the map $D \longmapsto \mathfrak{a}_{-D}$ induces an isomorphism of monoids between $\operatorname{Div}^{+}(Y)$ and the monoid of closed nonzero ideals of $B$, and $D \leq D^{\prime} \Longleftrightarrow \mathfrak{a}_{-D^{\prime}} \subseteq \mathfrak{a}_{-D}$,
- for $f, g \in B \backslash\{0\}$, we have $g \mid f \Longleftrightarrow \operatorname{div}(f) \geq \operatorname{div}(g)$; in particular, the map div: $B \backslash\{0\} / B^{\times} \longrightarrow \operatorname{Div}^{+}(Y)$ is an injection of monoids,
- $|Y|$ is the set of closed maximal ideals of $B$.


## 5 Divisors on $Y^{\text {ad }} / \varphi^{\mathbb{Z}}$

Motivation. Suppose we want to classify $\varphi$-modules over $B$, i.e. free $B$-modules equipped with a $\varphi$-semilinear automorphism (which is usually also denoted by $\varphi$ ). These should correspond to vector bundles on $Y^{\text {ad }} / \varphi^{\mathbb{Z}}$. We have a bijection

$$
\begin{aligned}
& \mathbb{Z}\text { isomorphism classes of rank } 1 \varphi \text {-modules }\}, \\
& d \longmapsto B \text {-module with basis } e \text { and } \varphi(e)=\pi^{d} e .
\end{aligned}
$$

Hence we should have an isomorphism

$$
\mathbb{Z} \longrightarrow \operatorname{Pic}\left(Y^{\mathrm{ad}} / \varphi^{\mathbb{Z}}\right), \quad d \longmapsto \mathcal{L}^{\otimes d}
$$

where $\mathcal{L}$ is a line bundle such that

$$
\mathrm{H}^{0}\left(Y^{\mathrm{ad}} / \varphi^{\mathbb{Z}}, \mathcal{L}^{\otimes d}\right)=B^{\varphi=\pi^{d}}
$$

This leads us to studying the scheme

$$
\operatorname{Proj}\left(\bigoplus_{d \geq 0} B^{\varphi=\pi^{d}}\right)
$$

In the next talk (FF-IV) we will see that this scheme is in fact an algebraic version of the Fontaine-Fargues curve. See $\left[\mathrm{FF}_{14}, \S 4.1\right]$ for a more detailed version of this motivation.

Definition: For $d \geq 0$, let $P_{d}=B^{\varphi=\pi^{d}}$ and

$$
P=\bigoplus_{d \geq 0} P_{d}
$$

Note that $P_{0}=E$, hence $P$ is a graded $E$-algebra. Further put

$$
\operatorname{Div}^{+}\left(Y / \varphi^{\mathbb{Z}}\right)=\operatorname{Div}^{+}(Y)^{\varphi=\mathrm{id}}=\left\{D \in \operatorname{Div}^{+}(Y): \varphi^{*} D=D\right\} .
$$

Lemma 9: The map

$$
|Y| / \varphi^{\mathbb{Z}} \longrightarrow \operatorname{Div}^{+}\left(Y / \varphi^{\mathbb{Z}}\right), \quad \mathfrak{m} \longmapsto \sum_{n \in \mathbb{Z}} \varphi^{n}(\mathfrak{m})
$$

is well-defined and injective and makes $\operatorname{Div}^{+}\left(Y / \varphi^{\mathbb{Z}}\right)$ a free abelian monoid over $|Y| / \varphi^{\mathbb{Z}}$.
Sketch of proof: For an interval $\left[\rho_{1}, \rho_{2}\right] \subseteq(0,1), \varphi$ maps $B_{\left[\rho_{1}, \rho_{2}\right]} \longrightarrow B_{\left[\rho_{1}^{q}, \rho_{2}^{q}\right]}$, hence the map is well-defined. The rest is easy.

Theorem 10: The map

$$
\operatorname{div}:\left(\cup_{d \geq 0} P_{d} \backslash\{0\}\right) / E^{\times} \longrightarrow \operatorname{Div}^{+}\left(Y / \varphi^{\mathbb{Z}}\right)
$$

is an isomorphism of graded monoids.
Sketch of proof: For the injectivity, let $x \in P_{d}, y \in P_{d^{\prime}}$ such that $\operatorname{div}(x)=\operatorname{div}(y)$, and assume without loss of generalty that $d^{\prime} \geq d$. Then $(x)=\mathfrak{a}_{-\operatorname{div}(x)}=\mathfrak{a}_{-\operatorname{div}(y)}=(y)$, so there is a $u \in B^{\times}=\left(B^{b}\right)^{\times}$such that $x=u y$. A calculation then shows that $u \in$ $\left(B^{\mathrm{b}}\right)^{\varphi=\pi^{d-d^{\prime}}}$. The latter is 0 if $d \neq d^{\prime}$ and $E$ else, as was explained in talk FF-I.
For the surjectivity, let $x \in W_{\mathcal{O}_{E}}\left(\mathcal{O}_{F}\right)$ be primitive of degree $d$ and $D=\operatorname{div}(x)$. We want to construct an $f \in P_{d} \backslash\{0\}$ such that

$$
\operatorname{div}(f)=\sum_{n \in \mathbb{Z}} \varphi^{n}(D) .
$$

Without loss of generality we can assume that $x \in \pi^{d}+W_{\mathcal{O}_{E}}\left(\mathfrak{m}_{F}\right)$, and one can show that then

$$
\Pi^{+}(x)=\prod_{n \geq 0} \frac{\varphi^{n}(x)}{\pi^{d}}
$$

converges, and obviously

$$
\operatorname{div}\left(\Pi^{+}(x)\right)=\sum_{n \geq 0} \varphi^{n}(D)
$$

The problem is that the product

$$
\prod_{n<0} \varphi^{n}(D)
$$

does not converge. If it would converge, it would have the property $\varphi\left(\Pi^{-}(x)\right)=x \Pi^{-}(x)$. However, using Kummer and Artin-Schreier equations, one can prove that for each primitive $z \in W_{\mathcal{O}_{E}}\left(\mathcal{O}_{F}\right)$ there exists a $\Pi^{-}(z) \in B^{b,+} \backslash\{0\}$ which is unique up to an $E^{\times}$-multiple, and such that $\varphi\left(\Pi^{-}(z)\right)=z \Pi^{-}(z)$.
For $\Pi^{-}(x)$, we then have

$$
\varphi\left(\operatorname{div}\left(\Pi^{-}(x)\right)\right)=\operatorname{div}\left(x \Pi^{-}(x)\right)=D+\operatorname{div}\left(\Pi^{-}(x)\right)
$$

which by applying $\varphi^{-1}$ yields $\operatorname{div}\left(\Pi^{-}(x)\right)=\varphi^{-1}(D)+\varphi\left(\operatorname{div}\left(\Pi^{-}(x)\right)\right)$. By repeatedly applying this, we see that

$$
\operatorname{div}\left(\Pi^{-}(x)\right)=\sum_{n<0} \varphi^{n}(D)
$$

Hence $f=\Pi^{+}(x) \Pi^{-}(x)$ does the job!
For details, see [FF14, Thm. 4.3, Prop. 4.5] or [FF13, Thm. 9.7 (3)].
Corollary 11: $P$ is a graded factorial algebra with irreducible elements of degree 1.

## References

[FF13] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique. 2013. URL: http://webusers.imj-prg.fr/~laurent. fargues/Prepublications.html.
[FF14] Laurent Fargues and Jean-Marc Fontaine. "Vector bundles on curves and p-adic Hodge theory". In: Automorphic Forms and Galois Representations. Proceedings of the Durham EPSRC Symposium. 2014. Forthcoming.

