Short Introduction to Adic Spaces

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Abstract.

This is a short introduction to adic spaces for the participants of the Workshop "$\mathrm{Gal}_{\mathbb{Q}_p}$ as a geometric fundamental group" in Neckarbischofsheim organized by Sujahta Ramdorai, Peter Schneider, and Otmar Venjakob. No originality is claimed and no proofs are given! As a reference we refer to [Hu1], [Hu2], [Hu3], [Hu4], [Co], [Wd], and [CW].

Notation

All rings are commutative with 1. Complete (uniform) spaces are always Hausdorff by definition. If R and S are subsets of a ring A, we denote by $R \cdot S$ the additive subgroup of A generated by $\{ rs \ ; \ r \in R, s \in S \}$.

All complete spaces are by definition Hausdorff.

1 Huber pairs

Definition 1.1 (Huber rings). Let A be a topological ring.

(1) A is called *adic* if there exists ideal $I \subseteq A$ such that $(I^n)_n$ is a basis of neighborhoods of 0 in A. Such an ideal I is called an *ideal of definition.*

Example: $\mathbb{Z}_p$, $I = (p)$.

(2) A is called *Huber ring*[1] if there exists an open adic subring $A_0 \subseteq A$ (called *ring of definition of A*) with finitely generated ideal of definition I (called also *ideal of definition of A* by abuse of language).

Example: $A = \mathbb{Q}_p$, $A_0 = \mathbb{Z}_p$ or $A = A_0 = \mathbb{Z}_p$.

(3) A subset $S \subseteq A$ is *bounded* if for every neighborhood U of 0 in A there exists a neighborhood V of 0 with $\{ vs \ ; \ v \in V, s \in S \} \subseteq U$. An element $a \in A$ is *power bounded* (resp. *topological nilpotent*) if $\{ a^n \ ; \ n \geq 1 \}$ is bounded (resp. if $\lim_{n\to\infty} a^n = 0$). Set

$$A^\circ := \{ a \in A \ ; a \text{ power bounded}\},$$
$$A^{\circ\circ} := \{ a \in A \ ; a \text{ is topologically nilpotent}\}.$$

[1] Huber calls such rings f-adic which might lead to confusion as there is no f. Moreover this notion is neither a generalization nor a specializiation of the notion of an adic ring. This may be a good reason to introduce this new terminology following Scholze.

(4) A is called *Tate ring* if A is a Huber ring and has a topologically nilpotent unit.
Example: $A = \mathbb{Q}_p$ with topological nilpotent unit p.

Note that every open subgroup of A (and in particular every open subring of A) is also closed in A.

Example 1.2. Let A be a ring endowed with a seminorm $|\cdot|: A \to \mathbb{R}_{\geq 0}$ (i.e., $|0| = 0$, $|a - b| \leq |a| + |b|$, $|ab| \leq |a||b|$ for all $a, b \in A$). Then there exists a unique topology on A making A into a topological ring such that $(\{ a \in A \ ; \ |a| < r \})_{r>0}$ is a fundamental system of neighborhoods of 0. Then $S \subseteq A$ is bounded if and only if $\{ |a| \ ; \ a \in S \}$ is bounded in $\mathbb{R}$.

Suppose that $|\cdot|$ is power-multiplicative ($|a^n| = |a|^n$ for all $a \in A$, $n \in \mathbb{N}$). Then

$$A^\circ = \{ a \in A \ ; \ |a| \leq 1 \}, \qquad A^{\circ\circ} = \{ a \in A \ ; \ |a| < 1 \}.$$

For instance the supremum seminorm on a k-affinoid algebra (see Example 1.5 below) is non-archimedean and power-multiplicative.

Remark 1.3. Let A be a Huber ring.

(1) Every ring of definition of A is bounded and A° is the filtered union of all rings of definition of A. It is an integrally closed open subring of A and $A^{\circ\circ}$ is an ideal of A°, equal to its radical.

(2) Let A be a Tate ring, A_0 a ring of definition, ϖ a topologically nilpotent unit of A. Then there exists $n \geq 1$ with $\varpi^n \in A_0$, A_0 is ϖ^n-adic, and $A = A_0[\varpi^{-n}]$. Moreover, a subset $S \subseteq A$ is bounded iff $S \subseteq \varpi^{-m} A_0$ for some $m \geq 1$.

In general, A° is not a ring of definition: Every ring of definition is clearly bounded but for $A = \mathbb{Q}_p[T]/(T^2)$ one has $A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p T$ which is not bounded.

Definition 1.4 (Huber pairs). A *Huber pair*[2] is a pair (A, A^+), where A is a Huber ring and $A^+ \subseteq A^\circ$ is an open subring of A that is integrally closed in A (such a ring is called a *ring of integral elements of* A).

A *morphism of Huber pairs* $(A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $\varphi: A \to B$ such that $\varphi(A^+) \subseteq B^+$.

It is called *adic* if there exist ring of definitions $A_0 \subseteq A$ and $B_0 \subseteq B$ and an ideal of definition $I \subseteq A_0$ such that $\varphi(A_0) \subseteq B_0$ and such that $\varphi(I)B_0$ is an ideal of definition of B_0.

We will very often consider the case that $A^+ = A^\circ$ but for some natural constructions it is necessary to have the added flexibility to consider other rings A^+.

Example 1.5. (1) Let A be an adic ring with finitely generated ideal of definition I. Then A is Huber with $A = A_0$ and ideal of definition I. Every subset of A is bounded and $A^\circ = A$. Hence (A, A) is a Huber pair.

A special case is a ring A endowed with the discrete topology, i.e. $I = 0$. Then for every subring A^+ of A that is integrally closed in A, (A, A^+) is a Huber pair.

[2] Huber uses the notion of an affinoid ring which might be confused with affinoid algebras from rigid geometry. Moreover we are speaking here really of a pair of rings.

(2) Let k be a *non-archimedean field* (i.e., k is a topological field whose topology is given by a nontrivial non-archimedean norm $|\cdot|\colon k \to \mathbb{R}_{\geq 0}$). Then $\mathcal{O}_k := \{\, a \in k \;;\; |a| \leq 1 \,\} = k^\circ$ and $(k, \mathcal{O}_k)$ is a Huber pair. Every element $0 \neq \varpi \in k$ with $|\varpi| < 1$ is a topologically nilpotent unit.

(3) More generally, let $(A, \|\cdot\|)$ be any k-Banach algebra with power-multiplicative norm. Then A is a Tate ring with topologically nilpotent unit ϖ. As a ring of definition one can take $A_0 = A^\circ = \{\, a \in A \;;\; \|a\| \leq 1 \,\}$ which is an $\mathcal{O}_k$-algebra.

(4) Suppose that k is complete. An example for a k-Banach algebra is the *Tate algebra*

$$T_{n,k} := \{ \sum_\nu a_\nu t^\nu \in k[\![t_1, \dots, t_n]\!] \;;\; a_\nu \to 0 \text{ for } \sum_i \nu_i \to \infty \}$$

with its *Gauß norm*

$$\| \sum_\nu a_\nu t^\nu \| := \max_\nu |a_\nu|.$$

This is a multiplicative norm ($\|ab\| = \|a\|\|b\|$). Then $T_{n,k}^\circ = \mathcal{O}_k\langle t_1, \dots, t_n \rangle$.

Every quotient $\pi\colon T_{n,k} \to A := T_{n,k}/\mathfrak{a}$ is a k-Banach algebra ($T_{n,k}$ can be shown to be noetherian, therefore every ideal $\mathfrak{a}$ of $T_{n,k}$ is closed and hence A is complete because $T_{n,k}$ is metrizable) with norm

$$\|a\|_\pi := \inf\{ \, \|f\| \;;\; f \in T_{n,k} \text{ with } \pi(f) = a \}.$$

Topological k-algebras of this form are called *k-affinoid algebras.* The equivalence class of $\|\cdot\|_\pi$ does not depend on the presentation of A as quotient of a Tate algebra. The norm $\|\cdot\|_\pi$ is in general not power-multiplicative.

But k-affinoid algebras A carry also a non-archimedean power-multiplicative semi-norm defined by

$$\|f\|_{\sup} := \sup_{x \in \operatorname{Max}(A)} |f(x)|,$$

where $|f(x)|$ denotes the absolute value of the image of f in the finite extension $A/\mathfrak{m}_x$ of k for a maximal ideal $x = \mathfrak{m}_x$ of A. The pair (A, A°) is a Huber pair. If $A = T_{n,k}$, then $\|\cdot\|_{\sup}$ is the Gauß norm.

Note that for non-reduced k-affinoid algebras this seminorm yields a coarser topology than the standard Banach space topology but that the sets of power bounded and of topologically nilpotent elements are the same for both topologies.

2 Construction with Huber pairs

Completion

Let $A = (A, A^+)$ be a Huber pair. Let $\hat{A}$ be the I-adic completion with respect to some ideal of definition (completion of A as additive topological group). Then the ring multiplication of A extends to $\hat{A}$ making $\hat{A}$ into a topological ring. Moreover $\hat{A}$ is complete with respect to the $\bar{I}$-adic topology (this is nontrivial because A has not necessarily a noetherian ring of definition; here it is important that I is finitely generated). Let $\hat{A}^+$ be the integral closure of the closure of A^+ in $\hat{A}$. Then $\hat{A} = (\hat{A}, \hat{A}^+)$ is a Huber pair, called the *completion of* (A, A^+).

The canonical morphism $A \to \hat{A}$ of Huber pairs is universal with respect to morphisms to complete Huber pairs B (a Huber pair (B, B^+) is called *complete* if B is complete; then B^+ (as a closed subspace of B) is also complete).

Quotients

Let $A = (A, A^+)$ be a Huber pair, let $\mathfrak{a} \subseteq A$ be an ideal, and endow $A/\mathfrak{a}$ with the quotient topology. Then the canonical map $A \to A/\mathfrak{a}$ is surjective, continuous, and open. Let $(A/\mathfrak{a})^+$ be the integral closure of $A^+/(A^+ \cap \mathfrak{a})$ in $A/\mathfrak{a}$. Then $A/\mathfrak{a} := (A/\mathfrak{a}, (A/\mathfrak{a})^+)$ is a Huber pair: If $A_0 \subseteq A$ is a ring of definition, its image in $A/\mathfrak{a}$ is a ring of definition in $A/\mathfrak{a}$.

If A is a complete Huber pair and $\mathfrak{a} \subseteq A$ is a closed ideal, then the Huber pair $A/\mathfrak{a}$ is complete (note that the topology of any Hausdorff Huber ring is metrizable).

A morphism $\varphi\colon A \to B$ of Huber pairs factors through the Huber pair morphism $A \to A/\mathfrak{a}$ if and only if $\varphi(f) = 0$ for all $f \in \mathfrak{a}$.

Restricted Power Series

Let $A = (A, A^+)$ be a Huber pair, let $T_1, \dots, T_n$ be finite subsets of A such that $T_i \cdot A$ is open in A for all i (equivalently, $T^\nu := T_1^{\nu_1} \cdot \dots \cdot T_n^{\nu_n} \cdot U$ is an open neighborhood of 0 for every open subgroup U and for all $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$). Define a subring of the ring of formal power series $\hat{A}[\![X]\!] := \hat{A}[\![X_1, \dots, X_n]\!]$ as follows

$$A\langle X\rangle_T := \Big\{ \sum_\nu a_\nu X^\nu \in \hat{A}[\![X]\!] \ ; \ \begin{matrix} a_\nu \in T^\nu \cdot U \text{ for all open subgroups} \\ U \text{ of } \hat{A} \text{ and for almost all } \nu \end{matrix} \Big\}.$$

We endow $A\langle X\rangle_T$ with the (unique) structure of a topological ring such that the subgroups (for U open subgroup in A)

$$U_{\langle X\rangle} := \{ \sum_\nu a_\nu X^\nu \in \hat{A}\langle X\rangle_T \ ; \ a_\nu \in T^\nu \cdot U \text{ for all } \nu \in \mathbb{N}_0^n \}$$

form a fundamental system of neighborhoods of 0 in $A\langle X\rangle_T$. We also write simply $A\langle X\rangle = A\langle X_1, \dots, X_n\rangle$ if $T_i = \{1\}$ for all $i \in 1, \dots, n$.

Then $A\langle X\rangle_T$ is a complete Huber ring such that tX_i is power bounded for all $t \in T_i$. The canonical morphism $\iota\colon A \to A\langle X\rangle_T$ has the follwing universal property. If $\varphi\colon A \to B$ is a morphism of Huber pairs with B complete and if $(x_i)_{1 \le i \le n}$ is a family of elements in B such that $\varphi(t)x_i \in B^\circ$ for all $i = 1, \dots, n$ and $t \in T_i$, then there exists a unique continuous ring homomorphism $\psi\colon A\langle X\rangle_T \to B$ with $\psi \circ \iota = \varphi$ and $\psi(X_i) = x_i$.

Let us now define a ring of integral element for $A\langle X\rangle_T$ depending on A^+:

$$\{ \sum_{\nu \in \mathbb{N}_0^n} a_\nu X^\nu \in A\langle X\rangle_T \ ; \ a_\nu \in T^\nu \cdot \widehat{(A^+)} \text{ for all } \nu \in \mathbb{N}_0^n \}$$

is a subring of $A\langle X\rangle_T$. Its integral closure in $A\langle X\rangle_T$ is a ring of integral elements of $A\langle X\rangle_T$ denoted by $A\langle X\rangle_T^+$, and

$$A\langle X_1,\dots,X_n\rangle_{T_1,\dots,T_n} := A\langle X\rangle_T := (A\langle X\rangle_T, A\langle X\rangle_T^+)$$

is a complete Huber pair.

The canonical morphism $\iota\colon (A,A^+) \to (A\langle X\rangle_T, A\langle X\rangle_T^+)$ of Huber pairs has the following universal property. If $\varphi\colon (A,A^+) \to (B,B^+)$ is a morphism of Huber pairs with B complete and if $(x_i)_{1\le i\le n}$ is a family of elements in B such that $\varphi(t)x_i \in B^+$ for all $i = 1,\dots,n$ and $t \in T_i$, then there exists a unique morphism of Huber pairs $\psi\colon (A\langle X\rangle_T, A\langle X\rangle_T^+) \to (B,B^+)$ with $\psi\circ\iota = \varphi$ and $\psi(X_i) = x_i$.

Example 2.1. Let $(k,|\cdot|)$ be a complete non-archimedean field. Then $k\langle X_1,\dots,X_n\rangle = T_{n,k}$ is the Tate algebra.

More generally, fix $r_1,\dots,r_n \in \mathbb{Z}$, set $\rho_i := |\pi^{-r_i}|$, $T_i := \{\pi^{r_i}\} \subseteq k \subseteq A$ for $i = 1,\dots,n$. As usual, write $\rho^\nu := \rho_1^{\nu_1}\cdots\rho_n^{\nu_n}$ for $(\nu_1,\dots,\nu_n) \in \mathbb{N}_0^n$. Then

$$k\langle X\rangle_T = \{\sum_\nu a_\nu X^\nu \in k[\![X]\!] \ ; \lim_{\sum\nu_i\to\infty} |a_\nu|\rho^\nu = 0\}$$

is a k-affinoid algebra whose k-Banach algebra structure is given by the norm

$$\|\sum_\nu a_\nu X^\nu\|_\rho := \max_\nu |a_\nu|\rho^\nu.$$

It is the algebra of power series $f \in k[\![X]\!]$ which converge on the polydisc

$$P_\rho := \{\, x \in k^n \ ; \ |x_i| \le \rho_i \text{ for } i = 1,\dots,n\}.$$

Localization

Let A be a Huber ring, $s \in A$, $\emptyset \ne T = \{t_1,\dots,t_n\} \subseteq A$ finite such that ideal of A generated by T is open in A. Choose (A_0, I) a pair of definition. Let $D := A_0[\frac{t_1}{s},\dots,\frac{t_n}{s}] \subseteq A_s = A[s^{-1}]$ and define a topology on A_s by choosing $(I^n\cdot D)_{n\ge1}$ as fundamental system of neighborhoods of 0. Denote this topological ring by $A(\frac{T}{s})$. It is a Huber ring with (D, ID) as pair of definition such that $\frac{t}{s}$ is power bounded for all $t \in T$. Let $A\langle\frac{T}{s}\rangle$ be its completion (again a Huber ring).

The canonical ring homomorphism $A \to A(\frac{T}{s})$ (resp. $A \to A\langle\frac{T}{s}\rangle$) is continuous and it is universal with respect to all continuous homomorphisms $\varphi\colon A \to B$ to Huber rings (resp. to complete Huber rings) with $\varphi(s) \in B^\times$ and such that $\varphi(t)/\varphi(s)$ is power bounded for all $t \in T$.

Now let $A = (A,A^+)$ be a Huber pair. Let $A(\frac{T}{s})^+$ be the integral closure of $A^+[\frac{t_1}{s},\dots,\frac{t_n}{s}]$ in $A(\frac{T}{s})$. Then $A(\frac{T}{s}) := (A(\frac{T}{s}), A(\frac{T}{s})^+)$ is a Huber pair. Let $A\langle\frac{T}{s}\rangle := (A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$ be its completion. The morphism of Huber pairs $(A,A^+) \to (A\langle\frac{T}{s}\rangle, A\langle\frac{T}{s}\rangle^+)$ is universal for morphism of Huber pairs $\varphi\colon A \to B$ with B complete, $\varphi(s) \in B^\times$ and $\frac{\varphi(t)}{\varphi(s)} \in B^+$.

One has an isomorphism of Huber pairs $A\langle\frac{T}{s}\rangle \cong A\langle X\rangle_T/\overline{(1-sX)}$ because both rings satisfy the same universal property with respect to morphisms to complete Huber pairs.

3 The adic spectrum of a Huber pair

Definition 3.1. Let A be a topological ring. A *valuation of A* is a map $|\cdot|\colon A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group (written multiplicatively) such that

$$|0| = 0, \quad |1| = 1, \quad |ab| = |a||b|, \quad |a+b| \leq \max\{|a|, |b|\}$$

for all $a, b \in A$. It is called *continuous* if $\{\, a \in A \;;\; |a| < \gamma \,\}$ is open in A for all $\gamma \in \Gamma$.

For every valuation $|\cdot|$ its *support* $\operatorname{supp}(|\cdot|) := \{\, a \in A \;;\; |a| = 0 \,\}$ is a prime ideal of A. A continuous valuation is called *analytic* if $\operatorname{supp}(|\cdot|)$ is not an open prime ideal in A.

Two such valuations $|\cdot|$ and $|\cdot|'$ are called *equivalent* if for all $a, b \in A$ one has $|a| \geq |b| \Leftrightarrow |a|' \geq |b|'$.

We can now define the underlying topological space of the adic spectrum. Let A be a Huber ring and let $\Sigma \subseteq A^\circ$ be any subset. Define

$$\begin{aligned}\operatorname{Spa}(A, \Sigma) := \{&\text{equivalence classes of} \\ &\qquad \text{continuous valuations } |\cdot| \text{ on } A\,;\, |f| \leq 1 \text{ for all } f \in \Sigma\}\end{aligned}$$

as a set. For $x \in \operatorname{Spa}(A, \Sigma)$ and $f \in A$ we write $|f(x)|$ instead of $x(f)$. We endow $\operatorname{Spa}(A, \Sigma)$ with the topology generated by the subsets

$$\{\, x \in \operatorname{Spa}(A, \Sigma) \;;\; |f(x)| \leq |g(x)| \neq 0 \,\}$$

with $f, g \in A$.

For a subset $\Sigma \subseteq A^\circ$ let A_+ be the integral closure of the subring of A generated by Σ and $A^{\circ\circ}$. Then $\operatorname{Spa}(A, \Sigma) = \operatorname{Spa}(A, A^+)$. Moreover, if A is a Huber ring, then (A, A^+) is a Huber pair.

Theorem 3.2. *Let (A, A^+) be a Huber pair. Then $X := \operatorname{Spa}(A, A^+)$ is a spectral space (i.e., $X \cong \operatorname{Spec}(R)$ for some ring R, equivalently, X is the limit of finite T_0-spaces).*

A basis of the topology consisting of open quasi-compact subspaces is given by rational subsets*, i.e. by subsets of the form*

$$X(\frac{T}{s}) := \{\, x \in X \;;\; \forall\, t \in T\colon |t(x)| \leq |s(x)| \neq 0 \,\},$$

where $s \in A$ and $\emptyset \neq T \subseteq A$ a finite subset such that TA is an open ideal of A. Finite intersections of rational subsets are again rational.

Remark 3.3. Let $\varphi\colon (A, A^+) \to (B, B^+)$ be a morphism of Huber pairs. Then $\gamma \mapsto \gamma \circ \varphi$ is a well defined continuous map $\operatorname{Spa}(\varphi)\colon X := \operatorname{Spa}(B, B^+) \to Y := \operatorname{Spa}(B, B^+)$.

If $\varphi\colon A \to B$ is adic, then the inverse image of a rational subset of Y is a rational subset of X.

Example 3.4. Let $A = (A, A^+)$ be a Huber pair, let $\mathfrak{a} \subseteq A$ be an ideal, and let $\pi\colon A \to A/\mathfrak{a}$ be the canonical morphism of Hubert pairs. Then $\operatorname{Spa}(\pi)\colon \operatorname{Spa} A/\mathfrak{a} \to \operatorname{Spa} A$ is a homeomorphism of $\operatorname{Spa} A/\mathfrak{a}$ onto the closed subset of points $x \in \operatorname{Spa} A$ with $\operatorname{supp}(x) \supseteq \mathfrak{a}$.

Proposition 3.5. *Let (A, A^+) be a Huber pair. The canonical map* $\mathrm{Spa}(\hat{A}, \hat{A}^+) \to \mathrm{Spa}(A, A^+)$ *is a homeomorphism respecting analytic points and rational subsets in both directions.*

4 Structure (Pre)Sheaf on $\mathrm{Spa}(A, A^+)$

Lemma 4.1. *The map* $\mathrm{Spa}(A\langle \frac{T}{s}\rangle, A\langle \frac{T}{s}\rangle^+) \to X := \mathrm{Spa}(A, A^+)$ *is an open embedding with image $X(\frac{T}{s})$. Via this map rational subsets of* $\mathrm{Spa}(A\langle \frac{T}{s}\rangle, A\langle \frac{T}{s}\rangle^+)$ *correspond to rational subsets in X that are contained in $X(\frac{T}{s})$.*

For $U = X(\frac{T}{s})$ we define $\mathscr{O}_X(U) := A\langle \frac{T}{s}\rangle$ and $\mathscr{O}_X^+(U) := A\langle \frac{T}{s}\rangle^+$. One shows that one obtains well defined presheaves $\mathscr{O}_X$ and $\mathscr{O}_X^+$ defined on the basis of rational subsets of X. Note that

$$(\mathscr{O}_X(X), \mathscr{O}_X^+(X)) = (\hat{A}, \hat{A}^+).$$

For every rational subset U of X one has

$$(*) \qquad \mathscr{O}_X^+(U) = \{\, f \in \mathscr{O}_X(U) \ ; \ |f(x)| \leq 1 \text{ for all } x \in U\}.$$

If $\mathscr{O}_X$ is a sheaf on the basis of rational subsets with values in the category of complete topological rings, it extends uniquely to a sheaf on X by setting for every open subset V of X

$$\mathscr{O}_X(V) = \lim_{U \subseteq V \text{ rational}} \mathscr{O}_X(U).$$

This is a sheaf with values in the category of complete topological rings. Then $\mathscr{O}_X^+$ is also a sheaf with values in the category of complete topological rings by (*). Moreover (*) holds for every open subset U of X.

Theorem 4.2. *Let (A, A^+) be a Huber pair, $X := \mathrm{Spa}(A, A^+)$. Then $\mathscr{O}_X$ is a sheaf (in which case we call (A, A^+)* sheafy*) and its higher cohomology vanishes on rational subsets in each of the following situations.*

(1) *$\hat{A}$ has a noetherian ring of definition.*
(2) *A is a Tate ring and $A\langle X_1, \dots, X_n\rangle$ is noetherian for all n.*
(3) *A is a Tate ring and for all rational subsets $U \subseteq X := \mathrm{Spa}(A, A^+)$ the ring $\mathscr{O}_X(U)^\circ$ is bounded.*
(4) *$\hat{A}$ has the discrete topology.*

In general $\mathscr{O}_X$ is not a sheaf. For a nice list of counterexamples see [BV] §4. But the conditions of Theorem 4.2 ensure that we do not run into this problem when considering the following categories as full subcategories of the category of adic spaces (see Definition 4.3 below): Condition (1) will ensure that one can consider every locally noetherian formal scheme as adic space, Condition (2) will ensure that one can consider every rigid analytic space as adic space, Condition (3) (proved in [BV]) will ensure that perfectoid spaces are adic spaces.

Let (A, A^+) be a Huber pair, $x \in X := \mathrm{Spa}(A, A^+)$. Let

$$\mathscr{O}_{X,x} := \operatorname*{colim}_{U \ni x \text{ open}} \mathscr{O}_X(U) = \operatorname*{colim}_{U \ni x \text{ rational}} \mathscr{O}_X(U)$$

be the stalk (colimit in the category of rings, hence $\mathscr{O}_{X,x}$ is not endowed with a topology). For every rational neighborhood U of x the valuation $x\colon A \to \Gamma_x \cup \{0\}$ extends uniquely to a valuation $v_U\colon \mathscr{O}_X(U) \to \Gamma_x \cup \{0\}$ by Lemma 4.1. Passing to the colimit we obtain a valuation $v_x\colon \mathscr{O}_{X,x} \to \Gamma_x \cup \{0\}$. One shows that for $U \ni x$ rational and $f \in \mathscr{O}_X(U)$ with $|f(x)| \neq 0$ the image of f in $\mathscr{O}_{X,x}$ is a unit and deduces that $\mathscr{O}_{X,x}$ is a local ring whose maximal ideal is the support of v_x.

Hence if (A, A^+) is a sheafy Huber pair, then $X := \mathrm{Spa}(A, A^+) = (X, \mathscr{O}_X, (v_x)_{x\in X})$ is an element of the category $\mathcal{V}$ whose objects consist of

(a) a topological space X,

(b) a sheaf $\mathscr{O}_X$ of complete topological rings on X such that the stalk $\mathscr{O}_{X,x}$ is a local ring,

(c) and for all $x \in X$ an equivalence class v_x of valuations on $\mathscr{O}_{X,x}$ whose support is the maximal ideal of $\mathscr{O}_{X,x}$.

We have the obvious notion of a morphism in the category $\mathcal{V}$. For every object $(X, \mathscr{O}_X, (v_x)_{x\in X})$ we set

$$\mathscr{O}_X^+(U) := \{\, f \in \mathscr{O}_X(U)\ ;\ |f(x)| \leq 1 \text{ for all } x \in U\}, \qquad U \subseteq X \text{ open.}$$

Definition 4.3. An *adic space* is an object of $\mathcal{V}$ that is locally isomorphic to $\mathrm{Spa}(A, A^+)$ for some sheafy Huber pair (A, A^+). The category of adic spaces is the full subcategory of $\mathcal{V}$ whose objects are the adic spaces. An adic space is called *affinoid*, if it is isomorphic to $\mathrm{Spa}(A, A^+)$ for some sheafy Huber pair (A, A^+).

We obtain a functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from the category of sheafy Huber pairs to the category of adic spaces. The canonical morphism of adic spaces $\mathrm{Spa}(\hat{A}, \hat{A}^+) \to \mathrm{Spa}(A, A^+)$ is an isomorphism of adic spaces. The functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from the category of sheafy *complete* Huber pairs to the category of adic spaces is fully faithful. More precisely one has for every adic space Y and every sheafy Huber pair (A, A^+) a bijection

$$\mathrm{Hom}(Y, \mathrm{Spa}(A, A^+)) \xrightarrow{\sim} \mathrm{Hom}((A, A^+), (\mathscr{O}_X(X), \mathscr{O}_X^+(X)),$$

where the right hand side denotes continuous ring homomorphisms $\varphi\colon A \to \mathscr{O}_X(X)$ such that $\varphi(A^+) \subseteq \mathscr{O}_X^+(X)$.

Example 4.4. (1) Let k be a non-archimedean field. Then $\mathrm{Spa}(k, k^\circ)$ consists of a single point x, the equivalence class of the valuation $|\cdot|\colon k \to \mathbb{R}_{\geq 0}$ defining the topology of k. One has $\kappa(x) = \hat{k}$.

(2) Let A be a valuation ring of height 1. Then $\mathrm{Spa}(A, A)$ consists of an open point η and a closed point s with $\kappa(\eta) = \mathrm{Frac}(A) =: k$ and $\kappa(s) = A/\mathfrak{m}_A$. The canonical morphism $S^0 = \mathrm{Spa}(k, A) \to S$ is an open immersion onto the open point.

Example 4.5. Endow $\mathbb{Z}$ and $\mathbb{Z}[t]$ with the discrete topology. Then $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ is the final object in the category of adic spaces and for every adic space X we find

$$\mathrm{Hom}(X, \mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z})) = \mathscr{O}_X(X),$$
$$\mathrm{Hom}(X, \mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z}[t])) = \mathscr{O}_X^+(X).$$

Endow $\mathbb{Z}[\![t]\!]$ with the t-adic topology and set $D^0 := \mathrm{Spa}(\mathbb{Z}[\![t]\!], \mathbb{Z}[\![t]\!])$. Then for every affinoid adic space $X = \mathrm{Spa}\, A$ one has

$$\mathrm{Hom}(X, D^0) = \mathscr{O}_X(X)^{\circ\circ} = \hat{A}^{\circ\circ}.$$

Indeed, first note that every integrally closed open subring of $\hat{A}$ contains $\hat{A}^{\circ\circ}$. In particular $(\hat{A}^+)^{\circ\circ} = \hat{A}^{\circ\circ}$. Every continuous homomorphism $\varphi\colon \mathbb{Z}[\![t]\!] \to \mathscr{O}_X^+(X) = \hat{A}^+$ is determined by the image a of t. As t is topologically nilpotent, a is topologically nilpotent. Conversely, let $a \in \mathscr{O}_X^+(X)$ be topologically nilpotent. As $\mathscr{O}_X^+(X)$ is complete and 0 has a fundamental system of neighborhood consisting of additive subgroups, a series $\sum_n \lambda_n a^n$ converges if and only if $\lim_n \lambda_n a^n = 0$. But this is the case if $\lambda_n \in \varphi(\mathbb{Z})$ because $\varphi(\mathbb{Z})$ is automatically bounded (as $\varphi(\mathbb{Z})$ is contained in every ring of definition of A) and a is topologically nilpotent.

We view D^0 as the "formal open unit disc".

Proposition and Definition 4.6. *Let X be an adic space. A point $x \in X$ is called* analytic *if the following equivalent conditions are satisfied.*

(i) *There exists an open neighborhood U of x such that $\mathscr{O}_X(U)$ contains a topologically nilpotent unit.*

(ii) *For every open affinoid neighborhood $U = \mathrm{Spa}\, A$ of x, the point* $\mathrm{supp}\, x \subset A$ *is not open in A (i.e., $x \in \mathrm{Spa}\, A$ is analytic as in Definition 3.1.)*

5 Morphisms locally of finite type and fiber products

Definition 5.1. A morphism $\varphi\colon A \to B$ of Huber pairs with B complete is called *topologically of finite type* if there exists an isomorphism of Huber pairs

$$B \cong A\langle X_1, \dots, X_n\rangle_{T_1,\dots,T_n}/\mathfrak{a}$$

for some $n \in \mathbb{N}_0$, finite subsets $T_1, \dots, T_n$ of A with $T_i \cdot A$ open in A, and $\mathfrak{a} \subseteq A\langle X\rangle_T$ a closed ideal.

Every morphism topologically of finite type is adic.

Example 5.2. Let k be a complete discretely valued field. Let A be a k-affinoid algebra. Then there is a unique ring of integral elements A^+ of A such that $(k, k^\circ) \to (A, A^+)$ is topologically of finite type, namely $A^+ = A^\circ$.

Definition 5.3. Let $f\colon X \to Y$ be a morphism of adic spaces. Then f is called *locally of finite type* if for every $x \in X$ there exists an open affinoid neighborhood $U = \mathrm{Spa}\, B$ of x in X and an open affinoid subspace $V = \mathrm{Spa}\, A$ of Y with $f(U) \subseteq V$ such that the induced homomorphism of complete Huber pairs $(A, A^+) \to (B, B^+)$ is topologically of finite type.

Proposition 5.4. *Let $f\colon X \to S$ and $g\colon Y \to S$ be morphisms of adic spaces. Then the fiber product of f and g exists in the category of adic spaces if f or g is locally of finite type.*

The explicit construction in [Hu4] 1.2.2 shows that even if S, X, and Y are affinoid, $X \times_S Y$ is not necessarily affinoid. Here we give only the following example.

Example 5.5. Let $\mathcal{O}_k$ be a complete discrete valuation ring, k its field of fractions and let $\pi \in k^\times$ be a uniformizing element. Then $k^\circ = \mathcal{O}_k$ and

$$\mathcal{O}_k\langle \frac{\pi}{\pi} \rangle = \mathcal{O}_k\langle X \rangle_{\{\pi\}}/(1 - \pi X) = k.$$

Hence the open immersion $\eta := \mathrm{Spa}(k, \mathcal{O}_k) \to S := \mathrm{Spa}(\mathcal{O}_k, \mathcal{O}_k)$ is locally of finite type. (In fact the open immersion of any rational subset is locally of finite type.)

Let $X := \mathrm{Spa}(A, A^+) \to S$ be a morphism of adic spaces with (A, A^+) complete and let $\varphi\colon \mathcal{O}_k \to A$ be the corresponding homomorphism of Huber pairs. Then the fiber product X_η is constructed as follows. Choose a finite set L of generators of an ideal of definition of A and let $B_m := A\langle \frac{\{\varphi(\pi)\} \cup L^m}{\varphi(\pi)} \rangle$, where $L^m = \{\, \ell_1 \cdots \ell_m \ ; \ \ell_i \in L \,\}$. Then

$$X_\eta = \operatorname*{colim}_m \mathrm{Spa}(B_m, B_m^+).$$

Below we will describe the generic fiber of $\mathrm{Spa}(\mathbb{Z}_p[\![t]\!], \mathbb{Z}_p[\![t]\!])$, the formal open p-adic unit disc, and see that it is the (adic space associated to the) rigid analytic open unit disc over $\mathbb{Q}_p$, which is not an affinoid space.

6 Formal Schemes and Rigid Spaces as Adic Spaces

Formal Schemes as Adic Spaces

For every complete noetherian adic ring A denote $\mathrm{Spf}(A)$ its formal spectrum. Then the functor $\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A)$ form noetherian affine formal schemes to the category of adic spaces can be globalized to a fully faithful functor $t\colon X \mapsto X^{\mathrm{ad}}$ from the category of locally noetherian formal schemes to the category of adic spaces. More precisely for every locally noetherian formal scheme X there exists an adic space X^{ad} and a morphism of locally and topologically ringed spaces $\pi\colon (X^{\mathrm{ad}}, \mathscr{O}^+_{X^{\mathrm{ad}}}) \to (X, \mathscr{O}_X)$ satisfying the following universal property. For every adic space Z and for every morphism $f\colon (Z, \mathscr{O}_Z^+) \to (X, \mathscr{O}_X)$ of locally and topologically ringed spaces there exists a unique morphism of adic spaces $g\colon Z \to X^{\mathrm{ad}}$ making the following diagram commutative

$$\begin{tikzcd}
(Z, \mathscr{O}_Z^+) \arrow[r, "f"] \arrow[dr, dotted, "g^+"'] & (X, \mathscr{O}_X) \\
 & (X^{\mathrm{ad}}, \mathscr{O}^+_{X^{\mathrm{ad}}}). \arrow[u, "\pi"']
\end{tikzcd}$$

For $X = \mathrm{Spf}(A)$ for a complete noetherian adic ring A, the underlying continuous map of π is given by

$$X^{\mathrm{ad}} = \mathrm{Spa}(A, A) \ni x \mapsto \{\, f \in A \ ; \ |f(x)| < 1 \,\},$$

which is an open prime ideal of A.

Rigid Spaces as Adic Spaces

Let k be a complete non-archimedean field. For every k-affinoid algebra A let $\mathrm{Sp}(A)$ be the attached rigid analytic k-space. Then the functor $\mathrm{Sp}(A) \mapsto \mathrm{Spa}(A, A^\circ)$ from affinoid rigid k-spaces to adic spaces can be globalized to a fully faithful functor $r_k\colon X \mapsto X^{\mathrm{ad}}$ from the category of rigid analytic k-spaces to the category of adic spaces preserving open immersions and intersections of open immersions. Moreover, a family $(U_i)_i$ of admissible open subspaces of X is an admissible cover of X if and only if $(r_k(U_i))_i$ is an open covering of X^{ad}. These properties characterize the functor r_k. The morphism of sites

$$X^{\mathrm{ad}} \longrightarrow X, \qquad r_k(U) \longmapsto U$$

induces an equivalence of toposes $(X^{\mathrm{ad}})^\sim \xrightarrow{\sim} X^\sim$.

For a point x of a rigid analytic space X let $i\colon \mathrm{Sp}\,\kappa(x) \to X$ be the canonical morphism (recall that $\kappa(x)$ is a finite extension of k by the rigid analytic Nullstellensatz). Let $e(x) \in X^{\mathrm{ad}}$ the image point of the morphism $r_k(i)\colon \mathrm{Spa}(\kappa(x), \kappa(x)^\circ) \to X^{\mathrm{ad}}$. This defines a bijective map

$$e\colon X \to \{\, x \in X^{\mathrm{ad}} \ ;\ \kappa(x) \text{ is finite over } k\}$$

and one considers X as a subset (but not a subspace) of X^{ad}. The map $U \mapsto r_k(U)$ yields an injective map from the set of admissible open subsets of X to the set of all open subsets of X^{ad}. This is almost never a bijection. If X is quasi-separated then this map restricts to a bijection

$$\left\{\begin{matrix}\text{quasi-compact admissible}\\ \text{open subsets of } X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{quasi-compact}\\ \text{open subsets of } X^{\mathrm{ad}}\end{matrix}\right\}$$

If $f\colon X \to Y$ is a morphism of rigid analytic spaces, then $f^{\mathrm{ad}} := r_k(f)\colon X^{\mathrm{ad}} \to Y^{\mathrm{ad}}$ is locally of finite type.

The Generic Fiber of a Formal Scheme

Let V be a complete discrete valuation ring with uniformizing element π and field of fractions k. The adic space $S := \mathrm{Spa}(V, V)$ attached to $\mathrm{Spf}\, V$ consists of an open point η and a closed point s with $\kappa(\eta) = k$ and $\kappa(s) = V/\mathfrak{m}_V$. The canonical morphism $S^0 = \mathrm{Spa}(k, V) \to S$ is an open immersion onto the open point. Clearly it is a morphism of adic spaces locally of finite type.

Let $\mathcal{F}_{ff}$ be the category of formal schemes X locally of formally finite type over V (i.e., X is locally of the form $\mathrm{Spf}\, A$ such that there exists a surjective continuous open ring homomorphism $V[\![T_1, \dots, T_n]\!]\langle X_1, \dots, X_n\rangle \to A$, where $V[\![T_1, \dots, T_n]\!]$ is the noetherian adic ring whose ideal of definition is $(\pi, T_1, \dots, T_n)$). Then Berthelot has extended Raynaud's generic fiber functor to a functor rig from $\mathcal{F}_{ff}$ to the category of rigid analytic spaces over k. If $X \to \mathrm{Spf}\, V$ is an object in $\mathcal{F}_{ff}$ we obtain an associated morphism of adic spaces $X^{\mathrm{ad}} \to S$ and we can form the fiber product $X^{\mathrm{ad}} \times_S S^0$ (Proposition 5.4). Then Berthelot's generic fiber functor and the functor $X \mapsto X^{\mathrm{ad}} \times_S S^0$ are isomorphic (see [Ka] §4).

The generic fiber of $\mathrm{Spf}\,\mathcal{O}_k[\![t]\!]$

As an example let $\mathcal{O}_k$ be a complete discrete valuation ring with uniformizing element π, field of fractions k, and absolute value $|\cdot|$. Let $\eta := \mathrm{Spa}(k, \mathcal{O}_k)$, $S := \mathrm{Spa}(\mathcal{O}_k, \mathcal{O}_k)$ and $\eta \to S$ be the canonical morphism identifying η with a rational subset of S. Let $X = \mathrm{Spa}(\mathcal{O}[\![t]\!], \mathcal{O}[\![t]\!])$ be the adic space attached to the formal scheme $\mathrm{Spf}(\mathcal{O}[\![t]\!])$. We visualize X as the open formal $\mathcal{O}_k$-adic unit disc (cf. Example 4.5). Then Example 5.5 (with $L = \{\pi, t\}$) shows that the generic fiber is

$$X_\eta = \operatorname*{colim}_{m \geq 1} \mathrm{Spa}\, B_m.$$

Here B_m is the Huber pair over $(k, \mathcal{O}_k)$ such that for every other Tate ring C over $(k, \mathcal{O}_k)$ the set of continuous homomorphisms $\varphi\colon B_m \to C$ correspond to topologically nilpotent elements $c = \varphi(t) \in C$ such that the elements $\pi^{m-1}, \pi^{m-2}c, \dots, c^{m-1}, \frac{c^m}{\pi} \in C^+$. But if c is topologically nilpotent, then $\pi^{m-1}, \pi^{m-2}c, \dots, c^{m-1}$ are topologically nilpotent and in particular in C^+. We conclude that continuous homomorphisms $B_m \to C$ correspond to elements $c^m \in \pi C^+$ (such elements are automatically topologically nilpotent). Hence $\mathrm{Spa}\, B_m$ is the closed disc of radius $|\pi|^{1/m}$ and X_η is the open unit disc, (the adic space associated to) a non-affinoid rigid analytic space. We write more suggestively $B_m = \mathrm{Spa}\, k\langle \frac{X^m}{\pi} \rangle$.

For a very detailed description of the closed disc $\mathrm{Spa}(k\langle X \rangle, \mathcal{O}_k\langle X \rangle)$ we refer to [Co] Lecture 11.

References

[BV] K. Buzzard, A. Verberkmoes, *Stably uniform affinoids are sheafy*, arXiv preprint (2014).

[Co] B. Conrad, *Seminaire Scholze*, Stanford Number theory learning seminar 2014–2015, http://math.stanford.edu/ conrad/Perfseminar

[CW] B. Conrad, T. Wedhorn, *Introduction to Adic Spaces*, in preparation

[Hu1] R. Huber, *Bewertungsspektrum und rigide Geometrie*, Regensburger Math. Schriften **23** (1993).

[Hu2] R. Huber, *Continuous valuations*, Math. Zeitschrift **212** (1993), 445–477.

[Hu3] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Zeitschrift **217** (1994), 513–551.

[Hu4] R. Huber, *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, Aspects of Mathematics **E30**, Vieweg (1996).

[Ka] C. Kappen, *Uniformly rigid spaces*, Algebra Number Theory **6** (2012), 341–388.

[Wd] T. Wedhorn, *Adic Spaces*, Unpublished script, http://www2.math.uni-paderborn.de/fileadmin/Mathematik/People/wedhorn/Lehre/AdicSpaces.pdf