ON A NONCOMMUTATIVE IWASAWA MAIN CONJECTURE FOR FUNCTION FIELDS

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ABSTRACT. We formulate and prove an analogue of the noncommutative Iwasawa main conjecture for ℓ-adic representations of the Galois group of a function field of characteristic p. As corollaries we obtain noncommutative generalisations of the main conjecture for Picard-1-motives of Greither and Popescu and a main conjecture for abelian varieties over function fields in precise analogy to the Gl_2 main conjecture of Coates, Fukaya, Kato, Sujatha and Venjakob.

1. INTRODUCTION

In [CFK$^+$05], Coates, Fukaya, Kato, Sujatha and Venjakob formulate a noncommutative Iwasawa main conjecture for ℓ-adic Lie extensions of number fields. Other, partly more general versions are formulated in [HK02], [RW04], and [FK06]. A geometric version for seperated schemes of finite type over a finite field is formulated and proved in [Wit10] and [Bur11].

Building on the ideas of [Wit10] we will consider in this article admissible ℓ-adic Lie extensions K_∞/K of a function field K of transcendence degree 1 over a finite field $\mathbb{F}$ of characteristic p. Here, an extension K_∞/K inside a fixed separable closure of K is called an *admissible ℓ-adic Lie extension* if

(1) K_∞/K is Galois and the Galois group $G = \mathrm{Gal}(K_\infty/K)$ is an ℓ-adic Lie group,
(2) K_∞/K contains the unique $\mathbb{Z}_\ell$-extension of $\mathbb{F}$,
(3) K_∞/K is unramified outside a finite set of places.

We will formulate and prove a noncommutative main conjecture for the extension K_∞/K and any continuous representation T of the absolute Galois group of K which is unramified outside a finite set of primes (Thm. 4.3, Thm. 4.4). From this main conjecture, we will deduce a main conjecture for Greenberg's Selmer group of T (Cor. 4.11), noncommutative generalisations of the main conjecture for the ℓ-adic Tate module of a Picard-1-motive from [GP12] (Cor. 4.18, Cor. 4.19), a noncommutative generalisation of the classical main conjecture for the Galois group of the maximal abelian ℓ-extension of K_∞ unramified outside a finite set of places (Cor. 4.21), and an analogue for abelian varieties over function fields of the Gl_2 main conjecture in [CFK$^+$05] for $\ell \neq p$ (Cor. 4.22).

Let $\mathbb{Z}_\ell[[G]]$ be the Iwasawa algebra of G and let $\mathbb{Z}_\ell[[G]]_S$ be its localisation at Venjakob's canonical Ore set S. Let C be the smooth and proper curve corresponding to K and assume for simplicity that C is geometrically connected. We also fix two disjoint, possibly empty sets Σ and Σ' of closed points in C. To the representation T we will associate a certain Selmer complex $C^\bullet_{\Sigma,\Sigma'}(T)$. The noncommutative main conjecture postulates that $C^\bullet_{\Sigma,\Sigma'}(T)$ is a perfect complex of $\mathbb{Z}_\ell[[G]]$-modules which is S-torsion and hence, gives rise to a class $[C^\bullet_{\Sigma,\Sigma'}(T)]$ in the relative K-group $\mathrm{K}(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$. Moreover, it postulates the existence of a noncommutative L-function $\mathcal{L}(T) \in \mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S)$ which maps to $-[C^\bullet_{\Sigma,\Sigma'}(T)]$ under the connecting

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homomorphism

$$\partial\colon \mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S) \to \mathrm{K}(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$$

and which satisfies an interpolation property with respect to values of the Σ-truncated Σ'-modified L-function of the twist $T \otimes \rho$ of T by representations ρ of G. Usually, one assumes that Σ contains the ramification locus of K_∞/K and of T, but we will show that this condition can be weakened a bit. Moreover, our main conjecture also applies to Iwasawa algebras over more general coefficient rings, including the power series ring $\mathbb{Z}_\ell/\ell\mathbb{Z}_\ell[[t]]$.

Greenberg's Selmer group may be identified with the Pontryagin dual of the second cohomology group of the complex $C^\bullet_{\Sigma,\emptyset}(T)$. The generalisation of the main conjecture for Picard-1-motives of Greither and Pofollows from the special cases $T = \mathbb{Z}_\ell$ and $T = \mathbb{Z}_\ell(1)$ with Σ and Σ' nonempty, the generalisation of the classical main conjecture follows from the case $T = \mathbb{Z}_\ell(1)$ and $\Sigma' = \emptyset$. For the noncommutative main conjecture for abelian varieties one choses T to be the Tate module of the dual abelian variety.

The article is structured as follows. In Section 2 we recall the K-theoretic framework. As in [Wit10] we will use the language of Waldhausen categories as a convenient tool to handle the relevant elements in K-groups. In particular, we recall the construction of the Waldhausen category of perfect complexes of adic sheaves and the construction of the total derived section functor as a Waldhausen exact functor. In Section 3 this will be used to construct the Selmer complex $C^\bullet_{\Sigma,\Sigma'}(T)$ and to analyse its properties. Section 4 contains the definition of the Σ-truncated Σ'-modified L-function as well as the formulation and proofs of the different main conjectures, including the description of the cohomology of $C^\bullet_{\Sigma,\Sigma'}(T)$.

Large parts of this article have been inspired by [FK06]. There has been previous work on a noncommutative main conjecture for elliptic curves over function fields in the case $\ell \neq p$ [Sec06] and independent work on the $\mathbf{M}_H(G)$-conjecture for ℓ-adic Selmer groups of abelian varieties over function function fields [BV]. We do not treat noncommutative main conjectures for abelian varieties in the case $\ell = p$, which necessitates different methods. There is work in progress on this conjecture by Trihan and Vauclair.

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2. Preliminaries

2.1. **The K-theoretic Framework.** We will recall some K-theoretic constructions from [Wit10]. All rings in this article will be associative with identity; a module over a ring will always refer to a left unitary module. For a ring R, R^{op} will denote the opposite ring. Recall that an *adic ring* is a ring Λ such that for each $n \geq 1$ the n-th power of the Jacobson radical $\operatorname{Jac}(\Lambda)^n$ is of finite index in Λ and

$$\Lambda = \varprojlim_{n\geq 1} \Lambda/\operatorname{Jac}(\Lambda)^n.$$

Write $\mathfrak{I}_\Lambda$ for the set of open two-sided ideals of Λ, partially ordered by inclusion. To compute the Quillen K-groups of Λ one can apply the Waldhausen S-construction to one of a variety of different Waldhausen categories associated to Λ. As in [Wit10], we will use the following, which turns out to be particularly convenient for our purposes.

We recall that for any ring R, a complex $M^\bullet$ of R-modules is called *DG-flat* if every module M^n is flat and for every acyclic complex $N^\bullet$ of R^{op}-modules, the total complex $(N \otimes_R M)^\bullet$ is acyclic. The complex $M^\bullet$ is called *perfect* if it is quasi-isomorphic to a complex $P^\bullet$ such that P^n is finitely generated and projective and $P^n = 0$ for almost all n.

Definition 2.1. Let Λ be an adic ring. We denote by $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ are inverse system $(P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ satisfying the following conditions:

(1) for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ is a DG-flat perfect complex of Λ/I-modules,
(2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism of the system

$$\varphi_{IJ} : P_I^\bullet \to P_J^\bullet$$

induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

A morphism of inverse systems $(f_I \colon P_I^\bullet \to Q_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ is a weak equivalence if every f_I is a quasi-isomorphism. It is a cofibration if every f_I is injective.

By [Wit10, Prop. 3.7] the complex

$$\varprojlim_{I\in\mathfrak{I}_\Lambda} Q_I^\bullet$$

is a perfect complex for every system $(Q_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ and the K-groups $\mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$ of the Waldhausen category coincide with the Quillen K-groups $\mathrm{K}_n(\Lambda)$ of the adic ring Λ.

If Λ' is a second adic ring and M is a Λ'-Λ-bimodule which is finitely generated and projective as Λ'-module, then the derived tensor product with M induces natural homomorphisms $\mathrm{K}_n(\Lambda) \to \mathrm{K}_n(\Lambda')$. It also follows from [Wit10, Prop. 3.7] that these homomorphisms coincide with the homomorphisms induced from the following Waldhausen exact functor.

Definition 2.2. For $(P_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ we define a Waldhausen exact functor

$$\Psi_M \colon \mathbf{PDG}^{\mathrm{cont}}(\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'), \qquad P^\bullet \to (\varprojlim_{J\in\mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P_J)^\bullet)_{I\in\mathfrak{I}_{\Lambda'}}.$$

Let now R be a commutative compact noetherian local ring with finite residue field of characteristic ℓ and G be a compact ℓ-adic Lie-group which is the semidirect product $H \rtimes \Gamma$ of a closed normal subgroup H and a closed subgroup Γ which isomorphic to $\mathbb{Z}_\ell$. Then the profinite group ring $R[[G]]$ is a noetherian adic ring and the set

$$S = \{f \in R[[G]] \colon R[[G]]/R[[G]]f \text{ is finitely generated as } R[[H]]\text{-module}\}$$

is a left denominator set in $R[[G]]$ such that the localisation $R[[G]]_S$ exists. Moreover, the K-theory localisation sequence for $S \subset R[[G]]$ splits into short split exact sequences [Wit11, Cor. 3.4]. In particular, we obtain a split exact sequence

$$0 \to \mathrm{K}_1(R[[G]]) \to \mathrm{K}_1(R[[G]]_S) \xrightarrow{\partial} \mathrm{K}_0(R[[G]], R[[G]]_S) \to 0.$$

We will describe this sequence in more detail, introducing the following Waldhausen categories.

Definition 2.3. We write $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ for the full Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ of objects $(P_J^\bullet)_{J\in\mathfrak{I}_{R[[G]]}}$ such that

$$\varprojlim_{J\in\mathfrak{I}_{R[[G]]}} P_J^\bullet$$

is a perfect complex of $R[[H]]$-modules.

We write $w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$, but with a new set of weak equivalences given by those morphisms whose cones are objects of the category $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$.

We may then identify for all $n \geq 0$

$$\mathrm{K}_n(R[[G]], R[[G]]_S) = \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])),$$
$$\mathrm{K}_{n+1}(R[[G]]_S) = \mathrm{K}_{n+1}(w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]]))$$

[Wit10, § 4]. In particular, $\mathrm{K}_0(R[[G]], R[[G]]_S)$ is the abelian group generated by the symbols $[P^\bullet]$ with $P^\bullet$ an object in $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ modulo the relations

$$[P^\bullet] = [Q^\bullet] \qquad \text{if } P^\bullet \text{ and } Q^\bullet \text{ are quasi-isomorphic,}$$
$$[P_2^\bullet] = [P_1^\bullet] + [P_3^\bullet] \qquad \text{if } 0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0 \text{ is an exact sequence.}$$

The groups $\mathrm{K}_1(R[[G]])$ and $\mathrm{K}_1(R[[G]]_S)$ are generated by the symbols $[f]$ where $f\colon P^\bullet \to P^\bullet$ is an endomorphism which is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ or in $w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$, respectively [Wit10, Prop. 3.10]. The map $\mathrm{K}_1(R[[G]]) \to \mathrm{K}_1(R[[G]]_S)$ is the obvious one; the boundary map

$$\partial\colon\ \mathrm{K}_1(R[[G]]_S) \to \mathrm{K}_0(R[[G]], R[[G]]_S)$$

is given by

$$\partial[f] = -[\mathrm{Cone}(f)^\bullet]$$

where $\mathrm{Cone}(f)^\bullet$ denotes the cone of f [Wit10, Thm. A.5]. (We note that other authors use $-\partial$ instead.)

Remark 2.4. Let M be an $R[[G]]$-module which has a resolution by a strictly perfect complex of $R[[G]]$-modules $P^\bullet$ as well as a resolution by a strictly perfect complex $Q^\bullet$ of $R[[H]]$-modules. We may identify $P^\bullet$ with the object

$$(R[[G]]/J \otimes_{R[[G]]} P^\bullet)_{J \in \mathfrak{I}_{R[[G]]}}$$

in $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ and set

$$[M] = [P^\bullet] \in \mathrm{K}_0(R[[G]], R[[G]]_S).$$

Note that $[M]$ does not depend on the particular choice of the resolutions $P^\bullet$ or $Q^\bullet$.

Consider a second commutative compact noetherian local ring R' with finite residue field of characteristic ℓ and another compact ℓ-adic Lie-group G' which is the semidirect product $H' \rtimes \Gamma'$ of a closed normal subgroup H' and a closed subgroup Γ' which isomorphic to $\mathbb{Z}_\ell$. Assume that M is a $R'[[G']]$-$R[[G]]$-bimodule which is finitely generated and projective as $R'[[G']]$-module. Assume further that there exists a $R'[[H']]$-$R[[H]]$-bimodule N which is finitely generated and projective as $R'[[H']]$-module and an isomorphism of $R'[[H']]$-$R[[G]]$-bimodules

$$N\hat{\otimes}_{R[[H]]}R[[G]] \cong M.$$

(Here, $N\hat{\otimes}_{R[[H]]}R[[G]]$ denotes the completed tensor product.) Then the Waldhausen exact functor

$$\Psi_M\colon \mathbf{PDG}^{\mathrm{cont}}(R[[G]]) \to \mathbf{PDG}^{\mathrm{cont}}(R'[[G']])$$

takes objects of $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ to objects of $\mathbf{PDG}^{\mathrm{cont},w_{H'}}(R'[[G']])$ and weak equivalences of $w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ to weak equivalences of $w_{H'}\mathbf{PDG}^{\mathrm{cont}}(R'[[G']])$ [Wit10, Prop. 4.6]. Hence, it also induces homomorphisms between the corresponding K-groups. In particular, this applies to

(1) $G' = G/U$, $H' = H/U$ for some closed subgroup U of H which is normal in G, $\Gamma = \Gamma'$, $R = R'$, $M = R[[G']]$;
(2) $G = G'$, $H = H'$, $\Gamma = \Gamma'$, $R = R$, $M = \rho \otimes_R R[[G]]$ for some $R[[G]]$-module ρ which is finitely generated and projective as R-module. Here, the left G-operation on M is given by the diagonal action, the right G-operation is the right operation on the second factor.

We obtain the evaluation map

$$\Phi_\rho\colon \mathrm{K}_1(R[[G]]_S) \to \mathrm{K}_1(R[[\Gamma]]_S) = R[[\Gamma]]_S^\times \tag{2.1}$$

from [CFK+05, (22)] as the composition of the maps induced by $\Psi_{\rho\otimes_R R[[G]]}$ and $\Psi_{R[[\Gamma]]}$ on the K-groups.

Assume now that $\mathcal{O}$ is a complete discrete valuation ring with finite residue field of characteristic ℓ and that the ℓ-adic Lie group $G = H \rtimes \Gamma$ has no element of order ℓ. Then the ring $\mathcal{O}[[G]]$ is both noetherian and of finite global dimension [Bru66, Thm. 4.1]. Let $\mathbf{N}_H(\mathcal{O}[[G]])$ denote the abelian category of finitely generated $\mathcal{O}[[G]]$-modules which are also finitely generated as $\mathcal{O}[[H]]$-modules. Note that

$$\begin{gathered}\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},w_H}(\mathcal{O}[[G]])) \to \mathrm{K}_0(\mathbf{N}_H(\mathcal{O}[[G]])),\\ [(P_I^\bullet)_{I\in\mathfrak{I}_{\mathcal{O}[[G]]}}] \mapsto \sum_{i=-\infty}^{\infty}(-1)^i[\mathrm{H}^i(\varprojlim_{I\in\mathfrak{I}_{\mathcal{O}[[G]]}} P_I^\bullet)]\end{gathered} \tag{2.2}$$

is an isomorphism since every N in $\mathbf{N}_H(\mathcal{O}[[G]])$ has a resolution $P^\bullet$ of finite length by finitely generated, projective $\mathcal{O}[[G]]$-modules and $(P^\bullet/IP^\bullet)_{I\in\mathfrak{I}_{\mathcal{O}[[G]]}}$ is an object in $\mathbf{PDG}^{\mathrm{cont},w_H}(\mathcal{O}[[G]])$.

If the quotient field of $\mathcal{O}$ is of characteristic 0, one may also consider the abelian category $\mathbf{M}_H(\mathcal{O}[[G]])$ of finitely generated $\mathcal{O}[[G]]$-modules whose ℓ-torsionfree part is finitely generated as $\mathcal{O}[[H]]$-module and the left denominator set

$$S^* = \bigcup_n \ell^n S \subset \mathcal{O}[[G]].$$

Still assuming that G has no element of order ℓ it is known that the natural maps

$$\mathrm{K}_1(\mathcal{O}[[G]]_S) \to \mathrm{K}_1(\mathcal{O}[[G]]_{S^*}), \qquad \mathrm{K}_0(\mathbf{N}_H(\mathcal{O}[[G]])) \to \mathrm{K}_0(\mathbf{M}_H(\mathcal{O}[[G]]))$$

are split injective [BV11, Prop. 3.4] and fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_1(\mathcal{O}[[G]]) \arrow[r] \arrow[d, "="] & \mathrm{K}_1(\mathcal{O}[[G]]_S) \arrow[r, "\partial"] \arrow[d] & \mathrm{K}_0(\mathbf{N}_H(\mathcal{O}[[G]])) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{K}_1(\mathcal{O}[[G]]) \arrow[r] & \mathrm{K}_1(\mathcal{O}[[G]]_{S^*}) \arrow[r, "\partial"] & \mathrm{K}_0(\mathbf{M}_H(\mathcal{O}[[G]])) \arrow[r] & 0
\end{tikzcd}$$

In particular, an identity of the type $f = \partial g$ in $\mathrm{K}_0(\mathbf{N}_H(\mathcal{O}[[G]]))$ will imply a corresponding identity in $\mathrm{K}_0(\mathbf{M}_H(\mathcal{O}[[G]]))$. We will work exclusively with $\mathbf{N}_H(\mathcal{O}[[G]])$.

2.2. **Perfect complexes of adic sheaves.** In this section we recall some constructions from [Wit08, § 5.4–5.5].

Let X be a separated scheme of finite type over a finite field $\mathbb{F}$ (We will only need the case $\dim X \leq 1$). Recall that for a finite ring R, a complex $\mathcal{F}^\bullet$ of étale sheaves of left R-modules on X is called *strictly perfect* if it is strictly bounded and each $\mathcal{F}^n$ is constructible and flat. It is *perfect* if it is quasi-isomorphic to a strictly perfect complex. We call it *DG-flat* if for each geometric point of X, the complex of stalks is *DG*-flat.

Definition 2.5. Let X be a separated scheme of finite type over a finite field and let Λ be an adic ring. The *category of perfect complexes of adic sheaves* $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$

is the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ are inverse systems $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ such that:

(1) for each $I \in \mathfrak{I}_\Lambda$, $\mathcal{F}_I^\bullet$ is DG-flat perfect complex of étale sheaves of Λ/I-modules on X,

(2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism
$$\varphi_{IJ}\colon \mathcal{F}_I^\bullet \to \mathcal{F}_J^\bullet$$
of the system induces an isomorphism
$$\Lambda/J \otimes_{\Lambda/I} \mathcal{F}_I^\bullet \xrightarrow{\sim} \mathcal{F}_J^\bullet.$$

Weak equivalences and cofibrations are those morphisms of inverse systems that are weak equivalences or cofibrations for each $I \in \mathfrak{I}_\Lambda$, respectively.

For an abitrary inverse system $\mathcal{F} = (\mathcal{F}_I)_{I\in\mathfrak{I}_\Lambda}$ of sheafs of Λ-modules on X we let $\mathrm{H}^k(X,\mathcal{F})$ denote the k-th continuous cohomology [Jan88, §3]. Likewise, we write $\mathrm{H}^k(\overline{X},\mathcal{F})$ for the continuous cohomology over the base change $\overline{X}$ of X with respect to a fixed algebraic closure of $\mathbb{F}$.

Assume now that either X is proper or that the characteristic of $\mathbb{F}$ is a unit in Λ. Using Godement resolution we constructed in [Wit08, Def. 5.4.13] Waldhausen exact functors
$$\mathrm{R}\,\Gamma(X,\cdot), \mathrm{R}\,\Gamma(\overline{X},\cdot)\colon \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$
such that for each $(\mathcal{F}_I)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ concentrated in degree 0 we have
$$\mathrm{H}^i(X,(\mathcal{F}_I)_{I\in\mathfrak{I}_\Lambda}) = \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{H}^i(X,\mathcal{F}_I) = \mathrm{H}^i(\varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(X,(\mathcal{F}_I)_{I\in\mathfrak{I}_\Lambda})),$$
$$\mathrm{H}^i(\overline{X},(\mathcal{F}_I)_{I\in\mathfrak{I}_\Lambda}) = \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{H}^i(\overline{X},\mathcal{F}_I) = \mathrm{H}^i(\varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\overline{X},(\mathcal{F}_I)_{I\in\mathfrak{I}_\Lambda})).$$

Moreover, there is an exact sequence
$$0 \to \mathrm{R}\,\Gamma(X,\cdot) \to \mathrm{R}\,\Gamma(\overline{X},\cdot) \xrightarrow{\mathrm{id}-\mathfrak{F}} \mathrm{R}\,\Gamma(\overline{X},\cdot) \to 0$$
where $\mathfrak{F}$ denotes the geometric Frobenius acting on $\overline{X}$ [Wit08, Prop. 6.1.2].

We also recall that given a second adic ring Λ' and a Λ'-Λ-bimodule M which is finitely generated and projective as Λ'-module, we may extend Ψ_M to a Waldhausen exact functor
$$\Psi_M\colon \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda'),$$
$$(\mathcal{P}_J^\bullet)_{J\in\mathfrak{I}_\Lambda} \mapsto (\varprojlim_{J\in\mathfrak{I}_\Lambda} M/IM \otimes_\Lambda \mathcal{P}_J{}^\bullet)_{I\in\mathfrak{I}_{\Lambda'}}$$
such that
$$\Psi_M \,\mathrm{R}\,\Gamma(X,\mathcal{P}^\bullet) \to \mathrm{R}\,\Gamma(X,\Psi_M(\mathcal{P}^\bullet)),$$
$$\Psi_M \,\mathrm{R}\,\Gamma(\overline{X},\mathcal{P}^\bullet) \to \mathrm{R}\,\Gamma(\overline{X},\Psi_M(\mathcal{P}^\bullet))$$
are quasi-isomorphisms in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda')$ [Wit08, Prop. 5.5.7].

3. Selmer Complexes

Selmer complexes have been introduced by Nekovář in [Nek06]. They are perfect complexes defined by modifying Galois cohomology by certain local conditions. In the geometric situation, one can identify the cohomology of the Selmer complexes with the étale cohomology of certain constructible sheaves. A nontrivial local condition in a point corresponds to the sheaf being not smooth in this point. (In the number field case, this description works as well for the local conditions away from the places over the chosen prime ℓ.) This is the point of view that we are adapting in the present article. In order to associate in a canonical manner classes in K-groups to these complexes, we will construct them as elements of the Waldhausen categories introduced in the previous section.

3.1. **Admissible Lie extensions and representations.** In the following, we let K denote a fixed function field of characteristic $p > 0$, i. e. a field of transcendence degree 1 over $\mathbb{F}_p$. We also fix a separable closure $\overline{K}$ of K and write Gal_K for the Galois group of $\overline{K}/K$. Let $\mathbb{F}$ denote the algebraic closure of $\mathbb{F}_p$ inside K and $\overline{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$ inside $\overline{K}$. We let

$$q = p^{[\mathbb{F}:\mathbb{F}_p]}$$

denote the number of elements of $\mathbb{F}$.

Write C for the smooth and projective curve over $\mathbb{F}$ whose closed points are the places of K. Note that by the definition of $\mathbb{F}$, C is geometrically connected. This is a convenient, but not necessary restriction. One can proceed as in [Wit10] to deal also with the non-connected case.

For any open dense subscheme U of C and any extension field L of K inside $\overline{K}$, we write U_L for the normalisation of U in L and $j_L\colon \operatorname{Spec} L \to U_L$ for the canonical inclusion of the generic point. More generally, we will write $j_U\colon U \to V$ for the inclusion of an open subscheme U of V and $i_\Sigma\colon \Sigma \to V$ for the inclusion of a closed subscheme Σ of V.

Furthermore, we fix for each closed point $v \in C$ a strict henselisation of the local ring at v inside $\overline{K}$. We write $\mathcal{I}_v \subset \mathcal{D}_v \subset \mathrm{Gal}_K$ for the corresponding inertia group and decomposition group, ξ_v for the geometric point over v corresponding to the residue field of the strict henselisation, and $\mathfrak{F}_v$ for the corresponding geometric Frobenius at ξ_v. We let $\mathbb{F}(v)$ denote the residue field of v and $\deg(v) = [\mathbb{F}(v) : \mathbb{F}]$ the degree of v over $\mathbb{F}$.

For any compact or discrete abelian group A we let

$$A^\vee = \mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{R}/\mathbb{Z})$$

denote its Pontryagin dual.

If $\ell \neq p$, let $\varepsilon_{\mathrm{cyc}}\colon \mathrm{Gal}_K \to \mathbb{Z}_\ell^\times$ denote the cyclotomic character:

$$\sigma(\zeta) = \zeta^{\varepsilon_{\mathrm{cyc}}(\sigma)}$$

for any ℓ^k-th root of unity $\zeta \in \overline{K}$ and any $\sigma \in \mathrm{Gal}_K$. If Λ is adic $\mathbb{Z}_\ell$-algebra, $k \in \mathbb{Z}$, and T is a Gal_K-module we let $T(k)$ denote the k-th Tate twist of T, i. e. the Gal_K-module obtained from T by multiplying the action of Gal_K by ${\varepsilon_{\mathrm{cyc}}}^k$.

Definition 3.1. An extension K_∞/K inside $\overline{K}$ is called an *admissible ℓ-adic Lie extension* if

(1) K_∞/K is Galois and the Galois group $\mathrm{Gal}(K_\infty/K)$ is an ℓ-adic Lie group,
(2) K_∞ contains the $\mathbb{Z}_\ell$-extension $\mathbb{F}_\infty$ of $\mathbb{F}$,
(3) K_∞/K is unramified outside a finite set of places.

If K_∞/K is an admissible ℓ-adic Lie extension, we let $G = \mathrm{Gal}(K_\infty/K)$ denote its Galois group and set $H = \mathrm{Gal}(K_\infty/\mathbb{F}_\infty K)$, $\Gamma = \mathrm{Gal}(\mathbb{F}_\infty/\mathbb{F})$. We may then choose a continuous splitting $\Gamma \to G$ to identify G with the corresponding semidirect product $G = H \rtimes \Gamma$.

Definition 3.2. Let Λ be an adic ring. We call a compact $\Lambda[[\mathrm{Gal}_K]]$-module T *admissible* if

(1) it is finitely generated and projective as Λ-module,

(2) T is unramified outside a finite set of places, i. e. the set
$$\{v \in C \mid T^{\mathcal{I}_v} \neq T\}$$
is finite.

3.2. **The adic sheaf associated to an admissible module.** Let R be a commutative compact noetherian local ring with finite residue field of characteristic ℓ and K_∞/K an admissible ℓ-adic Lie extension with Galois group G. We let $R[[G]]^\sharp$ denote the admissible $R[[G]][[\mathrm{Gal}_K]]$-module $R[[G]]$ with $g \in \mathrm{Gal}_K$ acting by the image of g^{-1} in G from the right.

Recall that for a finite ring Λ, taking the stalk in the geometric point $\operatorname{Spec} \overline{K}$ is an equivalence of categories between the category of étale sheaves of Λ-modules on $\operatorname{Spec} K$ and the category of discrete $\Lambda[[\mathrm{Gal}_K]]$-modules [Mil80, Thm. II.1.9]. In our notation, we will not distinguish between the discrete $R[[\mathrm{Gal}_K]]$-module and the corresponding sheaf on $\operatorname{Spec} K$.

An admissible $\Lambda[[\mathrm{Gal}_K]]$-module T then corresponds to a projective system of sheaves on $\operatorname{Spec} K$. We want to consider the system of direct image sheaves under the inclusion $j_K\colon \operatorname{Spec} K \to U \subset C$. However, the naive definition, applying j_{K*} to each element of the system, leads to an object with some bad properties. We will consider a stabilised version instead, redefining the direct image sheaf as follows.

Definition 3.3. Let Λ be an adic ring, $U \subset C$ an open nonempty subscheme and T an admissible $\Lambda[[\mathrm{Gal}_K]]$-module. We define an inverse system of étale sheaves of Λ-modules $j_{K*}T = (j_{K*}T_I)_{I \in \mathfrak{I}_\Lambda}$, where
$$j_{K*}T_I = \varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda j_{K*}T/JT.$$

Proposition 3.4. *Let Λ be a noetherian adic ring and T be an admissible $\Lambda[[\mathrm{Gal}_K]]$-module. For any $I \in \mathfrak{I}_\Lambda$, $j_{K*}T_I$ is a constructible étale sheaf of Λ/I-modules on U. If v is a closed point of U, then the stalk of $j_{K*}T_I$ in the geometric point ξ_v is given by*
$$(j_{K*}T_I)_{\xi_v} = \Lambda/I \otimes_\Lambda T^{\mathcal{I}_v}.$$
In particular, $j_{K*}T$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ if $T^{\mathcal{I}_v}$ is a finitely generated projective Λ-module for each closed point v of U.

Proof. Let V be the open complement of U by the set of points v with $T^{\mathcal{I}_v} \neq T$. Consider an open étale neighborhood W of V and let $L \subset \overline{K}$ be the function field of W. Then for any $J \subset I$
$$(\Lambda/I \otimes_\Lambda j_{K*}T/JT)(W) = (T/IT)^{\mathrm{Gal}_L} = (j_{K*}T/IT)(W).$$
In particular, the restriction of $\Lambda/I \otimes_\Lambda j_{K*}T/JT$ to V is a locally constant étale sheaf of Λ/I-modules which independent of J. Now the category of étale sheaves of Λ/I-modules on U which are locally constant on V is equivalent to the category of tuples $(M, (M_v, \phi_v)_{v \in U-V})$ where M is a discrete $\Lambda/I[[\mathrm{Gal}_K]]$-module unramified over V, the M_v are discrete $\mathrm{Gal}_{k(v)}$-modules, and $\phi_v\colon M_v \to M^{\mathcal{I}_v}$ are homomorphisms of discrete $\mathrm{Gal}_{k(v)}$-modules [Mil80, Ex. II.3.16].

By the above considerations, it is clear that the projective limit of the system $(\Lambda/I \otimes_\Lambda j_{K*}T/JT)_{J \in \mathfrak{I}_\Lambda}$ exists in the category of étale sheaves of Λ/I-modules which are locally constant on V and coincides with the projective limit taken in the category of all étale sheaves of Λ/I-modules. Moreover, it corresponds to the tuple
$$(T/IT, (\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (T/JT)^{\mathcal{I}_v}, \phi_v\colon \varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (T/JT)^{\mathcal{I}_v} \to (T/IT)^{\mathcal{I}_v})_{v \in U-V}).$$

Beware that the projective limit

$$\varprojlim_{J\in\mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (T/JT)^{\mathcal{I}_v}$$

is a priori taken in the category of discrete $\Lambda/I[[\mathrm{Gal}_{k(v)}]]$-modules.

In the category of abstract $\Lambda/I[[\mathrm{Gal}_{k(v)}]]$-modules, we have

$$\varprojlim_{J\in\mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (T/JT)^{\mathcal{I}_v} = \Lambda/I \otimes_\Lambda \varprojlim_{J\in\mathfrak{I}_\Lambda} (T/JT)^{\mathcal{I}_v} = \Lambda/I \otimes_\Lambda T^{\mathcal{I}_v}.$$

Here, the first equality is justified because Λ/I is finitely presented as Λ^{op}-module and because projective limits of finite Λ/I-modules are exact. Since Λ is noetherian, $T^{\mathcal{I}_v}$ is finitely generated and $\Lambda/I \otimes_\Lambda T^{\mathcal{I}_v}$ is finite. Hence, the equality

$$\varprojlim_{J\in\mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (T/JT)^{\mathcal{I}_v} = \Lambda/I \otimes_\Lambda T^{\mathcal{I}_v}$$

also holds in the category of discrete $\Lambda/I[[\mathrm{Gal}_{k(v)}]]$-modules. This shows that $j_{K*}T_I$ is constructible and that the stalks have the given form.

From the description of the stalks it is also immediate that

$$\Lambda/I \otimes_{\Lambda/J} j_{K*}T_J \cong j_{K*}T_I$$

such that under the given condition, $j_{K*}T$ is indeed an object of $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. $\square$

Remark 3.5. Note that in any adic ring Λ, the Jacobson radical $\mathrm{Jac}(\Lambda)$ is finitely generated both as left and as right ideal [War93, Thm. 36.39]. Therefore, the same is true for all open ideals $I \in \mathfrak{I}_\Lambda$ and thus, Λ/I is a finitely presented Λ^{op}-module even if Λ is an adic ring which is not noetherian. In this case, the conclusion of the above proposition still holds under the assumption that $T^{\mathcal{I}_v}$ is a finitely presented Λ-module for all closed points $v \in U$. Nevertheless, we will restrict to noetherian adic rings in the following to avoid technicalities.

Proposition 3.6. *Let Λ be an noetherian adic ring, T be an admissible $\Lambda[[\mathrm{Gal}_K]]$-module, and $U \subset V$ be open dense subschemes of C. If $\ell = p$, assume that $V = C$. For all $k \in \mathbb{Z}$*

$$\mathrm{H}^k(V, j_{U!}j_{K*}T) = \varprojlim_{J\in\mathfrak{I}_\Lambda} \mathrm{H}^k(V, j_{U!}j_{K*}T/JT).$$

Proof. Let $(\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$ and $(\mathcal{C}_J)_{J\in\mathfrak{I}_\Lambda}$ denote the kernel and cokernel of the natural morphism of systems

$$(j_{K*}T_J)_{J\in\mathfrak{I}_\Lambda} \to (j_{K*}T/JT)_{J\in\mathfrak{I}_\Lambda}$$

of étale sheaves on U. The restriction of $(\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$ to the complement of

$$\Sigma = \left\{ v \in U \mid T^{\mathcal{I}_v} \neq T \right\}$$

in U is 0. For $v \in \Sigma$ the stalk of $\mathcal{K}_J$ in the geometric point ξ_v is a finite abelian group for each $J \in \mathfrak{I}_\Lambda$ and since

$$T = \varprojlim_{J\in\mathfrak{I}_\Lambda} (j_{K*}T_J)_{\xi_v} = \varprojlim_{J\in\mathfrak{I}_\Lambda} (j_{K*}T/JT)_{\xi_v}$$

we see that the projective limit of the system $((\mathcal{K}_J)_{\xi_v})_{J\in\mathfrak{I}_\Lambda}$ is 0. Hence, the system must be Mittag-Leffler zero in the sense of [Jan88, Def. 1.10]: For each natural number n there exists a $m > n$ such that the transition maps

$$(\mathcal{K}_{\mathrm{Jac}(\Lambda)^m})_{\xi_v} \to (\mathcal{K}_{\mathrm{Jac}(\Lambda)^n})_{\xi_v}$$

is the zero map. We conclude that the system of sheaves $(\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$ are also Mittag-Leffler zero. The same remains true for $(j_{U!}\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$ with $j_U\colon U\to V$ denoting the natural inclusion. Now [Jan88, Lemma 1.11] implies

$$\mathrm{H}^k(V,(j_{U!}\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda})=0.$$

The same argumentation also shows

$$\mathrm{H}^k(V,(j_{U!}\mathcal{C}_J)_{J\in\mathfrak{I}_\Lambda})=0.$$

Since the cohomology groups $\mathrm{H}^k(V,j_{U!}j_{K*}T/JT)$ are finite if $\ell\neq p$ [Del77, Th. finitude, Cor. 1.10] or if $V=C$ [Mil80, Cor. VI.2.8] we conclude

$$\mathrm{H}^k(V,j_{U!}j_{K*}T)=\varprojlim_{J\in\mathfrak{I}_\Lambda}\mathrm{H}^k(V,j_{U!}j_{K*}T/JT).$$

□

3.3. **Properties of Selmer complexes.** We will now fix an admissible ℓ-adic Lie extension K_∞/K with Galois group $G=H\rtimes\Gamma$. For a closed point v of C we will write $\mathcal{K}_v$ and $\mathcal{J}_v$ for the kernel and the image of the homomorphism $\mathcal{I}_v\to G$, respectively. We also fix an open dense subscheme U of C, a commutative compact noetherian local ring R with finite residue field of characteristic ℓ, and an admissible $R[[\mathrm{Gal}_K]]$-module T.

Proposition 3.7. *Let v be a closed point of U.*

(1) *For every $I\in\mathfrak{I}_{R[[G]]}$,*

$$(j_{K*}(R[[G]]^\sharp\otimes_R T)_I)_{\xi_v}=R[[G]]/I\otimes_{R[[G]]}(R[[G]]^\sharp\otimes_R T^{\mathcal{K}_v})^{\mathcal{J}_v}.$$

(2) *If $\mathcal{J}_v$ contains an element of infinite order then*

$$(j_{K*}(R[[G]]^\sharp\otimes_R T)_I)_{\xi_v}=0.$$

(3) *Assume that for every closed point v of U one of the following conditions is satisfied:*
   (a) *$T^{\mathcal{K}_v}=0$,*
   (b) *$\mathcal{J}_v$ contains an element of infinite order,*
   (c) *$T^{\mathcal{K}_v}$ is a projective R-module and $\mathcal{J}_v$ contains no element of order ℓ.*

   Then $j_{K*}(R[[G]]^\sharp\otimes_R T)$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(U,R[[G]])$.

Proof. Since R is noetherian, arbitrary direct products of copies of R are flat as R-modules. Let $G_n=\ker G\to(R[[G]]/\operatorname{Jac}(R[[G]])^n)^\times$. Since

$$\varprojlim_n R[G/G_n]=R[[G]],\qquad \varprojlim_n{}^1 R[G/G_n]=0$$

the sequence

$$0\to R[[G]]\to\prod_n R[G/G_n]\xrightarrow{\alpha}\prod_n R[G/G_n]\to 0$$

(with α being the difference of id and the transition morphisms) is exact. We conclude that $R[[G]]$ is flat as R-module. Hence, $R[[G]]\otimes_R T^{\mathcal{K}_v}=(R[[G]]\otimes_R T)^{\mathcal{K}_v}$. This and Proposition 3.4 proves the first assertion.

To prove the second assertion, note that

$$(R[[G]]^\sharp\otimes_R T^{\mathcal{K}_v})^{\mathcal{J}_v}=\varprojlim_{J\in\mathfrak{I}_R}(R/J[[G]]^\sharp\otimes_{R/J}T^{\mathcal{K}_v}/JT^{\mathcal{K}_v})^{\mathcal{J}_v}.$$

In particular, we may assume that R is a finite ring. We can then find an element τ of infinite order in an ℓ-Sylow subgroup of $\mathcal{J}_v$ which operates trivially on $T^{\mathcal{K}_v}$. We may replace $\mathcal{J}_v$ by the subgroup $\Upsilon\cong\mathbb{Z}_\ell$ topologically generated by τ. We now note that the $R[[\Upsilon]]$-module $R[[G]]^\sharp$, with τ acting by right multiplication by τ^{-1}, is the projective limit of the free $R[[\Upsilon]]$-modules $R[[\Upsilon]][U\backslash G/\Upsilon]$ generated by the elements of the double quotient $U\backslash G/\Upsilon$, with U running through the open subgroups of G. Since $\tau - 1$ is a nonzero divisor in $R[[\Upsilon]]$ we obtain for each such U an exact sequence of flat R-modules

$$0 \to R[[\Upsilon]][U\backslash G/\Upsilon] \xrightarrow{\tau-1} R[[\Upsilon]][U\backslash G/\Upsilon] \to R[U\backslash G/\Upsilon] \to 0.$$

The sequence remains exact after tensoring with $T^{\mathcal{K}_v}$ and passing to the projective limit over the open subgroups U. We conclude that

$$(R[[G]]^\sharp \otimes_R T^{\mathcal{K}_v})^\Upsilon = \ker(\tau - 1) \otimes \mathrm{id} = 0$$

as desired.

For the third assertion it remains to verify that $(R[[G]]^\sharp \otimes_R T^{\mathcal{K}_v})^{\mathcal{J}_v}$ is a projective $R[[G]]$-module if $T^{\mathcal{K}_v}$ is a projective R-module and $\mathcal{J}_v$ contains no element of order ℓ. By the second assertion we may assume that $\mathcal{J}_v$ contains no element of infinite order. This implies that the ℓ-Sylow-subgroups of $\mathcal{J}_v$ must be trivial. However, the ℓ-Sylow subgroups of compact ℓ-adic Lie groups are open subgroups. Therefore, $\mathcal{J}_v$ is a finite group of order prime to ℓ and the invariants under $\mathcal{J}_v$ are given as the kernel of the projector

$$\mathrm{id} - \frac{1}{\sharp \mathcal{J}_v} \sum_{g \in \mathcal{J}_v} g.$$

Hence, $(R[[G]]^\sharp \otimes_R T^{\mathcal{K}_v})^{\mathcal{J}_v}$ is a projective $R[[G]]$-module. By Proposition 3.4 we then conclude that $j_{K*}(R[[G]]^\sharp \otimes_R T)$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(U, R[[G]])$. □

Remark 3.8. If K_∞/K is unramified over U and $f\colon U_\infty \to U$ is the principal covering with Galois group G in the sense of [Wit10, Def. 2.1] then

$$j_{K*}(R[[G]]^\sharp \otimes_R T) = f_! f^* j_{K*} T$$

in the notation of [Wit10, §6], see also [Wit10, Rem. 6.10].

We distinguish the following sets of bad primes.

Definition 3.9. In the situation of Prop. 3.7 we set

$\Xi_0 = \left\{ v \in C \mid T^{\mathcal{I}_v} \text{ is not projective} \right\}$,
$\Xi_1 = \{ v \in C \mid v \text{ satisfies none of the conditions of Prop. 3.7.(3)} \}$,
$\Xi_2 = \Xi_1 \cup \{ v \in C \mid \mathcal{J}_v \text{ has a nontrivial } \ell\text{-Sylow subgroup} \}$,
$\Xi_3 = \Xi_2 \cup \{ v \in C \mid \mathrm{im}(\mathcal{K}_v \to \mathrm{Aut}_R T) \text{ has a nontrivial } \ell\text{-Sylow subgroup} \}$.

Note that we have

$$\Xi_0 \cup \Xi_1 \subset \Xi_2 \subset \Xi_3.$$

From now on, we fix two open dense subschemes V and W of C and set

$$\Sigma_V = C - V, \qquad \Sigma_W = C - W, \qquad U = V \cap W.$$

The complexes $\mathrm{R}\,\Gamma(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T))$ are the right analogue of Nekovar's Selmer complexes. Below, we will investigate their properties.

Proposition 3.10. *Assume that $U \cap \Xi_1 = \emptyset$. If $\ell = p$ we also assume $V = C$. Then the complex*

$$\mathrm{R}\,\Gamma(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T))$$

is an object of $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ and the canonical morphism

$$\mathrm{R}\,\Gamma(V, j_{(U-\Xi_2)!} j_{K*}(R[[G]]^\sharp \otimes_R T)) \to \mathrm{R}\,\Gamma(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T))$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$. The same is true for V replaced by $\overline{V}$.

Proof. We let $j'\colon U - \Xi_2 \to U$ denote the inclusion map. By Prop. 3.7 we have $j_{K*}(R[[G]]^\sharp \otimes_R T)_{\xi_v} = 0$ for each $v \in U \cap \Xi_2$. Hence, the canonical morphism

$$j'_! j_{K*}(R[[G]]^\sharp \otimes_R T) \to j'_* j_{K*}(R[[G]]^\sharp \otimes_R T)$$

is an isomorphism in $\mathbf{PDG}^{\mathrm{cont}}(U, [[G]])$. The assertion of the proposition is an immediate consequence. □

Proposition 3.11. *Assume that $U \cap \Xi_2 = \emptyset$. If $\ell = p$ we will also assume $V = C$. Let Λ be a noetherian adic $\mathbb{Z}_\ell$-algebra and M be a Λ-$R[[G]]$-bimodule which is finitely generated and projective as Λ-module.*

(1) *Let M be flat as R^{op}-module. Then the canonical morphism*

$$\Psi_M \operatorname{R}\Gamma(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) \to \operatorname{R}\Gamma(V, j_{U!} j_{K*}(M \otimes_{R[[G]]} R[[G]]^\sharp \otimes_R T))$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

(2) *If $U \cap \Xi_3 = \emptyset$, then*

$$\Psi_M \operatorname{R}\Gamma(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) \to \operatorname{R}\Gamma(V, j_{U!} j_{K*}(M \otimes_{R[[G]]} R[[G]]^\sharp \otimes_R T))$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ without any further condition on M.

The same is true for V replaced by $\overline{V}$.

Proof. As explained in Section 2.2, Ψ_M commutes with $\operatorname{R}\Gamma(V, \cdot)$ and $\operatorname{R}\Gamma(\overline{V}, \cdot)$. So, we need to prove that

$$\Psi_M j_{K*}(R[[G]]^\sharp \otimes_R T) \to j_{K*}(M \otimes_{R[[G]]} R[[G]]^\sharp \otimes_R T)$$

is an isomorphism in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$. Since this can be checked on stalks we need to prove that

$$M \otimes_{R[[G]]} (R[[G]]^\sharp \otimes_R T^{\mathcal{K}_v})^{\mathcal{J}_v} = M \otimes_{R[[G]]} (R[[G]]^\sharp \otimes_R T)^{\mathcal{I}_v}$$

for all closed points $v \in U$. Since $U \cap \Xi_2 = \emptyset$ the ℓ-Sylow subgroup of $\mathcal{J}_v$ is trivial such that taking invariants under $\mathcal{J}_v$ is an exact functor on the category of compact $\mathbb{Z}_\ell[[\mathcal{J}_v]]$-modules. Moreover, $T^{\mathcal{K}_v}$ is finitely generated and projective as R-module. Hence

$$M \otimes_{R[[G]]} (R[[G]]^\sharp \otimes_R T^{\mathcal{K}_v})^{\mathcal{J}_v} = (M \otimes_{R[[G]]} R[[G]]^\sharp \otimes_R T^{\mathcal{K}_v})^{\mathcal{J}_v}.$$

If M is flat as R^{op}-module or if the image of $\mathcal{K}_v$ in $\operatorname{Aut}_R(T)$ is of order prime to ℓ, then also

$$M \otimes_R T^{\mathcal{K}_v} = (M \otimes_R T)^{\mathcal{K}_v}.$$

This completes the proof of the proposition. □

Corollary 3.12. *Let $\ell \neq p$. For each $k \in \mathbb{Z}$*

$$\operatorname{H}^k(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) = \varprojlim_{K \subset_f L \subset K_\infty} \operatorname{H}^k(V_L, j_{U_L!} j_{L*} T)$$

where L runs through the finite Galois extensions of K inside K_∞. If $\ell = p$ then this is true for $V = C$.

Proof. By Proposition 3.6 we have

$$\operatorname{H}^k(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) = \varprojlim_{K \subset_f L \subset K_\infty} \operatorname{H}^k(V, j_{U!} j_{K*}(R[\operatorname{Gal}(L/K)]^\sharp \otimes_R T)).$$

Let $f\colon V_L \to V$ denote the finite morphism of schemes corresponding to the finite extension L/K. Then

$$\operatorname{H}^k(V, j_{U!} j_{K*}(R[\operatorname{Gal}(L/K)]^\sharp \otimes_R T)) = \operatorname{H}^k(V, f_* j_{U_L!} j_{L*} T) = \operatorname{H}^k(V_L, j_{U_L!} j_{L*} T)$$

by [Mil80, Cor. II.3.6]. □

Proposition 3.13. *Assume that $H = \mathrm{Gal}(K_\infty/\mathbb{F}_\infty K)$ is finite. If $\ell = p$, we also assume that $V = C$. Then for each $k \in \mathbb{Z}$*

$$\begin{aligned}\mathrm{H}^k(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T)) &= \mathrm{H}^{k-1}(V_{K_\infty}, j_{U_{K_\infty}}{}_!j_{K_\infty *}T)\\ &= \mathrm{H}^{k-1}(V_{\overline{\mathbb{F}}K_\infty}, j_{U_{\overline{\mathbb{F}}K_\infty}}{}_!j_{\overline{\mathbb{F}}K_\infty *}T)^{\mathrm{Gal}(\overline{\mathbb{F}}K_\infty/K_\infty)}.\end{aligned}$$

In particular, $\mathrm{H}^k(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$ is finitely generated as R-module.

Proof. We use Cor. 3.12 and then proceed as in the proof of [Wit13, Prop. 13]. □

4. Noncommutative main conjectures

As before, we let R denote a commutative compact noetherian local ring with finite residue field of characteristic ℓ and fix an admissible ℓ-adic Lie extension K_∞/K with Galois group $G = H \rtimes \Gamma$, an admissible $R[[\mathrm{Gal}_K]]$-module T, and two open dense subschemes V and W of the proper smooth curve C with function field K. Recall that

$$U = V \cap W, \qquad \Sigma_V = C - V, \qquad \Sigma_W = C - W.$$

We will also continue to use the general conventions from the beginning of Section 3.1 and the sets of bad primes Ξ_i $(i = 0, 1, 2, 3)$ from Def. 3.9.

4.1. ***L*-functions.** Below, we will define a Σ_W-truncated Σ_V-modified L-function of T. For obviuous reasons we need to assume that $U \cap \Xi_0 = \emptyset$. Usually, one also requires that Σ_V does not contain any v for which T is ramified. If $\ell \neq p$ however, there exists a natural extension of the usual definition. In the case $\ell = p$ one can still give a sensible definition under the condition that T is at most tamely ramified for all $v \in \Sigma_V$.

Under the assumption $U \cap \Xi_0 = \emptyset$ we know by Prop. 3.4 that $j_{K*}T$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(U, R)$. Hence, assuming $\ell \neq p$, the complex $i_v^* \mathrm{R}\, j_{V*}j_{U!}j_{K*}T$ is in $\mathbf{PDG}^{\mathrm{cont}}(v, R)$ and we may define its L-function $L(i_v^* \mathrm{R}\, j_{V*}j_{U!}j_{K*}T, t) \in R[[t]]^\times$ as in [Wit10, §8].

In more concrete terms, we can proceed as follows. Let $\mathcal{R}_v \subset \mathcal{I}_v$ denote the ramification subgroup and fix a choice of $\tau, \varphi \in \mathcal{D}_v/\mathcal{R}_v$ such that τ topologically generates the tame inertia group $\mathcal{I}_v/\mathcal{R}_v$ and φ is a lift of the geometric Frobenius $\mathfrak{F}_v$. It is well known that τ and φ topologically generate $\mathcal{D}_v/\mathcal{R}_v$ and that

$$\varphi\tau\varphi^{-1} = \tau^{-q^{\deg(x)}}$$

[NSW00, Thm. 7.5.2]. We define a complex $C^\bullet(T)$ of admissible $R[[\mathrm{Gal}_{\mathbb{F}(v)}]]$-modules as follows. For $k \neq 1, 2$ we set $C^k(T) = 0$. As R-modules we have $C^0(T) = C^1(T) = T^{\mathcal{R}_v}$ and the differential is given by $\mathrm{id} - \tau$. The geometric Frobenius acts on $C^0(T)$ via φ and on $C^1(T)$ via

$$\varphi\left(\sum_{m=0}^{q^{\deg(v)}-1} \tau^m\right) \in R[[\mathcal{D}_v/\mathcal{R}_v]]^\times.$$

The complex $C^\bullet(T)$ may be viewed as an object in $\mathbf{PDG}^{\mathrm{cont}}(v, R)$ which is weakly equivalent to $i_v^*(\mathrm{R}\, j_{V*}j_{U!}j_{K*}T)$ and we have

$$L(i_v^* \mathrm{R}\, j_{W*}j_{K*}T, t) = \det(1 - t^{\deg(v)}\mathfrak{F}_v : C^0(T))^{-1} \det(1 - t^{\deg(v)}\mathfrak{F}_v : C^1(T)).$$

Note that if the ℓ-Sylow-subgroups of $\mathrm{im}(\mathcal{I}_v \to \mathrm{Aut}_R(T))$ are trivial, then

$$L(i_v^* \mathrm{R}\, j_{V*}j_{U!}j_{K*}T, t) = \det(1 - t^{\deg v}\mathfrak{F}_v : T^{\mathcal{I}_v})^{-1} \det(1 - (tq)^{\deg(v)}\mathfrak{F}_v : T^{\mathcal{I}_v}).$$

Definition 4.1. Assume that $U \cap \Xi_0 = \emptyset$. If $\ell = p$ we must also assume that T is at most tamely ramified at v for all $v \in \Sigma_V$. We set

$$f_v(T,t) = \begin{cases} L(v, i_v^* \mathrm{R}\, j_{V*} j_{U!} j_{K*} T) & \text{if } \ell \neq p, \\ \det(1 - t^{\deg v} \mathfrak{F}_v : T^{\mathcal{I}_v})^{-1} \det(1 - (tq)^{\deg(v)} \mathfrak{F}_v : T^{\mathcal{I}_v}) & \text{if } \ell = p. \end{cases}$$

The Σ_W*-truncated* Σ_V*-modified* L*-function of* T is given by

$$L_{\Sigma_W,\Sigma_V}(T,t) = \prod_{v \in U} \det(1 - \mathfrak{F}_v t^{\deg(v)} : T^{\mathcal{I}_v})^{-1} \prod_{v \in \Sigma_V} f_v(T,t)$$

We need a variant of this definition. Still assuming that $U \cap \Xi_0 = \emptyset$, we note that $\mathrm{R}\,\Gamma(\overline{C}, j_{U!} j_{K*} T)$ is in $\mathbf{PDG}^{\mathrm{cont}}(R)$, see Prop. 3.4. Hence, $\Psi_{R[[t]]} \mathrm{R}\,\Gamma(\overline{C}, j_{U!} j_{K*} T)$ in an object in $\mathbf{PDG}^{\mathrm{cont}}(R[[t]])$ on which $\mathrm{id} - \mathfrak{F}t$ acts as an automorphism. We thus obtain a class

$$[\mathrm{id} - \mathfrak{F}t \colon \Psi_{R[[t]]} \mathrm{R}\,\Gamma(\overline{C}, j_{U!} j_{K*} T)] \in \mathrm{K}_1(\mathbf{PDG}^{\mathrm{cont}}(R[[t]])) \cong \mathrm{K}_1(R[[t]]).$$

Recall further that the determinant induces an isomorphism

$$\det \colon \mathrm{K}_1(R[[t]]) \to R[[t]]^\times.$$

Definition 4.2. Assume that $U \cap \Xi_0 = \emptyset$. If $\ell = p$ we must also assume that T is at most tamely ramified at v for all $v \in \Sigma_V$. We set

$$\tilde{L}_{\Sigma_W,\Sigma_V}(T,t) = \det[\mathrm{id} - \mathfrak{F}t \colon \Psi_{R[[t]]} \mathrm{R}\,\Gamma(\overline{C}, j_{U!} j_{K*} T)]^{-1} \prod_{v \in \Sigma_V} f_v(T,t).$$

We note that if $\ell \neq p$ then $\mathrm{R}\,\Gamma(\overline{V}, j_{U!} j_{K*} T)$ is also in $\mathbf{PDG}^{\mathrm{cont}}(R)$ and

$$\tilde{L}_{\Sigma_W,\Sigma_V}(T,t) = \det[\mathrm{id} - \mathfrak{F}t \colon \Psi_{R[[t]]} \mathrm{R}\,\Gamma(\overline{V}, j_{U!} j_{K*} T)]^{-1}.$$

Set

$$R\langle t \rangle = \varprojlim_{I \in \mathfrak{I}_R} R/I[t]$$

and consider the multiplicatively closed subset

$$P = \left\{ f(t) \in R\langle t \rangle \mid f(0) \in R^\times \right\} \subset R\langle t \rangle.$$

It follows as in [Wit13, Thm. 1.9] that

$$\tilde{L}_{\Sigma_W,\Sigma_V}(T,t) \in R\langle t \rangle_P^\times$$

and that

$$\frac{L_{\Sigma_W,\Sigma_V}(T,t)}{\tilde{L}_{\Sigma_W,\Sigma_V}(T,t)}$$

is equal to 1 if $\ell \neq p$ [Del77, Fonctions L mod ℓ, Thm. 2.2] and to a unit in $R\langle t \rangle$ if $\ell = p$ [EK01, Cor. 1.8].

Let γ be the image of $\mathfrak{F}$ in $\Gamma = \mathrm{Gal}(\mathbb{F}_\infty/\mathbb{F})$. We recall from [Wit13, Lemma 1.10] that

$$R\langle t \rangle_P \to R[[\Gamma]]_S, \qquad t \mapsto \gamma^{-1}$$

is a ring homomorphism such that we obtain elements

$$L_{\Sigma_W,\Sigma_V}(T,\gamma^{-1}), \tilde{L}_{\Sigma_W,\Sigma_V}(T,\gamma^{-1}) \in R[[\Gamma]]_S^\times.$$

These elements are the natural analogues of the classical ℓ-adic L-function in the number field case.

4.2. **The main conjecture for ℓ-adic representations.** As before, we fix an admissible ℓ-adic Lie extension K_∞/K with Galois group $G = H \rtimes \Gamma$ as well as two open dense subsets V and W of the smooth, projective and geometrically connected curve C over $\mathbb{F}$ with function field K. In this section we will discuss non-commutative Iwasawa main conjectures for the admissible $R[[\mathrm{Gal}_K]]$-module T, with R any commutative compact local noetherian ring with finite residue field of characteristic ℓ. We will need the Waldhausen categories $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ and $w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ from Def. 2.3. Recall also the evaluation map Φ_ρ from (2.1).

First, we treat the case $\ell \neq p$.

Theorem 4.3. *Assume that $\ell \neq p$ and that $U \cap \Xi_1 = \emptyset$. Then*

(1) $\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$ *is in* $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ *and the endomorphism* $\mathrm{id} - \mathfrak{F}_\mathbb{F}$ *of* $\mathrm{R}\,\Gamma(\overline{V}, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$ *is a weak equivalence in* $w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$.

(2) *Set*

$$\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T) = [\mathrm{id} - \mathfrak{F}_\mathbb{F} \colon \mathrm{R}\,\Gamma(\overline{V}, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))]^{-1}$$

in $\mathrm{K}_1(R[[G]]_S)$. *Then*

(a) $\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T) = -[\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))]$

(b) *For any commutative compact noetherian local ring R' which is a finite flat extension of R and any admissible $R'[[\mathrm{Gal}_K]]$ module ρ such that the Gal_K-action factors through G*

$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)) = L_{\Sigma_W\cup(\Xi_2\cap V),\Sigma_V}(T \otimes_R \rho, \gamma^{-1}).$$

Proof. Set

$$\mathcal{F} = j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T).$$

We begin by showing (1). By Prop. 3.10 the complex $\mathrm{R}\,\Gamma(V, \mathcal{F})$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ and we may assume that $U \cap \Xi_2 = \emptyset$. We can hence find a strictly perfect complex $C^\bullet$ of $R[[G]]$-modules and a quasi-isomorphism

$$C^\bullet \to \varprojlim_{I\in\mathrm{Jac}(R[[G]])} \mathrm{R}\,\Gamma(V, \mathcal{F}_I).$$

In the case that H is finite, then the cohomology groups of $C^\bullet$ are finitely generated R-modules by Prop. 3.13. Recall that $R[[H]]$ is noetherian. Since every finitely generated projective $R[[G]]$-module is flat as $R[[H]]$-module , the complex $C^\bullet$ is also of finite flat dimension over $R[[H]]$. Hence, it is a perfect complex of $R[[H]]$-modules and $\mathrm{R}\,\Gamma(V, \mathcal{F})$ is in $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$.

For general H we may choose an open pro-ℓ-subgroup H' in H which is normal in G. By [Wit10, Prop. 4.8] a complex in $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$ is in $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ if its image under $\Psi_{R[[G/H']]}$ is in $\mathbf{PDG}^{\mathrm{cont},w_{H/H'}}(R[[G/H']])$. But

$$\Psi_{R[[G/H']]}\,\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T)) \to \mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G/H']]^\sharp \otimes_R T))$$

is a weak equivalence by Prop. 3.11.

Moreover, recall from Section 2.2 that we have an exact sequence

$$0 \to \mathrm{R}\,\Gamma(V, \mathcal{F}) \to \mathrm{R}\,\Gamma(\overline{V}, \mathcal{F}) \xrightarrow{\mathrm{id}-\mathfrak{F}} \mathrm{R}\,\Gamma(\overline{V}, \mathcal{F}) \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$. In particular, the endomorphism $\mathrm{id} - \mathfrak{F}$ is a weak equivalence in $w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$. This finishes the proof of (1).

By the explicit descripition of the boundary homomorphism ∂ from Section 2.1 we conclude

$$\partial[\mathrm{id} - \mathfrak{F}] = [\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))].$$

This proves the first part of (2).

For the second part we we may still assume that $U \cap \Xi_2 = \emptyset$. From Prop. 3.11 we conclude

$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)) = \mathcal{L}_{\mathbb{F}_\infty K/K,\Sigma_W,\Sigma_V}(T \otimes_R \rho)$$

Let $(f\colon C_{\mathbb{F}_\infty K} \to C, \Gamma)$ denote the principal covering with Galois group Γ in the sense of [Wit10, Def. 2.1] and consider the Waldhausen exact functor

$$f_! f^*\colon \mathbf{PDG}^{\mathrm{cont}}(C, R) \to \mathbf{PDG}^{\mathrm{cont}}(C, R[[\Gamma]])$$

from [Wit10, Def. 6.1]. Since f is everywhere unramified one checks easily on the stalks that the natural morphism

$$f_! f^*(\mathrm{R}\, j_{V*}) j_{U!} j_{K*}(T) \to (\mathrm{R}\, j_{V*}) j_{U!} j_{K*}(R[[\Gamma]]^\sharp \otimes_R T)$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(C, R[[\Gamma]])$. From [Wit10, Thm. 8.6] we thus conclude

$$\mathcal{L}_{\mathbb{F}_\infty K/K,\Sigma_W,\Sigma_V}(T) = L((\mathrm{R}\, j_{V*}) j_{U!} j_{K*} T, \gamma^{-1}) = L_{\Sigma_W,\Sigma_V}(T, \gamma^{-1})$$

and hence,

$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)) = L_{\Sigma_W,\Sigma_V}(T \otimes_R \rho, \gamma^{-1})$$

as claimed. □

Similarly, we can derive in the case $\ell = p$ a main conjecture featuring a non-commutative L-function $\tilde{\mathcal{L}}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)$ that satisfies an interpolation property with respect to the functions $\tilde{L}_{\Sigma_W \cup \Xi_2,\Sigma_V}(T \otimes_R \rho, \gamma^{-1})$.

Theorem 4.4. *Assume that $\ell = p$, that $U \cap \Xi_1 = \emptyset$, and that $\Sigma_V \cap \Xi_3 = \emptyset$, i. e. $R[[G]]^\sharp \otimes_R T$ is at most tamely ramified for each $v \in \Sigma_V$. Then*

(1) $\mathrm{R}\,\Gamma(C, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T))$ *is in* $\mathbf{PDG}^{\mathrm{cont},w_H}(R[[G]])$ *and the endomorphism* $\mathrm{id} - \mathfrak{F}_{\mathbb{F}}$ *of* $\mathrm{R}\,\Gamma(\overline{C}, j_{V!} j_{K*}(R[[G]]^\sharp \otimes_R T))$ *is a weak equivalence in* $w_H\mathbf{PDG}^{\mathrm{cont}}(R[[G]])$.

(2) *Set*

$$\begin{aligned}\tilde{\mathcal{L}}_{K_\infty/K,\Sigma_W,\Sigma_V}(T) =& [\mathrm{id} - \mathfrak{F}_{\mathbb{F}}\colon\ \mathrm{R}\,\Gamma(\overline{C}, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T))]^{-1} \\ & \prod_{v \in \Sigma_V} [\mathrm{id} - \mathfrak{F}_v q^{\deg(v)}\colon (R[[G]]^\sharp \otimes_R T)^{\mathcal{I}_v}]\end{aligned}$$

in $\mathrm{K}_1(R[[G]]_S)$. *Then*

(a) $\partial \tilde{\mathcal{L}}_{K_\infty/K,\Sigma_W,\Sigma_V}(T) = -[\mathrm{R}\,\Gamma(C, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T))]$

(b) *For any commutative compact noetherian local ring R' which is a finite flat extension of R and any admissible $R'[[\mathrm{Gal}_K]]$ module ρ such that the Gal_K-action factors through G*

$$\Phi_\rho(\tilde{\mathcal{L}}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)) = \tilde{L}_{\Sigma_W \cup \Xi_2,\Sigma_V}(T \otimes_R \rho, \gamma^{-1}).$$

Proof. If $\Sigma_V = \emptyset$, then one proceeds exactly as in the proof of Theorem 4.3. For $\Sigma_V \neq \emptyset$ it remains to notice that $\mathrm{id} - \mathfrak{F}_v q^{\deg(v)}$ is an automorphism of the finitely generated projective $R[[G]]$-module $(R[[G]]^\sharp \otimes_R T)^{\mathcal{I}_v}$ such that its class lies in $\mathrm{K}_1(R[[G]]) \subset \mathrm{K}_1(R[[G]]_S)$ and hence, has trivial image under the boundary homomorphism ∂. Moreover,

$$\Phi_\rho([\mathrm{id} - \mathfrak{F}_v q^{\deg(v)}\colon (R[[G]]^\sharp \otimes_R T)^{\mathcal{I}_v}]) = \det[\mathrm{id} - \gamma^{-1}\mathfrak{F}_v q^{\deg(v)}\colon R[[\Gamma]] \otimes_R (T \otimes_R \rho)^{\mathcal{I}_v}]$$

such that

$$\Phi_\rho(\tilde{\mathcal{L}}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)) = \Phi_\rho\left(\tilde{\mathcal{L}}_{K_\infty/K,\Sigma_W\cup\Xi_2,\emptyset}(T)\prod_{v\in\Sigma_V}[\mathrm{id}-\mathfrak{F}_v q^{\deg(v)}\colon (R[[G]]^\sharp\otimes_R T)^{\mathcal{I}_v}]\right) = \tilde{L}_{\Sigma_W\cup\Xi_2,\Sigma_V}(T\otimes_R\rho,\gamma^{-1}).$$

□

We conjecture that there also exists an element $\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)$ interpolating the actual L-functions $L_{\Sigma_W\cup(\Xi_2\cap V),\Sigma_V}(T\otimes_R\rho,\gamma^{-1})$.

Conjecture 4.5. *Assume that $\ell = p$, that $U\cap\Xi_1=\emptyset$, and that $\Sigma_V\cap\Xi_3=\emptyset$. There exists a $\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)\in\mathrm{K}_1(R[[G]]_S)$ such that*

1. $\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T) = -[\mathrm{R}\,\Gamma(C, j_{U!}j_{K*}(R[[G]]^\sharp\otimes_R T))]$
2. *For any commutative compact noetherian local ring R' which is a finite flat extension of R and any admissible $R'[[\mathrm{Gal}_K]]$-module ρ such that the Gal_K-action factors through G*

$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)) = L_{\Sigma_W\cup(\Xi_2\cap V),\Sigma_V}(T\otimes_R\rho,\gamma^{-1}).$$

So far, however, only a partial result regarding this conjecture is known by previous work of Burns, Crew, Emmerton, and Kisin.

Theorem 4.6. *Conjecture 4.5 holds if G is commutative. If G is arbitrary, $p>2$, and $R=\mathbb{Z}_p$ then there exists a $\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)\in\mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S)$ satisfying* (1) *and the possibly weaker interpolation property*

(2') *Let $\mathcal{O}$ be the ring of integers of a finite extension field of $\mathbb{Q}_p$ and ρ an Artin representation of Gal_K which factors through G and takes values in $\mathcal{O}$. Then*

$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(T)) = L_{\Sigma_W\cup(\Xi_2\cap V),\Sigma_V}(T\otimes_R\rho,\gamma^{-1}).$$

Proof. It suffices to show that there is an element $u\in\mathrm{K}_1(R[[G]])$ such that

$$\Phi_\rho(u) = \frac{L_{\Sigma_W\cup(\Xi_2\cap V),\Sigma_V}(T\otimes_R\rho,\gamma^{-1})}{\tilde{L}_{\Sigma_W\cup\Xi_2,\Sigma_V}(T\otimes_R\rho,\gamma^{-1})}.$$

Note that the right hand side of the equality does not depend on Σ_W and Σ_V. So, we may assume that $\Sigma_V=\emptyset$ and that Σ_W contains all points that ramify in K_∞/K and all points v with $T\neq T^{\mathcal{I}_v}$. The existence of u then follows from [Bur11, Thm. 4.6] in the abelian case and from [Bur11, Thm. 1.1] for arbitrary G (see also [Wit13, Prop. 2.3 and Thm. 1.5]. □

4.3. **Calculation of the cohomology.** In this section, we will give a description of the cohomology groups of the complex $\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp\otimes_R T))$. We will need the following generalisation of [Mil80, Prop. V.2.2.(b)].

Lemma 4.7. *Assume that $\ell\neq p$ and that R and H are finite. Then*

$$\mathrm{H}^k(V_{K_\infty}, j_{U_{K_\infty}!}j_{K_\infty *}T) = \mathrm{H}^{2-k}(W_{K_\infty}, j_{U_{K_\infty}!}j_{K_\infty *}T^\vee(1))^\vee.$$

Proof. In the following, we drop the index K_∞ and assume that C is a curve over $\mathbb{F}_\infty$ to ease the notation. For any étale sheaf $\mathcal{F}$ of R-modules on U, the canonical morphism

$$j_{V!}\,\mathrm{R}\,j_{U*}\mathcal{F}\to\mathrm{R}\,j_{W*}j_{U!}\mathcal{F}$$

is a isomorphism in the derived category of complexes of étale sheaves of R-modules on C. To see this, we note that

$$
\begin{gathered}
j_U^* j_{V!} \mathrm{R}\, j_{U*}\mathcal{F} \cong \mathcal{F} \cong j_U^* \mathrm{R}\, j_{W*} j_{U!}\mathcal{F}, \\
i_{\Sigma_V}^* \mathrm{R}\, j_{W*} j_{U!}\mathcal{F} \cong i_{\Sigma_V}^* j_W^* \mathrm{R}\, j_{W*} j_{U!}\mathcal{F} \cong 0 \cong i_{\Sigma_V}^* j_{V!} \mathrm{R}\, j_{U*}\mathcal{F}, \\
i_{\Sigma_W}^* i_{\Sigma_V *} i_{\Sigma_V}^* \mathrm{R}\, j_{V*} \mathrm{R}\, j_{U*}\mathcal{F} \cong 0 \cong i_{\Sigma_W}^* \mathrm{R}\, j_{W*} i_{\Sigma_V *} i_{\Sigma_V}^* \mathrm{R}\, j_{U*}\mathcal{F}, \\
i_{\Sigma_W}^* j_{V!} \mathrm{R}\, j_{U*}\mathcal{F} \cong i_{\Sigma_W}^* \mathrm{R}\, j_{V*} \mathrm{R}\, j_{U*}\mathcal{F} \cong i_{\Sigma_W}^* \mathrm{R}\, j_{W*} \mathrm{R}\, j_{U*}\mathcal{F} \cong i_{\Sigma_W}^* \mathrm{R}\, j_{W*} j_{U!}\mathcal{F}.
\end{gathered}
$$

Let R_0 be the image of $\mathbb{Z}$ in R, let $\mathcal{HOM}_U(\mathcal{F}, R_0(1))$ denote the sheaf of R_0-homomorphisms $\mathcal{F} \to R_0(1)$ on U, and set

$$D_U(\mathcal{F}) = \mathrm{R}\,\mathcal{HOM}_U(\mathcal{F}, R_0(1)).$$

From the relative Poincaré duality theorem and the biduality theorem we obtain natural isomorphisms

$$j_{U!} D_U(\mathcal{F}) \cong D_V(\mathrm{R}\, j_{U*}\mathcal{F}), \qquad \mathrm{R}\, j_{U*} D_U(\mathcal{F}) \cong D_V(j_{U!}\mathcal{F})$$

in the derived category [KW01, Cor. II.7.3, Thm II.10.3, Cor. II.7.5]. Let $f\colon C \to \operatorname{Spec} \mathbb{F}_\infty$ denote the structure map. Applying the Poincaré duality theorem again we get

$$
\begin{aligned}
\operatorname{Hom}_{R_0}(\mathrm{R}\,\Gamma(W, j_{U!}\mathcal{F}), R_0[2]) &\cong \operatorname{Hom}_{R_0}(\mathrm{R}\,\Gamma(C, j_{V!} \mathrm{R}\, j_{U*}\mathcal{F}), R_0[2]) \\
&\cong \mathrm{R}\,\Gamma(C, D_C(j_{V!} \mathrm{R}\, j_{U*}\mathcal{F})) \\
&\cong \mathrm{R}\,\Gamma(V, j_{U!} D_U(\mathcal{F})).
\end{aligned}
$$

To finish the proof, we note that

$$D_U(j_* T) \cong j_* T^\vee(1)$$

[Mil80, Proof of Prop. V.2.2.(b)] and take cohomology. □

Theorem 4.8. *We assume either $\ell \neq p$ or $V = C$. Then*

1. $\mathrm{H}^k(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) = 0$ *for* $k \notin \{1, 2, 3\}$.
2. $$\mathrm{H}^1(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) = \begin{cases} T^{\mathrm{Gal}_{K_\infty}} & \text{if } U = V \text{ and } H \text{ is finite,} \\ 0 & \text{else.} \end{cases}$$
3. *If $\ell \neq p$ then*
$$\mathrm{H}^2(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) = \mathrm{H}^1(W_{K_\infty}, j_{U_{K_\infty}!} j_{K_\infty *} T^\vee(1))^\vee.$$
4. $$\mathrm{H}^3(V, j_{U!} j_{K*}(R[[G]]^\sharp \otimes_R T)) = \begin{cases} T(-1)_{\mathrm{Gal}_{K_\infty}} & \text{if } \ell \neq p \text{ and } V = C, \\ 0 & \text{else.} \end{cases}$$

Proof. In the view of Prop. 3.6 we may assume that R is a finite ring. We will first consider the case that H is finite. Using Prop. 3.13 and the fact that the cohomology of an étale sheaf of R-modules on the curve $V_{\overline{\mathbb{F}}K_\infty}$ over the algebraically closed field $\overline{\mathbb{F}}$ is concentrated in degrees 0 up to 2 if $\ell \neq p$ and $V = C$ and up to 1 if $\ell = p$ [Mil80, Cor. VI.2.5] or $V \neq C$ [Mil80, Rem. V.2.4] we deduce Assertion (1) and the second case of Assertion (4). Assertion (2) for H finite follows since

$$\mathrm{H}^0(V_{K_\infty}, j_{U_{K_\infty}!} j_{K_\infty *} T) = \begin{cases} T^{\mathrm{Gal}_{K_\infty}} & \text{if } U = V \\ 0 & \text{else.} \end{cases}$$

We now assume $\ell \neq p$. Assertion (3) is a direct consequence of Lemma 4.7. Moreover, the lemma implies

$$\begin{aligned}\mathrm{H}^3(C, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T)) = \mathrm{H}^2(C_{K_\infty}, j_{U_{K_\infty}!}j_{K_\infty *}T) = \mathrm{H}^0(U_{K_\infty}, j_{K_\infty *}T^\vee(1))^\vee \\ = (T^\vee(1))^{\mathrm{Gal}_{K_\infty}\,\vee} = T(-1)_{\mathrm{Gal}_{K_\infty}}.\end{aligned}$$

This proves Assertion (4) in the case $\ell \neq p$.

Finally, we use Cor. 3.12 to deduce the assertions for general H. In the case of Assertion (2) it remains to notice that since T is finite there exists a finite extension $L/\mathbb{F}_\infty K$ inside K_∞ with $T = T^{\mathrm{Gal}_L}$ and such that $\mathrm{Gal}(K_\infty/L)$ is pro-ℓ. Hence, the norm map $N_{L''/L'}\colon T \to T$ is multiplication by a power of ℓ for $L \subset L' \subset L'' \subset K_\infty$. We conclude that

$$\mathrm{H}^1(V, j_{K*}(R[[G]]^\sharp \otimes_R T)) = \varprojlim_{\mathbb{F}_\infty K \subset L \subset K_\infty} T^{\mathrm{Gal}_L} = 0$$

if H is infinite. □

As we will explain in Section 4.5, the following theorem may be viewed as a vast generalisation of the main result of [GP12].

Theorem 4.9. *Assume that $U \cap \Xi_1 = \emptyset$, $\Sigma_W \neq \emptyset$, and that either $\ell \neq p$ and $\Sigma_V \neq \emptyset$ or $\ell = p$ and $V = C$. Then $\mathrm{H}^2(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$ is finitely generated and projective as $R[[H]]$-module and has a strictly perfect resolution as $R[[G]]$-module. Moreover,*

$$[\mathrm{H}^2(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))] = [\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))]$$

in $\mathrm{K}_0(R[[G]], R[[G]]_S)$.

Proof. By Thm. 4.3 and Thm. 4.4 we may find strictly perfect complexes $P^\bullet$ and $Q^\bullet$ of $R[[G]]$-modules and $R[[H]]$-modules, respectively, which are quasi-isomorphic to $\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$. By Thm. 4.8 the only cohomology group of these complexes that does not vanish is $\mathrm{H}^2(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$. Hence, the above equality holds by Rem. 2.4. Since $R[[H]]$ is noetherian, we also see that $\mathrm{H}^2(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$ is finitely generated as $R[[H]]$-module. It remains to prove that $\mathrm{H}^2(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$ is projective as $R[[H]]$-module.

We first assume that R and H are finite. By Prop. 3.13 we know that

$$\begin{aligned}\mathrm{H}^2(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T)) &= \mathrm{H}^1(V_{K_\infty}, j_{U_{K_\infty}!}j_{K_\infty *}T) \\ &= \mathrm{H}^1(V_{\mathbb{F}_\infty K}, j_{U_{\mathbb{F}_\infty K}!}j_{\mathbb{F}_\infty K*}R[H]^\sharp \otimes_R T))\end{aligned}$$

is the only non-vanishing cohomology group of $\mathrm{R}\,\Gamma(V_{\mathbb{F}_\infty K}, j_{U_{\mathbb{F}_\infty K}!}j_{\mathbb{F}_\infty K*}R[H]^\sharp \otimes_R T)$. The projectivity of $\mathrm{H}^2(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R T))$ follows since for every finitely generated $R[H]^{\mathrm{op}}$-module M, the canonical morphism

$$\begin{gathered}M \otimes_{R[H]} \mathrm{R}\,\Gamma(V_{\mathbb{F}_\infty K}, j_{U_{\mathbb{F}_\infty K}!}(j_{\mathbb{F}_\infty K*}R[H]^\sharp \otimes_R T)) \\ \downarrow \\ \mathrm{R}\,\Gamma(V_{\mathbb{F}_\infty K}, j_{U_{\mathbb{F}_\infty K}!}M \otimes_{R[H]} (j_{\mathbb{F}_\infty K*}R[H]^\sharp \otimes_R T))\end{gathered}$$

is a quasi-isomorphism and the right-hand complex is still concentrated in degree 1. We may then use Prop. 3.6 to deduce the general case. □

4.4. **The main conjecture for Selmer groups.** In this section we will assume that $R = \mathcal{O}$ is a complete discrete valuation ring with finite residue field of characteristic $\ell \neq p$. Furthermore, we assume that $\Sigma_V = \emptyset$. Let T be an admissible $\mathcal{O}[[\mathrm{Gal}_K]]$-module. For $K \subset L \subset \overline{K}$ we may define as in [Gre89, §5] a Selmer group

$$\mathrm{Sel}_{\Sigma_W}(L, T^\vee(1)) = \ker \mathrm{H}^1(\mathrm{Gal}_L, T^\vee(1)) \to \bigoplus_{v \in W_L} \mathrm{H}^1(\mathcal{I}_v, T^\vee(1)),$$

for T.

If K_∞/K is an admissible ℓ-adic Lie extension we set

$$\mathcal{X}_{K_\infty/K,\Sigma_W}(T) = \mathrm{Sel}_{\Sigma_W}(K_\infty, T^\vee(1))^\vee.$$

Lemma 4.10. *For any extension L/K inside $\overline{K}$,*

$$\mathrm{Sel}_{\Sigma_W}(L, T^\vee(1)) = \mathrm{H}^1(W_L, j_{L*}T^\vee(1)).$$

In particular,

$$\mathrm{H}^2(C, j_{W!}j_{K*}(\mathcal{O}[[G]]^\sharp \otimes_{\mathcal{O}} T)) = \mathcal{X}_{K_\infty/K,\Sigma_W}(T)$$

for any admissible ℓ-adic Lie extension K_∞/K.

Proof. Without loss of generality we assume that $L = K$. According to [Mil80, Lemma III.1.6] we have for every integer k

$$\mathrm{H}^k(\mathrm{Gal}_K, T^\vee(1)) = \varinjlim_U \mathrm{H}^k(U, j_{K*}T^\vee(1)).$$

Here, U runs through the open dense subschemes of W. For any such U, the Leray spectral sequence shows

$$\mathrm{H}^1(W, j_{K*}T^\vee(1)) = \ker \mathrm{H}^1(U, j_{K*}T^\vee(1)) \to \bigoplus_{v\in W-U} \mathrm{H}^0(v, i_v^* \mathrm{R}^1 j_{K*}T^\vee(1)).$$

Recall that for any discrete Gal_K-module M one has $i_v^* j_{K*} M_{\xi_v} = M^{\mathcal{I}_v}$. By considering an injective resolution of $T^\vee(1)$ we conclude

$$i_v^* \mathrm{R}^1 j_{K*}T^\vee(1)_{\xi_v} = \mathrm{H}^1(\mathcal{I}_v, T^\vee(1)).$$

The first equality in the lemma follows after passing to the direct limit over U. The second equality follows from the first equality and Thm. 4.8. □

We may thus deduce the following reformulation of the non-commutative main conjecture in terms of the $\mathcal{O}[[G]]$-module $\mathcal{X}_{K_\infty/K,\Sigma_W}(T)$.

Corollary 4.11. *Assume that G has no element of order ℓ. Then*

1. *$\mathcal{X}_{K_\infty/K,\Sigma_W}(T)$ is in $\mathbf{N}_H(\mathcal{O}[[G]])$.*
2. *In $\mathrm{K}_0(\mathcal{O}[[G]], \mathcal{O}[[G]]_S)$ we have*

$$\partial \mathcal{L}_{K_\infty/K,\Sigma_W,\emptyset} = -[\mathcal{X}_{K_\infty/K,\Sigma_W}(T)] + [T(-1)_{\mathrm{Gal}_{K_\infty}}] + \begin{cases} [T^{\mathrm{Gal}_{K_\infty}}] & \text{if } \Sigma_W = \emptyset \text{ and } H \text{ is finite,} \\ 0 & \text{else.} \end{cases}$$

Proof. This is a direct consequence of (2.2), Thm. 4.3, and Thm. 4.8. □

One might wonder under which conditions the class $[T(-1)_{\mathrm{Gal}_{K_\infty}}]$ is zero in $\mathrm{K}_0(\mathcal{O}[[G]], \mathcal{O}[[G]]_S)$. This is not always the case. Since the forgetful functor from $\mathbf{N}_H(\mathcal{O}[[G]])$ to the category of finitely generated $\mathcal{O}[[H]]$-modules induces a homomorphism $\mathrm{K}_0(\mathcal{O}[[G]], \mathcal{O}[[G]]_S) \to \mathrm{K}_0(\mathcal{O}[[H]])$ a necessary condition is that $[T(-1)_{\mathrm{Gal}_{K_\infty}}]$ is zero in $\mathrm{K}_0(\mathcal{O}[[H]])$. By a result of Serre [Ser98, Cor. to Thm. C] (see also [AW08, §1.3]) the class $[T]$ of every $\mathcal{O}[[H]]$-module T which is finitely generated as $\mathcal{O}$-module is zero in $\mathrm{K}_0(\mathcal{O}[[H]])$ precisely if the centraliser of every element in H has infinitely many elements. This condition is for example not satisfied by the group $H = \mathbb{Z}_\ell^d \rtimes \mu_{\ell-1}$ with the group of $\ell-1$-th roots of units acting by multiplication on $\mathbb{Z}_\ell^d$ ($\ell > 2$).

Assume now that $G = <\tau, \gamma> \cong \mathbb{Z}_\ell \rtimes \mathbb{Z}_\ell$ with $\gamma^{-1}\tau\gamma = \tau^{1+\ell}$. Set $H = <\tau>$ and consider the constant $\mathbb{Z}_\ell[[G]]$-module $\mathbb{Z}_\ell$. Clearly, $[\mathbb{Z}_\ell] = 0$ in $\mathrm{K}_0(\mathbb{Z}_\ell[[H]])$ according to the above result of Serre. However, $[\mathbb{Z}_\ell] \neq 0$ in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$. Indeed, the complex

$$\mathbb{Z}_\ell[[G]] \xrightarrow{v \mapsto \left(v - v\tau^{1+\ell},\, v - v\left(\sum_{i=0}^{\ell} \tau^i\right)\gamma\right)} \mathbb{Z}_\ell[[G]]^2 \xrightarrow{(v,w) \mapsto v - v\gamma - w + w\tau^{1+\ell}} \mathbb{Z}_\ell[[G]]$$

is a projective resolution of $\mathbb{Z}_\ell$. Hence, the image of $[\mathbb{Z}_\ell]$ in $\mathrm{K}_0(\mathbb{Z}_\ell[[\Gamma]], \mathbb{Z}_\ell[[\Gamma]]_S) = \mathbb{Z}_\ell[[\Gamma]]_S^\times / \mathbb{Z}_\ell[[\Gamma]]^\times$ under the natural projection map is given by the class of

$$\frac{1 - (\ell + 1)\gamma}{1 - \gamma} \in \mathbb{Z}_\ell[[\Gamma]]_S^\times,$$

which is not a unit in $\mathbb{Z}_\ell[[\Gamma]]^\times$.

A sufficient criterion for the vanishing of the class $[T(-1)_{\mathrm{Gal}_{K_\infty}}]$ in the group $\mathrm{K}_0(\mathcal{O}[[G]], \mathcal{O}[[G]]_S)$ is given in [FK06, Prop. 4.3.17]. Here is a variant of it, inspired by [Záb10, Prop. 4.2].

Proposition 4.12. *Let $G = H \rtimes \Gamma$ be an ℓ-adic Lie group without elements of order ℓ. Assume that there exists an ℓ-adic Lie group H' and a homomorphism $\rho\colon G \to H' \times \Gamma$ such that*

1. *The image of G is open in $H' \times \Gamma$.*
2. *$\rho(H) \cap H'$ is open in H'.*
3. *H' has no elements of order ℓ.*
4. *The centraliser of every element in H' has infinitely many elements.*

Let further $\mathcal{O}$ be a complete discrete valuation ring with finite residue field of characteristic ℓ. Then the class of every $\mathcal{O}[[G]]$-module which is finitely generated as $\mathcal{O}$-module is zero in $\mathrm{K}_0(\mathcal{O}[[G]], \mathcal{O}[[G]]_S)$.

Proof. By the result of Serre the constant $\mathcal{O}[[H']]$-module $\mathcal{O}$ has trivial class in $\mathrm{K}_0(\mathcal{O}[[H']])$. Since every $\mathcal{O}[[H']]$-module may be considered as $\mathcal{O}[[G]]$-module by letting Γ act trivially, we see that $[\mathcal{O}] = 0$ in $\mathrm{K}_0(\mathbf{N}_{H'}(\mathcal{O}[[H' \times \Gamma]]))$, as well. By assumption, every finitely generated $\mathcal{O}[[H' \times \Gamma]]$-module which is finitely generated as $\mathcal{O}[[H']]$-module may be considered via ρ as a finitely generated $\mathcal{O}[[G]]$-module which is also finitely generated as $\mathcal{O}[[H]]$-module. This induces an exact functor $\mathbf{N}_{H'}(\mathcal{O}[[H' \times \Gamma]]) \to \mathbf{N}_H(\mathcal{O}[[G]])$ and hence, a homomorphism between the corresponding K-groups. We conclude that $[\mathcal{O}] = 0$ also in $\mathrm{K}_0(\mathbf{N}_H(\mathcal{O}[[G]]))$. If T is any $\mathcal{O}[[G]]$-module which is finitely generated and free as $\mathcal{O}$-module and M is any module in $\mathbf{N}_H(\mathcal{O}[[G]])$, then $T \otimes_{\mathcal{O}} M$ with the diagonal action of G is again in $\mathbf{N}_H(\mathcal{O}[[G]])$. This induces an endomorphism of $\mathrm{K}_0(\mathbf{N}_H(\mathcal{O}[[G]]))$ mapping $[\mathcal{O}]$ to $[T]$. If T is a finite $\mathcal{O}[[G]]$-module, then the image of G in $\mathrm{Aut}_{\mathcal{O}}(T)$ is a finite group Δ and we get an exact sequence

$$0 \to T'' \to T' \to T \to 0$$

with T'' and T' finitely generated $\mathcal{O}[\Delta]$-modules which are free as $\mathcal{O}$-modules. So, all three modules have trivial class in $\mathrm{K}_0(\mathbf{N}_H(\mathcal{O}[[G]]))$. If finally T is any $\mathcal{O}[[G]]$-module which is finitely generated but not necessarily free as $\mathcal{O}$-module then its submodule $T^\#$ of $\mathcal{O}$-torsion elements is a finite $\mathcal{O}[[G]]$-module and the quotient $T/T^\#$ is free as $\mathcal{O}$-module. This shows that $[T] = 0$ in general. $\square$

4.5. **The main conjecture for Picard 1-motives.** In this section, we will clarify the relation of the main conjectures in Section 4.2 with the main conjecture for ℓ-adic realisations of Picard 1-motives considered in [GP12].

We recall the notion of Picard 1-motives introduced by Deligne [Del74]. For this, we need some more notation. Let $\mathbb{G}_{\mathrm{m}X}$ denote the étale sheaf defined by the group of units of a scheme X. For any closed immersion $i_Z\colon Z \to X$ we let $\mathbb{G}_{\mathrm{m}X,Z}$ denote the kernel of the surjection

$$\mathbb{G}_{\mathrm{m}X} \to i_{Z*}\mathbb{G}_{\mathrm{m}Z}.$$

From now on, we assume that X is a smooth and proper curve over a field k and let $K(X)$ denote its function field. We let $j_{K(X)*}\mathbb{G}_{\mathrm{m}K(X)}$ denote the étale sheaf of invertible rational functions on X and $\mathcal{P}_Z$ its subsheaf of rational functions which are congruent to 1 modulo the canonical divisor of Z in the sense of [Ser88, Ch. III, §1]. For any locally closed subscheme Y of X we let Div_Y denote the étale sheaf on X of divisors with support on Y and Div^0_Y denote the kernel of the degree map $\mathrm{Div}_Y \to \mathbb{Z}$.

Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{G}_{\mathrm{m}X,Z} \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}X} \arrow[r] \arrow[d] & i_{Z*}\mathbb{G}_{\mathrm{m}Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{P}_Z \arrow[r] \arrow[d] & j_{K(X)*}\mathbb{G}_{\mathrm{m}K(X)} \arrow[r] \arrow[d] & (j_{K(X)*}\mathbb{G}_{\mathrm{m}K(X)})/\mathcal{P}_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Div}_{X-Z} \arrow[r] \arrow[d] & \mathrm{Div}_X \arrow[r] \arrow[d] & \mathrm{Div}_Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of étale sheaves on X. One checks easily by taking stalks that all the rows and columns are exact. The third column and the second and third row are exact even in the category of presheaves. Hence,

$$\mathrm{H}^1(X, \mathcal{P}_Z) \subset \mathrm{H}^1(X, j_{K(X)*}\mathbb{G}_{\mathrm{m}K(X)})$$

and the latter group is zero by Hilbert 90 and the Leray spectral sequence. In particular, we have for any open dense subscheme Y of X:

$$\mathrm{H}^1(Y, \mathbb{G}_{\mathrm{m}X,Z}) = \mathrm{Div}_{X-Z}(Y)/\left\{\mathrm{div}(f)\colon f \in \mathcal{P}_Z(Y)\right\} =: \mathrm{Pic}(Y, Z \cap Y).$$

This group is usually called the Picard group of Y relative to the effective divisor corresponding to $Z \cap Y$. If $Y = X$ and k is finite, then it is also known as the ray class group for the modulus Z. We let $\mathrm{Pic}^0(X, Z) \subset \mathrm{Pic}(X, Z)$ denote the subgroup of elements of degree 0. If k is algebraically closed, then it can be identified with k-valued points of the generalised Jacobian variety of X with respect to Z [Ser88, Ch. V, Thm. 1].

We will now assume that $k = \overline{\mathbb{F}}$. As before, we let $p > 0$ denote the characteristic of $\mathbb{F}$. One deduces from [Mil80, Ex. III.2.22] that for all étale open neighbourhoods of X the higher cohomology groups of the sheaves in the third column and the second and third row vanish, i. e. the sheaves are flabby in the sense of [Mil80, Ex. III.1.9.(c)].

Consider two reduced subschemes Z_1 and Z_2 of X with empty intersection. The Picard 1-motive for Z_1 and Z_2 is defined to be the complex of abelian groups

$$\mathcal{M}_{Z_1,Z_2}\colon \mathrm{Div}^0_{Z_1}(X) \to \mathrm{Pic}^0(X, Z_2)$$

concentrated in degrees 0 and 1 [GP12, Def. 2.3]. Its group of n-torsion points for a number $n > 0$ is given by

$$\mathcal{M}_{Z_1,Z_2,n} = \mathrm{H}^0(\mathcal{M}_{Z_1,Z_2} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n))$$

and its ℓ-adic Tate module is given by

$$\mathrm{T}_\ell\, \mathcal{M}_{Z_1,Z_2} = \varprojlim_{k>0} \mathcal{M}_{Z_1,Z_2,\ell^k}$$

[Del74, §10.1.5].

Lemma 4.13. *We have for all numbers $n > 0$*

$$\mathcal{M}_{Z_1,Z_2,n} = \mathrm{H}^0(\mathrm{R}\,\Gamma(X - Z_1, \mathbb{G}_{\mathrm{m}\,X,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n))$$

where $\mathrm{R}\,\Gamma(X-Z_1, \mathbb{G}_{\mathrm{m}\,X,Z_2})$ denotes the total derived section functor and $\otimes^{\mathbb{L}}_{\mathbb{Z}}$ denotes the total derived tensor product in the derived category of abelian groups.

Proof. Consider the complexes

$$A^\bullet\colon\ \mathrm{Div}_{Z_1}(X) \to \mathrm{Pic}(X, Z_2),$$
$$B^\bullet\colon\ \mathrm{Div}_{Z_1}(X) \oplus \mathcal{P}_{Z_2}(X) \to \mathrm{Div}_{X-Z_2}(X),$$

and

$$E^\bullet = \begin{cases} \mathbb{Z}[-1] & \text{if } Z_1 = \emptyset, \\ 0 & \text{else,} \end{cases} \qquad F^\bullet = \begin{cases} \overline{\mathbb{F}}^\times & \text{if } Z_2 = \emptyset, \\ 0 & \text{else.} \end{cases}$$

We obtain two obvious distinguished triangles

$$\mathcal{M}_{Z_1,Z_2} \to A^\bullet \to E^\bullet, \qquad F^\bullet \to B^\bullet \to A^\bullet.$$

Moreover, the obvious map from $B^\bullet$ to the complex

$$\mathcal{P}_{Z_2}(X - Z_1) \to \mathrm{Div}_{X-Z_2}(X - Z_1)$$

is a quasi-isomorphism. The latter complex may be identified with the complex $\mathrm{R}\,\Gamma(X - Z_1, \mathbb{G}_{\mathrm{m}\,X,Z_2})$. The claim follows since

$$\mathrm{H}^{-1}(E^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = \mathrm{H}^0(E^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = \mathrm{H}^0(F^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = \mathrm{H}^1(F^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = 0.$$

□

Lemma 4.14. *If $p \nmid n$, then*

$$\mathrm{R}\,\Gamma(X - Z_1, \mathbb{G}_{\mathrm{m}\,X,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathrm{R}\,\Gamma(X - Z_1, j_{(X-Z_2)!}\mu_n)[1]$$

with μ_n the sheaf of n-th roots of unity on $X - Z_2$. In particular,

$$\mathcal{M}_{Z_1,Z_2,n} \cong \mathrm{H}^1(X - Z_1, j_{(X-Z_2)!}\mu_n) \cong \mathrm{H}^1(X - Z_2, j_{(X-Z_1)!}\mathbb{Z}/(n))^\vee$$

Proof. The first statement follows from the Kummer sequences for $\mathbb{G}_{\mathrm{m}\,X}$ and $\mathbb{G}_{\mathrm{m}\,Z_2}$ and the exactness of the sequence

$$0 \to j_{(X-Z_2)!}\mu_n \to i^*_{Z_2}\mu_n \to 0.$$

The second statement follows from Lemma 4.13 and Lemma 4.7. □

Lemma 4.15. *For all numbers $k > 0$ the canonical morphism*

$$\mathrm{R}\,\Gamma(X - Z_1, \mathbb{G}_{\mathrm{m}\,X,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^k) \to \mathrm{R}\,\Gamma(X - Z_1, \mathbb{G}_{\mathrm{m}\,X}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^k) \cong \mathrm{R}\,\Gamma(X - Z_1, \nu^1_k)$$

is an isomorphism. Here, $\nu^1_k = \mathrm{W}_k\,\Omega^1_{X,\log}$ is the logarithmic De Rham-Witt sheaf on X. In particular,

$$\mathcal{M}_{Z_1,Z_2,p^k} \cong \mathrm{H}^0(X - Z_1, \nu^1_k) \cong \mathrm{H}^1(X, j_{(X-Z_1)!}\mathbb{Z}/(p^k))^\vee.$$

Proof. Since we assume Z_2 to be reduced, we have

$$\mathrm{R}\,\Gamma(Z_2, \mathbb{G}_{\mathrm{m}\,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^k) \cong 0.$$

This explains the first isomorphism in the first part of the statement. For the second isomorphism we may use [Gei10, Prop. 2.2] together with the identifications

$$\mathbb{Z}^c_X \cong \mathbb{Z}_X(1)[2] \cong \mathbb{G}_{\mathrm{m}\,X}[1]$$

in the notation of *loc. cit.*. The duality statement

$$\mathrm{H}^0(X - Z_1, \nu^1_k) \cong \mathrm{H}^1(X, j_{(X-Z_1)!}\mathbb{Z}/(p^k))^\vee$$

can be deduced from [Gei10, Thm. 4.1]:

$$\begin{aligned}\mathrm{R}\,\Gamma(X - Z_1, \nu_k^1) &\cong \mathrm{R}\,\Gamma(X - Z_1, \mathbb{Z}_X^c) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^k)[-1] \\ &\cong \mathrm{R}\operatorname{Hom}_{X-Z_1}(\mathbb{Z}/(p^k), \mathbb{Z}_X^c) \\ &\cong \mathrm{R}\operatorname{Hom}(\mathrm{R}\,\Gamma(X, j_{(X-Z_1)!}\mathbb{Z}/(p^k)), \mathbb{Z}) \\ &\cong \mathrm{R}\,\Gamma(X, j_{(X-Z_1)!}\mathbb{Z}/(p^k))^{\vee}[-1].\end{aligned}$$

□

For a free $\mathbb{Z}_\ell$-module M we set

$$M^* = \operatorname{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) = (M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\vee}.$$

If M is a $\mathbb{Z}_\ell[[G]]$-module for some profinite group G, then M^* becomes a $\mathbb{Z}_\ell[[G]]$-module via the operation defined by

$$(g\alpha)(m) = \alpha(g^{-1}m)$$

for $g \in G$, $m \in M$ and $\alpha \in M^*$.

As before, we consider an admissible ℓ-adic Lie extension K_∞/K with Galois group $G = H \rtimes \Gamma$ and two open dense subschemes V and W of the proper smooth curve C over $\mathbb{F}$ with function field K. We will assume that $H = \operatorname{Gal}(K_\infty/\mathbb{F}_\infty K)$ is finite. Set $U = V \cap W$ and $\Upsilon = \operatorname{Gal}(\overline{\mathbb{F}}K_\infty/K_\infty)$. Recall that Υ is of order prime to ℓ.

Proposition 4.16. *Assume that H is finite.*

(1) *If $\ell \neq p$, then $\mathrm{H}^2(W, j_{U!}j_{K*}\mathbb{Z}_\ell[[G]]^\sharp)$ is a finitely generated free $\mathbb{Z}_\ell$-module and*

$$\begin{aligned}(\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{V_{\overline{\mathbb{F}}K_\infty}},\Sigma_{W_{\overline{\mathbb{F}}K_\infty}}})^{\Upsilon} &\cong \mathrm{H}^2(V, j_{U!}j_{K*}\mathbb{Z}_\ell[[G]]^\sharp(1)) \\ &\cong \mathrm{H}^2(W, j_{U!}j_{K*}\mathbb{Z}_\ell[[G]]^\sharp)^*;\end{aligned}$$

(2) *if $\ell = p$, then $\mathrm{H}^2(C, j_{U!}j_{K*}\mathbb{Z}_\ell[[G]]^\sharp)$ is a finitely generated free $\mathbb{Z}_\ell$-module and*

$$(\mathrm{T}_p\,\mathcal{M}_{\Sigma_{V_{\overline{\mathbb{F}}K_\infty}},\Sigma_{W_{\overline{\mathbb{F}}K_\infty}}})^{\Upsilon} \cong \mathrm{H}^2(C, j_{V!}j_{K*}\mathbb{Z}_p[[G]]^\sharp)^*.$$

Proof. We first assume that $U \neq W$ (and $W = C$ if $\ell = p$). Then the complex $\mathrm{R}\,\Gamma(W_{K_\infty}, j_{U_{K_\infty}!}j_{K_\infty *}\mathbb{Z}_\ell)$ is quasi-isomorphic to a strictly perfect complex of $\mathbb{Z}_\ell$-modules $P^\bullet$ whith $P^i = 0$ for $i \notin \{1,2\}$. In particular, $\mathrm{H}^1(P^\bullet)$ is a submodule of the finitely generated and free $\mathbb{Z}_\ell$-module P^1 and therefore, finitely generated and free, as well. If $U = W$ we let $j'\colon W' \to W_{K_\infty}$ be the inclusion of the complement of a single closed point. We then see that

$$\mathrm{H}^1(W_{K_\infty}, j_{K_\infty *}\mathbb{Z}_\ell) = \mathrm{H}^1(W_{K\infty}, j'_! j_{K_\infty *}\mathbb{Z}_\ell)$$

is still a finitely generated and free $\mathbb{Z}_\ell$-module. By Prop. 3.13, this module is isomorphic to $\mathrm{H}^2(W, j_{U!}j_{K*}\mathbb{Z}_\ell[[G]]^\sharp)$. Then we apply Lemma 4.14 and Lemma 4.15. □

Remark 4.17. Note that the image of Υ in $\operatorname{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{V_{\overline{\mathbb{F}}K_\infty}},\Sigma_{W_{\overline{\mathbb{F}}K_\infty}}})$ is finite. Hence, we can always choose the admissible extension K_∞/K large enough such that

$$(\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{V_{\overline{\mathbb{F}}K_\infty}},\Sigma_{W_{\overline{\mathbb{F}}K_\infty}}})^{\Upsilon} = \mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{V_{\overline{\mathbb{F}}K_\infty}},\Sigma_{W_{\overline{\mathbb{F}}K_\infty}}}.$$

With K_∞/K as above, consider the sets Ξ_i from Def. 3.9 for the admissible $\mathbb{Z}_\ell[[\operatorname{Gal}_K]]$-modules $\mathbb{Z}_\ell$ and (if $\ell \neq p$) $\mathbb{Z}_\ell(1)$. We note that $\Xi_0 = \emptyset$ and that

$$\Xi_1 = \Xi_2 = \Xi_3 = \{v \in C \mid \mathcal{J}_v \text{ contains an element of order } \ell\}.$$

Recall that q denotes the number of elements in $\mathbb{F}$ and that γ is the image of the geometric Frobenius in Γ. The following two corollaries are a non-commutative generalisation of Greither's and Popescu's main conjecture for Picard-1-motives [GP12, Cor. 4.13].

Corollary 4.18. *Assume that $\ell \neq p$, that H is finite, that both Σ_V and Σ_W are non-empty, and that $\Xi_1 \subset \Sigma_V \cup \Sigma_W$. Then*

(1) $(\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{V_{\overline{\mathbb{F}}K_\infty}},\Sigma_{W_{\overline{\mathbb{F}}K_\infty}}})^{\Upsilon}$ *has a strictly perfect resolution as a $\mathbb{Z}_\ell[[G]]$-module and is finitely generated and projective as a $\mathbb{Z}_\ell[[H]]$-module. In particular, it has a well-defined class in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]],\mathbb{Z}_\ell[[G]]_S)$.*

(2) *We have*
$$\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(\mathbb{Z}_\ell(1)) = -\left[(\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{V_{\overline{\mathbb{F}}K_\infty}},\Sigma_{W_{\overline{\mathbb{F}}K_\infty}}})^{\Upsilon}\right]$$
in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]],\mathbb{Z}_\ell[[G]]_S)$.

(3) *Let $\mathcal{O}$ be the ring of integers of a finite extension field of $\mathbb{Q}_\ell$ and ρ an Artin representation of Gal_K which factors through G and takes values in $\mathcal{O}$. Then*
$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(\mathbb{Z}_\ell(1))) = L_{\Sigma_W,\Sigma_V}(\rho, q^{-1}\gamma^{-1})$$

Proof. This follows from Thm. 4.3 with $T = \mathbb{Z}_\ell(1)$ together with Thm. 4.9 and Prop. 4.16. $\square$

Corollary 4.19. *Assume that H is finite and that Σ_W is not empty. If $\ell \neq p$ we also assume that Σ_V is not empty and that $\Xi_1 \subset \Sigma_V \cup \Sigma_W$. If $\ell = p$ we assume that $\Xi_1 \subset \Sigma_W$. Then*

(1) $\left((\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{W_{\overline{\mathbb{F}}K_\infty}},\Sigma_{V_{\overline{\mathbb{F}}K_\infty}}})^{\Upsilon}\right)^*$ *has a strictly perfect resolution as a $\mathbb{Z}_\ell[[G]]$-module and is finitely generated and projective as a $\mathbb{Z}_\ell[[H]]$-module. In particular, it has a well-defined class in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]],\mathbb{Z}_\ell[[G]]_S)$.*

(2) *We have*
$$\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(\mathbb{Z}_\ell) = -\left[\left((\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{W_{\overline{\mathbb{F}}K_\infty}},\Sigma_{V_{\overline{\mathbb{F}}K_\infty}}})^{\Upsilon}\right)^*\right]$$
in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]],\mathbb{Z}_\ell[[G]]_S)$.

(3) *Let $\mathcal{O}$ be the ring of integers of a finite extension field of $\mathbb{Q}_\ell$ and ρ an Artin representation of Gal_K which factors through G and takes values in $\mathcal{O}$. Then (except possibly if $p = \ell = 2$)*
$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\Sigma_V}(\mathbb{Z}_\ell)) = L_{\Sigma_W,\Sigma_V}(\rho, \gamma^{-1}).$$

Proof. This follows from Thm. 4.3 and Thm. 4.6 with $T = \mathbb{Z}_\ell$ together with Thm. 4.9 and Prop. 4.16. $\square$

Remark 4.20. Note that our definition of $L_{\Sigma_W,\Sigma_V}(\rho,t)$ follows Grothendieck's convention. If one uses Artin's original definition then it corresponds to the L-function of the dual representations of ρ, see [Mil80, Ex. V.2.20].

4.6. **The main conjecture for function fields.** In this section, we will deduce a non-commutative function field analogue of the most classical formulation of the Iwasawa main conjecture. We retain the notation of the previous sections. In particular, we fix an admissible ℓ-adic Lie extension K_∞/K with Galois group $G = H \rtimes \Gamma$. Different from Section 4.5, we do no longer assume the group H to be finite, but we will assume that $\Sigma_V = \emptyset$.

Corollary 4.21. *Assume that $\ell \neq p$ and that G does not contain any element of order ℓ. Let M be the maximal abelian ℓ-extension of K_∞ which is unramified outside Σ_W. Then*

(1) $\mathrm{Gal}(M/K_\infty)$ *is in* $\mathbf{N}_H(\mathbb{Z}_\ell[[G]])$ *and*

$$\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\emptyset}(\mathbb{Z}_\ell(1)) = -[\mathrm{Gal}(M/K_\infty)] + [\mathbb{Z}_\ell] + \begin{cases} [\mathbb{Z}_\ell(1)] & \text{if } \Sigma_W = \emptyset,\ H \text{ is finite, and } \mu_\ell \subset K_\infty \\ 0 & \text{else.} \end{cases}$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$

(2) *Let* $\mathcal{O}$ *be the ring of integers of a finite extension field of* $\mathbb{Q}_\ell$ *and* ρ *a continuous representation of* Gal_K *which factors through* G *and takes values in* $\mathcal{O}$*. Then*

$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\emptyset}(\mathbb{Z}_\ell(1))) = L_{\Sigma_W\cup\Xi_2,\emptyset}(\rho, q^{-1}\gamma^{-1}).$$

Proof. From Thm. 4.8 and from the equality

$$\mathrm{H}^1(W_{K_\infty}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathrm{Gal}(M/K_\infty)^\vee.$$

we deduce

$$\mathrm{H}^2(C, j_{W!}j_{K*}\mathbb{Z}_\ell[[G]]^\sharp(1)) = \mathcal{X}_{K_\infty/K,\Sigma_W}(\mathbb{Z}_\ell(1)) = \mathrm{Gal}(M/K_\infty)$$

. We then apply Thm. 4.3 and Cor. 4.11. Finally, we remark that $\mathbb{Z}_\ell(1)^{\mathrm{Gal}_{K_\infty}} = 0$ if K_∞ does not contain any ℓ-th root of unity. If K_∞ does contain an ℓ-th root of unity, then it contains also all ℓ^n-th roots of unity for any n, and therefore, $\mathbb{Z}_\ell(1)^{\mathrm{Gal}_{K_\infty}} = \mathbb{Z}_\ell(1)$ in this case. □

If G does contain elements of order ℓ then Thm. 4.3 applied to $\mathbb{Z}_\ell(1)$ is still a sensible main conjecture; however, we can no longer replace the class of the complex

$$[\mathrm{R}\,\Gamma(V, j_{U!}j_{K*}(R[[G]]^\sharp \otimes_R \mathbb{Z}_\ell(1)))] = -\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\emptyset}(\mathbb{Z}_\ell(1))$$

by the classes of its cohomology modules. One may also apply Thm. 4.3 and Thm. 4.6 to $\mathbb{Z}_\ell$ resulting in a main conjecture for every ℓ. Main conjectures of this type have already been discussed in [Bur11].

4.7. **The main conjecture for abelian varieties.** In this section we let A be an abelian variety over $\operatorname{Spec} K$. We continue to assume $\ell \neq p$ and that $\Sigma_V = \emptyset$. Our aim is to deduce a precise function field analogue of the Gl_2 main conjecture in [CFK+05].

Let $\mathcal{O}$ be the valuation ring of a finite extension of $\mathbb{Q}_\ell$ and ρ an admissible $\mathcal{O}[[\mathrm{Gal}_K]]$-module. The Σ_W-truncated L-function of A twisted by ρ is given by

$$L_{\Sigma_W}(A, \rho, t) = \prod_{v\in W} \det(1 - (\mathfrak{F}_v t)^{\deg(v)} \colon (\mathrm{H}^1(A \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}_\ell} \rho)^{\mathcal{I}_v}).$$

If ρ is an Artin representation of Gal_K then $L_{\Sigma_W}(A, \rho, q^{-s})$ is the Hasse-Weil L-function of A twisted by ρ.

We will write $\breve{A}$ for the dual abelian variety,

$$A(\overline{K})_n = \ker A(\overline{K}) \xrightarrow{n} A(\overline{K})$$

for the group of n-torsion points and

$$\mathrm{T}_\ell\, A = \varprojlim_k A(\overline{K})_{\ell^k}$$

for the ℓ-adic Tate module of A. It is well-known that $\mathrm{T}_\ell\, A$ is an admissible $\mathbb{Z}_\ell[[\mathrm{Gal}_K]]$-module. Moreover, the argument of [Sch82, §1] shows that for any closed point $v \in C$

$$(j_{K*}(\mathrm{T}_\ell\, \breve{A}(-1) \otimes_{\mathbb{Z}_\ell} \rho))_v \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong (\mathrm{H}^1(A \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}_\ell} \rho)^{\mathcal{I}_v}$$

such that

$$L_{\Sigma_W}(A, \rho, q^{-1}t) = L_{\Sigma_W,\emptyset}(\mathrm{T}_\ell\, \breve{A} \otimes_{\mathbb{Z}_\ell} \rho, t).$$

As an immediate consequence of Thm. 4.3 we obtain:

Corollary 4.22. *Let K_∞/K be an admissible ℓ-adic Lie extension with $\ell \neq p$. Assume that $\Xi_1 \subset \Sigma_W$. Then:*

(1) *We have*
$$\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\emptyset}(\mathrm{T}_\ell\,\check{A}) = -[\mathrm{R}\,\Gamma(C, j_{W!}j_{K*}(R[[G]]^\sharp \otimes_R \mathrm{T}_\ell\,\check{A}))]$$
in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$.

(2) *Let $\mathcal{O}$ be the ring of integers of a finite extension field of $\mathbb{Q}_\ell$ and ρ a continuous representation of Gal_K which factors through G and takes values in $\mathcal{O}$. Then*
$$\Phi_\rho(\mathcal{L}_{K_\infty/K,\Sigma_W,\emptyset}(\mathrm{T}_\ell\,\check{A})) = L_{\Sigma_W}(A, \rho, q^{-1}\gamma^{-1}).$$

For any extension L/K inside $\overline{K}$ we let
$$\mathrm{Sel}_{\Sigma_W}(L, A) = \varinjlim_k \ker \mathrm{H}^1(\mathrm{Gal}_L, A(\overline{K})_{\ell^k}) \to \bigoplus_{v\in W_L} \mathrm{H}^1(\mathcal{D}_v, A(\overline{K}))$$
be the Σ_W-truncated Selmer group of A over L. Here, $\mathcal{D}_v$ denotes a choice of a decomposition subgroup of v inside Gal_L as at the beginning of Section 3.1.

Lemma 4.23. *For every admissible ℓ-adic Lie extension K_∞/K with $\ell \neq p$ we have*
$$\mathrm{Sel}_{\Sigma_W}(K_\infty, A) \cong \mathrm{Sel}_{\Sigma_W}(K_\infty, \mathrm{T}_\ell(\check{A})^\vee(1))$$

Proof. Let L be a finite extension of K and let L_v be the completion of L at $v \in W_L$. According to Greenberg's approximation theorem we have
$$\mathrm{H}^1(\mathcal{D}_v, A(\overline{K})) = \mathrm{H}^1(\mathrm{Gal}_{L_v}, A(\overline{L_v}))$$
[Mil06, Rem. I.3.10]. Since the points of the formal group of A form an open pro-p-subgroup of $A(L_v)$ we conclude from the Kummer sequence that
$$\mathrm{Sel}_{\Sigma_W}(L, A) = \varinjlim_k \ker \mathrm{H}^1(\mathrm{Gal}_L, A(\overline{K})_{\ell^k}) \to \bigoplus_{v\in W_L} \mathrm{H}^1(\mathcal{D}_v, A(\overline{K})_{\ell^k})$$
for all extensions L/K inside $\overline{K}$. If $\mathbb{F}_\infty \subset L$ then $\mathcal{D}_v/\mathcal{I}_v$ is a profinite group of order prime to ℓ and the Hochschild-Serre spectral sequence shows that
$$\mathrm{H}^1(\mathcal{D}_v, A(\overline{K})_{\ell^k}) \to \mathrm{H}^1(\mathcal{I}_v, A(\overline{K})_{\ell^k})$$
is an injection. Furthermore,
$$\mathrm{T}_\ell(\check{A})^\vee(1) = \varinjlim_k A(\overline{K})_{\ell^k}$$
[Sch82, §1] such that indeed $\mathrm{Sel}_{\Sigma_W}(K_\infty, A) \cong \mathrm{Sel}_{\Sigma_W}(K_\infty, \mathrm{T}_\ell(\check{A})^\vee(1))$. □

In particular, we deduce the following function field analogue of the Gl_2 main conjecture of [CFK+05] as a special case of Cor. 4.11.

Corollary 4.24. *Let $\ell \neq p$, K_∞/K be an admissible ℓ-adic Lie extension with Galois group G, and A an abelian variety over $\operatorname{Spec} K$. We assume that G does not contain any element of order ℓ. Then $\mathrm{Sel}_{\Sigma_W}(K_\infty, A)^\vee$ is in $\mathbf{N}_H(\mathbb{Z}_\ell[[G]])$ and*
$$\begin{aligned}\partial\mathcal{L}_{K_\infty/K,\Sigma_W,\emptyset}(\mathrm{T}_\ell(\check{A})) = -[\mathrm{Sel}_{\Sigma_W}(K_\infty, A)^\vee] + [\mathrm{T}_\ell(\check{A})(-1)_{\mathrm{Gal}_{K_\infty}}]\\ + \begin{cases}[\mathrm{T}_\ell(\check{A})^{\mathrm{Gal}_{K_\infty}}] & \text{if } \Sigma_W = \emptyset \text{ and } H \text{ is finite,}\\ 0 & \text{else.}\end{cases}\end{aligned}$$
in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$.

The terms $[\mathrm{T}_\ell(\check{A})(-1)_{\mathrm{Gal}_{K_\infty}}]$ and $[\mathrm{T}_\ell(\check{A})^{\mathrm{Gal}_{K_\infty}}]$ disappear in the following situation.

Proposition 4.25. *Let $p \neq 2$ and let A be an abelian variety over* $\operatorname{Spec} K$ *of dimension $g \geq 1$ which is not of CM-type over $\overline{K}$. Assume that $\ell - 1 > 2g$ and let K_∞ be the extension of K obtained by adjoining the coordinates of all ℓ^n-torsion points of A. Then K_∞/K is an admissible ℓ-adic Lie extension,* $\operatorname{Gal}(K_\infty/K)$ *does not contain any element of order ℓ and there exists a finite subextension K'/K of K_∞/K such that*

$$\partial\mathcal{L}_{K_\infty/K',\Sigma_W,\emptyset}(\mathrm{T}_\ell(\check{A})) = -[\mathrm{Sel}_{\Sigma_W}(K_\infty, A)^\vee].$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[\operatorname{Gal}(K_\infty/K')]], \mathbb{Z}_\ell[[\operatorname{Gal}(K_\infty/K')]]_S)$.

Proof. It is well known that $\operatorname{Gal}(K_\infty/K)$ is the image of Gal_K in $\operatorname{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\check{A})$, that $\mathrm{T}_\ell(\check{A})$ is a free $\mathbb{Z}_\ell$-module of rank $2g$, and that Gal_K acts on the determinant of $\mathrm{T}_\ell(\check{A})$ via the cyclotomic character $\varepsilon_{\mathrm{cyc}}$. This shows that K_∞/K is an admissible ℓ-adic Lie extension. Since $\ell - 1 > 2g$, the group $\operatorname{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\check{A})$ does not contain any element of order ℓ. By a result of Zarhin [Zar77, §4] one can find a finite extension K'/K inside K_∞/K such that

$$\operatorname{Gal}(K_\infty/K') = \operatorname{Gal}(K_\infty/\mathbb{F}_\infty K) \times \operatorname{Gal}(\mathbb{F}_\infty K/K).$$

with $\operatorname{Gal}(K_\infty/\mathbb{F}_\infty K)$ pro-ℓ. Since A is not of CM-type over $\overline{K}$, $\operatorname{Gal}(K_\infty/\mathbb{F}_\infty K)$ is not finite. In particular,

$$[\mathrm{T}_\ell(\check{A}(-1)] = 0$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[\operatorname{Gal}(K_\infty/K')]], \mathbb{Z}_\ell[[\operatorname{Gal}(K_\infty/K')]]_S)$ by Prop. 4.12. □

Remark 4.26. If $g = 1$ in Prop. 4.25 then one can take $K' = K$. Indeed, $\operatorname{Gal}(K_\infty/K)$ must be open in $\operatorname{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\check{A})) = \mathrm{Gl}_2(\mathbb{Z}_\ell)$ and the intersection of $\operatorname{Gal}(K_\infty/\mathbb{F}_\infty K)$ with $\mathrm{Sl}_2(\mathbb{Z}_\ell)$ is open in $\mathrm{Sl}_2(\mathbb{Z}_\ell)$. Otherwise, $\operatorname{Gal}(K_\infty/K)$ would contain a commutative open subgroup by the above result of Zarhin, which is not possible since A is not of CM-type over $\overline{K}$ (This was also observed in the thesis [Sec06], using a different argument). By the assumption on ℓ we may write

$$\mathrm{Gl}_2(\mathbb{Z}_\ell) = H' \times \mathbb{Z}_\ell$$

and one checks that the centraliser of each element in H' is infinite. So we may apply Prop. 4.12.

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