Non-Commutative Iwasawa Theory for Global Fields

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Contents

CHAPTER 1

Introduction

In this work, we will prove various results concerning the non-commutative main conjecture of Iwasawa theory both for function fields and for totally real fields.

We build upon the framework introduced by Fukaya and Kato in their formulation of the ζ-isomorphism conjecture. In particular, we consider representations of the absolute Galois group of a global field F over an adic $\mathbb{Z}_\ell$-algebra Λ. Adding a bit more generality, we will use a formulation of the non-commutative main conjecture in terms of perfect complexes of Λ-adic sheaves over the Dedekind schemes associated to F.

As a central new result in the number field case, we prove a unicity statement for the ℓ-adic L-functions appearing in the non-commutative Iwasawa main conjecture over totally real fields under the assumption that the Iwasawa μ-invariant vanishes, improving on the work of Kakde. Using this unicity result, we show that there exists a unique sensible definition of a non-commutative L-function for a perfect complex $\mathcal{F}^\bullet$ of Λ-adic sheaves. However, we still need to assume that the representations corresponding to the stalks of $\mathcal{F}^\bullet$ factor through a totally real extension of F.

For a function field of characteristic p, we formulate and prove an analogue of the non-commutative Iwasawa main conjecture for complexes of Λ-adic sheaves without further assumptions. We also prove a functional equation for the resulting non-commutative L-functions.

As a corollary, we obtain non-commutative generalisations of the main conjectures for Picard 1-motives and abstract 1-motives of Greither and Popescu. In the case $\ell \neq p$, another corollary is a main conjecture for abelian varieties over function fields in precise analogy to the Gl_2 main conjecture of Coates, Fukaya, Kato, Sujatha and Venjakob.

1.1. Relation to Previous Work

The mysterious connection between special values of L-functions and algebraic invariants of a global field — such as its group of units and its class group, more generally the Galois cohomology of representations of its absolute Galois group with restricted ramification — is one of the central topics of the current research in number theory. With the Tamagawa number conjecture of Bloch and Kato [**BK90**] and its equivariant refinements of Burns and Flach [**BF01**], [**BF03**], Huber and Kings [**HK02**], and finally, Fukaya and Kato [**FK06**], we have a very precise and general conjectural description of this connection at our disposal. As a special case, it includes the conjecture of Birch and Swinnerton-Dyer from the list of the Millennium problems [**CJW06**]. However, only a few special instances of the Tamagawa number conjecture have been verified so far [**HK03**], [**BG03**], [**BF06**].

Non-commutative Iwasawa theory is concerned with the part of the Tamagawa number conjecture that involves ℓ-adic representations. The central aim is the formulation and the proof of non-commutative analogues of the classical Iwasawa main conjecture [**Coa77**] for abelian and totally real number fields that were proved in the works of Mazur und Wiles [**MW86**] and Wiles [**Wil90**]. The seminal work of

Coates, Fukaya, Kato, Sujatha and Venjakob [**CFK⁺05**] on the *non-commutative* Gl_2 *main conjecture* for elliptic curves created a blueprint for much of the current work.

The interest in the non-commutative main conjecture is based on the fundamental insight, first described in [**HK02**], that the ℓ-adic part of the Tamagawa number conjecture can be deduced from a sufficiently general version of the main conjecture. In particular, this version should infer a *strong interpolation property* of the corresponding non-commutative ℓ-adic L-functions in the sense that they are compatible under changes of the coefficient rings induced by appropriate bimodules. The strong interpolation property (Huber and Kings talk of *twist invariance*) permits the reduction of the existence of non-commutative ℓ-adic L-functions of an arbitrary motive to the case of the motive $\mathbb{Z}(1)$, which corresponds in turn to an equivariant refinement of the class number formula.

This insight was fully accounted for in the formulation of the ζ*-isomorphism conjecture* [**FK06**] of Fukaya and Kato by integrating the corresponding Iwasawa main conjecture into the general formalism. Whereas Burns and Flach [**BF01**], [**BF03**] formulate their equivariant Tamagawa number conjecture for motives with coefficients in the group ring of a Galois extension of number fields, Fukaya and Kato extend the point of view even further and consider representations of the absolute Galois groups over a certain class of profinite rings, which we shall refer to as *adic rings*.

Building upon unpublished ideas of Kato [**Kat93**] and the seminal work of Burns [**Bur15**], Kakde proves in [**Kak13**] a non-commutative main conjecture for admissible ℓ-adic Lie extension F_∞/F of totally real number fields, formulated in the style of [**CFK⁺05**]. In particular, he proves the existence of a non-commutative ℓ-adic ζ-function $\zeta_{F_\infty/F}$ in the first K-group of the localisation of the Iwasawa algebra of F_∞/F at Venjakob's canonical Ore set, such that $\zeta_{F_\infty/F}$ verifies the weaker interpolation property with respect to Artin representations. However, he can prove uniqueness of the element $\zeta_{F_\infty/F}$ only modulo the first special K-group $\mathrm{SK}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]])$. The strong interpolation property is not investigated.

Independently and slightly earlier, Ritter and Weiss also completed the proof of their formulation of the main conjecture in this case [**RW02**], [**RW04**], [**RW05**], [**RW11**]. Again, the uniqueness and the strong interpolation property remain open. The two versions of the main conjecture differ in some details, but may be translated into each other. Their precise connection is investigated in [**Ven13**] and [**Nic13**].

An essential prerequisite of both approaches is the vanishing of the Iwasawa μ-invariant for the cyclotomic extension of the relevant number fields. According to a conjecture that goes back to Iwasawa himself [**Iwa71**], [**Iwa73**], this is always the case. However, this conjecture is only known for pro-ℓ Galois extensions of abelian number fields [**FW79**] and in a few other special cases. In [**Mih16**], Mihăilescu attempts to settle the conjecture in the case of CM fields, but so far, he has not been able to convince the peers.

Fukaya and Kato also sketch in [**FK06**] an analogue of their conjecture for curves over a finite field $\mathbb{F}$ of characteristic p different from ℓ. In his dissertation [**Wit08**], the author proves an extension of this analogue to separated schemes of finite type over $\mathbb{F}$. Different from the number field case, the relevant ζ-isomorphism can be constructed explicitly and without detour via a corresponding non-commutative main conjecture.

In the case $p \neq \ell$, the author proves in [**Wit14**] an analogue for separated schemes over finite fields of the non-commutative main conjecture in the style of [**CFK⁺05**]. This result already includes the strong interpolation property and the vanishing of the μ-invariant. On this basis, Burns proves in [**Bur11**] a non-commutative main conjecture in the case $\ell = p$, but only with the interpolation property with respect to Artin representations; see [**Wit13a**] for a survey on both articles. The strong interpolation property in this case is again proved by the author [**Wit16**]. Note however, that the two latter works only give an interpolation of the L-values at $s = 0$. So far, higher Tate twists can not be treated due to the lack of an appropriate equivariant integral p-adic cohomology theory for varieties over $\mathbb{F}_p$.

Another central tool in this context is the author's article [**Wit13b**], which generalises previous results of Schneider and Venjakob [**SV10**]. It is shown that the localisation sequence of higher algebraic K-theory for the localisation of the Iwasawa algebras with respect to Venjakob's canonical Ore set splits into short split exact sequences. In particular, this allows the general definition of a *non-commutative algebraic L-function* in the sense of [**Bur09**]. This non-commutative algebraic L-function satisfies a similar interpolation property as the true non-commutative L-function, but it lacks the connection to the special values of classical L-functions. In particular, the algebraic and the true non-commutative L-function differ in general.

Main conjectures for the Tate module of abelian varieties over function fields in the case $\ell = p$ are considered among others in the articles [**OT09**], [**LLTT16**]. The lack of a suitable ℓ-adic cohomology theory is compensated by the use of flat cohomology. A proof of a analogue of the non-commutative Gl_2 main conjecture [**CFK$^+$05**] in this case has been announced by Vauclair and Trihan [**VT17**]. The results are summarised in [**BT15**]. Previous to the present work, there existed only isolated partial results treating the case $p \neq \ell$ [**Sec06**], [**BV15**], [**Pal14**], although it is much simpler than the case $\ell = p$.

In the works [**GP12**] and [**GP15**], Greither and Popescu formulate and prove a commutative equivariant main conjecture for the S-truncated, T-modified ζ-function both in the case of function fields and in the case of CM extensions of totally real number fields. The corresponding Iwasawa modules are constructed as ℓ-adic realisations of so-called *Picard 1-motives* in the function field case and *abstract 1-motives* in the number field case. In the number field case, Nickel [**Nic13**] has already formulated a non-commutative generalisation of the latter conjecture. He also describes how to deduce it from the main conjecture of Ritter and Weiss.

1.2. The Non-Commutative Main Conjecture of Iwasawa Theory

Let F_∞/F be an admissible ℓ-adic Lie extension of a totally real field F in the sense of [**Kak13**] that is unramified over an open dense subscheme U of the spectrum X of the algebraic integers of F and write $G = \mathrm{Gal}(F_\infty/F)$ for its Galois group. We further assume that ℓ is invertible on U. The non-commutative main conjecture of Iwasawa theory for F_∞/F predicts the existence of a non-commutative ℓ-adic L-function $\mathcal{L}_{F_\infty/F}(U, \mathbb{Z}_\ell(1))$ living in the first algebraic K-group $\mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S)$ of the localisation at Venjakob's canonical Ore set S of the profinite group ring

$$\mathbb{Z}_\ell[[G]] = \varprojlim_{\substack{H \triangleleft G \\ \text{open}}} \mathbb{Z}_\ell[[G/H]].$$

This L-function is supposed to satisfy the following two properties:

(1) It is a characteristic element for the total complex $\mathrm{R}\,\Gamma_c(U, f_! f^* \mathbb{Z}_\ell(1))$ of étale cohomology with proper support with values in the sheaf $f_! f^* \mathbb{Z}_\ell(1)$ corresponding to the the first Tate twist of the Galois module $\mathbb{Z}_\ell[[G]]^\sharp$, on which an element σ of the absolute Galois group Gal_F acts by right multiplication with σ^{-1}.
(2) It interpolates the values of the complex L-functions $L_{X-U}(\rho, s)$ for all Artin representations ρ factoring through G.

Under the assumption that

(a) $\ell \neq 2$,

(b) the Iwasawa μ-invariant of any totally real field is zero,

the non-commutative main conjecture is now a theorem, first proved by Ritter and Weiss [**RW11**]. Almost simultaneously, Kakde [**Kak13**] published an alternative proof, building upon unpublished work of Kato and the seminal article [**Bur15**] by Burns. We refer to Theorem 4.2.1 for a more precise formulation of Kakde's result.

It turns out that properties (1) and (2) are not sufficient to guarantee the uniqueness of $\mathcal{L}_{F_\infty/F}(U, \mathbb{Z}_\ell(1))$. It is only well-determined up to an element of a subgroup

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]]) \subset \mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S).$$

A first objective of this volume is to eradicate this indeterminacy. Under the assumptions (a) and (b) we show in Theorem 4.2.2 that if one lets F_∞ vary over all admissible extensions of F and requires a natural compatibility property for the elements $\mathcal{L}_{F_\infty/F}(U, \mathbb{Z}_\ell(1))$, there is indeed a unique choice of such a family.

In the course of their formulation of a very general version of the equivariant Tamagawa number conjecture, Fukaya and Kato introduced in [**FK06**] a certain class of coefficient rings which we call adic rings for short. Adic rings are precisely those compact, semi-local rings whose Jacobson radical is finitely generated as left or right ideal. In particular, for every adic $\mathbb{Z}_\ell$-algebra Λ, the compact group ring $\Lambda[[G]]$ of G with coefficients in Λ will again be an adic $\mathbb{Z}_\ell$-algebra. Other examples are finite rings and ℓ-adic group rings of finite groups. Moreover, note that this class also contains the coefficient rings of big Galois representations considered in Hida theory.

Our second objective concerns continuous representations $\mathcal{T}$ of the absolute Galois group Gal_F over an adic $\mathbb{Z}_\ell$-algebra Λ. Assume that $\mathcal{T}$ is *smooth over* U and *smooth at* ∞ in the sense that it factors through the Galois group of some (possibly infinite) totally real extension of F unramified over U. As a consequence of Theorem 4.2.2, we show in Theorem 4.2.4 and Corollary 4.2.9 that there exists a unique sensible assignment of a non-commutative L-function

$$\mathcal{L}_{F_\infty/F}(U, \mathcal{T}(1)) \in \mathrm{K}_1(\Lambda[[G]]_S)$$

to any such $\mathcal{T}$. In the sequel [**Wit**] to the present volume, we will use our result to prove the existence of the ζ-isomorphism for such $\mathcal{T}$ as predicted by Fukaya's and Kato's central conjecture [**FK06**, Conj. 2.3.2].

In fact, Corollary 4.2.9 applies more generally to perfect complexes $\mathcal{F}^\bullet$ of Λ-adic sheaves on U which are smooth at ∞. Moreover, we also consider the total derived direct image $\mathrm{R}\,k_*\mathcal{F}^\bullet$ for the open immersion $k\colon U \to W$ into another dense open subscheme W of X. The extension F_∞/F may be ramified over $W - U$, but we do assume that ℓ is invertible on W. We also prove the existence of a dual non-commutative L-function $\mathcal{L}^\circledast_{F_\infty/F}(W, k_!\mathcal{F}^\bullet)$ such that $\mathcal{L}^\circledast_{F_\infty/F}(W, k_!\mathcal{F}^\bullet)^{-1}$ is a characteristic element for the complex $\mathrm{R}\,\Gamma(W, k_!\mathcal{F}^\bullet)$ and satisfies the appropriate interpolation property. If $\mathcal{T}$ is a continuous representation as above and $\mathcal{T}^*$ the dual representation over the opposite ring Λ^{op}, then $\mathcal{L}^\circledast_{F_\infty/F}(U, \mathcal{T})$ is defined as the image of $\mathcal{L}_{F_\infty/F}(U, \mathcal{T}^*(1))$ under the canonical isomorphism

$$\circledast\colon \mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]_S) \cong \mathrm{K}_1(\Lambda[[G]]_S)$$

induced by mapping an invertible matrix A to the *inverse* of its transpose. As we explain in Corollary 4.3.3, all of this can be easily extended to the case that F_∞ is a CM field.

If F is a function field of positive characteristic p and $U \subset W$ are open, dense subschemes of the associated smooth and proper curve X over the finite field of constants $\mathbb{F} \subset F$, the formulation of the non-commutative main conjecture is basically the same, with some extra twists if $\ell = p$. However, the proof is much simpler, as there exists an explicit construction of the non-commutative L-functions $\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet)$ for all perfect complexes $\mathcal{F}^\bullet$ of Λ-adic sheaves on U. Moreover, this result is unconditional, as the vanishing of the Iwasawa μ-invariant is known in the function field case. We refer to Section 4.4 for the details. If $\ell = p$, there are again some extra twists.

Write $j\colon U \to V$ for the open immersion into $V \coloneqq U \cup (X - W)$. In the function field case, the dual non-commutative L-function $\mathcal{L}^{\circledast}_{F_\infty/F}(W, k_! \mathcal{F}^\bullet)$ is the same as the inverse of the product of $\mathcal{L}_{F_\infty/F}(V, \mathrm{R}\, j_* \mathcal{F}^\bullet)$ with a global ε-factor. For a continuous representation $\mathcal{T}$ as above, we obtain in Theorem 5.3.6 the functional equation

$$(\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\, k_* \mathcal{T}^*(1)))^{\circledast} = \varepsilon(V, \mathrm{R}\, j_* f_! f^* \mathcal{T})^{-1} \mathcal{L}_{F_\infty/F}(V, \mathrm{R}\, j_* \mathcal{T})^{-1}.$$

The unusual appearance of the reciprocal on the right-hand side is explained by the normalisation of $\circledast$.

Assume that F is a global field. If one restricts to Gal_F-representations $\mathcal{T}$, one can also allow F_∞/F to have some ramification over U. In particular, one can define extra Euler factors for the non-commutative L-function in ramified points. If the ramification indexes of F_∞/F in the closed point of U are prime to ℓ, these Euler factors are in most aspects well-behaved. If the inertia group of F_∞/F in a point x contains an element of infinite order, then the corresponding Euler factor is trivial. In these cases, one does not obtain any extra information. Points of finite ramification index divisible by ℓ cannot be dealt with. However, the construction of these extra non-commutative Euler factors necessitates some technical considerations. The main idea is to consider the constructible Λ-adic sheaf $\eta_* \mathcal{T}$ for $\eta\colon \operatorname{Spec} F \to U$ the inclusion of the generic point, in the spirit of the intermediate image of a perverse sheaf. The technical issues arise from the non-exactness of η_*. We will deal with these issues in Chapter 5.

As an application, we may choose $\mathcal{T}$ to be the ℓ-adic Tate module of an abelian variety over a function field F of characteristic $p \neq \ell$. The corresponding non-commutative main conjecture formulated in Corollary 5.6.1 is the direct analogue of the non-commutative Gl_2 main conjecture [**CFK⁺05**]. Finally, we will show in Chapter 6 that in the special cases $\mathcal{T} = \mathbb{Z}_\ell$ and $\mathcal{T} = \mathbb{Z}_\ell(1)$, the complexes $\mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* \mathcal{T})$ are directly related to the ℓ-adic realisations of Picard 1-motives and abstract 1-motives considered by Greither and Popescu. In particular, we identify their versions of the Iwasawa main conjecture as special instances of the type of main conjectures considered above.

1.3. Notational Conventions

All rings will be associative with identity; a module over a ring will always refer to a left unitary module. If R is a ring, R^{op} will denote the opposite ring and $R^\times$ its group of units. We will sometimes write $f \circlearrowright M$ for an endomorphism f of an object M. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ $\mathbb{C}$ have their usual meanings. For a prime number ℓ, $\mathbb{Z}_\ell$ denotes the ring of ℓ-adic integers and $\mathbb{Q}_\ell$ its fraction field. We write $\coloneqq$ do denote the definition of a symbol, reserving the symbol $=$ for expressing an identity. Isomorphisms are denoted by $\cong$, weak equivalences and quasi-isomorphisms by $\sim$. Cofibrations and quotient maps in Waldhausen categories are denoted by $\rightarrowtail$ and $\twoheadrightarrow$. Graded objects are denoted by $P^\bullet$ or $P_\bullet$, with P^n and P_n referring to the component in degree n, respectively. For any compact or discrete abelian group A we let

$$A^\vee \coloneqq \mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{R}/\mathbb{Z})$$

denote its Pontryagin dual. If G is a group and A is a G-module, then A^G denotes the invariants under G and A_G the coinvariants.

CHAPTER 2

Algebraic Preliminaries

The main purpose of this chapter is to present some algebraic and K-theoretic results that we will need for the formulation and the proof of the main conjecture. A central input to the proof of the unicity of the non-commutative L-function is Section 2.1, in which we show that the inverse limit

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]) := \varprojlim_{F'} \mathrm{SK}_1(\mathbb{Z}_\ell[\mathrm{Gal}(F'/F)])$$

of the first special K-groups of the group rings $\mathbb{Z}_\ell[\mathrm{Gal}(F'/F)]$ vanishes when F' runs through the Galois subextensions of a sufficiently large extension F_∞/F of a global field F. The results of this section are far more general than what is needed for our later applications and might be useful in other contexts as well.

In Section 2.4 we recall the essences of the K-theoretic machinery behind the formulation of the main conjecture. We briefly recall that notion of a Waldhausen category and how it may be used to compute the K-groups of a ring. We also recall the construction by Muro and Tonks of the one-type of the K-theory spectrum that gives an algebraic model for Deligne's category of virtual objects.

In Section 2.3, we explain how to identify the K-groups of a biWaldhausen category with those of its opposite category and show that this identification is compatible with localisation sequences. Then, we specialise to the case of rings and construct the identification

$$\mathrm{K}_1(R^{\mathrm{op}}) \cong \mathrm{K}_1(R)$$

of the first K-groups of a ring R and its opposite ring R^{op} on the level of Waldhausen categories.

Next, we introduce in Section 2.4 *adic rings* and discuss certain Waldhausen categories associated with them. Examples are the profinite group rings $\Lambda[[G]]$ over any adic ring Λ, with $G = H \rtimes \Gamma$ a semi-direct product of $\Gamma \cong \mathbb{Z}_\ell$ and a topologically finitely generated profinite group H which contains an open pro-ℓ-subgroup.

We are particularly interested in perfect complexes of $\Lambda[[G]]$-modules which are also perfect as complexes of $\Lambda[[H]]$-modules. If $\Lambda[[H]]$ is noetherian, then these complexes can be characterised as those perfect complexes of $\Lambda[[G]]$-modules whose cohomology is S-torsion, where S denotes Venjakob's canonical Ore set. The K-groups of the corresponding Waldhausen category may be identified with the relative K-groups $\mathrm{K}_n(\Lambda[[G]], S)$ and we may consider the long exact localisation sequence

$$\ldots \to \mathrm{K}_1(\Lambda[[G]]_S) \xrightarrow{d} \mathrm{K}_0(\Lambda[[G]], S) \to \mathrm{K}_0(\Lambda[[G]]) \to \mathrm{K}_0(\Lambda[[G]]_S).$$

These complexes will be studied in Section 2.5 and the succeeding sections of this chapter. We investigate the behaviour of the complexes under base change with complexes of bimodules in Section 2.6 and under duality in Section 2.7. Section 2.8 contains the proof of another presentation of the complexes that will turn out to be useful.

In Section 2.9 we investigate the base change properties of certain splittings of the boundary map
$$d\colon \mathrm{K}_1(\Lambda[[G]]_S) \to \mathrm{K}_0(\Lambda[[G]], S),$$
extending results from [**Bur09**] and [**Wit13b**]. With the help of these splittings we are able produce characteristic elements with good functorial properties, which we call non-commutative algebraic L-functions.

The final Section 2.10 deals with the K-theory of $\Lambda[[G]]$ in the classical case that the coefficient ring Λ is a commutative regular noetherian local ring and that G is an ℓ-adic Lie group without elements of order ℓ.

2.1. On the First Special K-group of a Profinite Group Algebra

Let ℓ be a fixed prime number. For any profinite group G, we write $\mathfrak{N}(G)$ for its lattice of open normal subgroups and $G_r \subset G$ for subset of ℓ-regular elements, i. e. the union of all q-Sylow-subgroups for all primes $q \neq \ell$. Note that
$$G_r = \varprojlim_{U \in \mathfrak{N}(G)} (G/U)_r$$
is closed subset of G. The group G acts continuously on G_r by conjugation. For any profinite G-set S we write $\mathbb{Z}_\ell[[S]]$ for the compact G-module which is freely generated by S as compact $\mathbb{Z}_\ell$-module.

We want to analyse the completed first special K-group
$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{SK}_1(\mathbb{Z}_\ell[G/U])$$
of the profinite group algebra
$$\mathbb{Z}_\ell[[G]] \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathbb{Z}_\ell[G/U].$$
Note that $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]])$ is a subgroup of the completed first K-group
$$\widehat{\mathrm{K}}_1(\mathbb{Z}_\ell[[G]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{K}_1(\mathbb{Z}_\ell[G/U]).$$
If G has an open pro-ℓ-subgroup which is topologically finitely generated, then
$$\widehat{\mathrm{K}}_1(\mathbb{Z}_\ell[[G]]) = \mathrm{K}_1(\mathbb{Z}_\ell[[G]])$$
by [**FK06**, Prop. 1.5.3]. In the case that G is a pro-ℓ ℓ-adic Lie group a thorough analysis of $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]])$ has been carried out in [**SV13**]. Note in particular that there are examples of torsion-free ℓ-adic Lie groups with non-trivial first special K-group. Some of the results of *loc. cit.* can certainly be extended to the case that G admits elements of order prime to ℓ. We will not pursue this further. Instead, we limit ourselves to the following results relevant to our application.

Recall from [**Oli88**, Thm. 10.12] that there is a canonical surjective homomorphism
$$\mathrm{H}_2(G, \mathbb{Z}_\ell[[G_r]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]]).$$
where
$$\mathrm{H}_2(G, \mathbb{Z}_\ell[[G_r]]) \coloneqq \varprojlim_{U \in \mathfrak{N}(G)} \mathrm{H}_2(G/U, \mathbb{Z}_\ell[(G/U)_r])$$
denotes the second continuous homology group of $\mathbb{Z}_\ell[[G_r]]$. We write $X(G_r) \coloneqq \mathrm{Map}(G_r, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for the Pontryagin dual of $\mathbb{Z}_\ell[[G_r]]$, such that the Pontryagin dual of $\mathrm{H}_2(G, \mathbb{Z}_\ell[[G_r]])$ is $\mathrm{H}^2(G, X(G_r))$.

LEMMA 2.1.1. *Let $G = H \rtimes \Gamma$ be a semi-direct product of a finite normal subgroup $H \subset G$ and $\Gamma \cong \mathbb{Z}_\ell$. Then $\mathrm{H}^2(G, X(G_r))$ and $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]])$ are finite.*

PROOF. Note that $X(G_r) = X(H_r)$ is of finite corank over $\mathbb{Z}_\ell$. The Hochschild-Serre spectral sequence induces an exact sequence

$$0 \to \mathrm{H}^1(\Gamma, H^1(H, X(H_r))) \to \mathrm{H}^2(G, X(H_r)) \to \mathrm{H}^0(\Gamma, \mathrm{H}^2(H, X(H_r))) \to 0$$

where both $\mathrm{H}^1(H, X(H_r))$ and $\mathrm{H}^2(H, X(H_r))$ are finite ℓ-groups. The lemma is an immediate consequence. □

We are interested in the following number theoretic situation. Assume that K is a *global field*. This means, either K is a *number field*, i. e. a finite extension of $\mathbb{Q}$ or K is a *function field*, i. e. a finite extension of the field of rational functions $\mathbb{F}_p(z)$ over the finite field $\mathbb{F}_p$ with p elements for some prime number p.

Assume further that K_∞ is a $\mathbb{Z}_\ell$-extension of K. In particular, if K is a number field, then K_∞/K is unramified in the places (including the archimedean places) of K that do not lie over ℓ. If K is a function field of characteristic p different from ℓ, then there exists only one $\mathbb{Z}_\ell$-extension of K, namely the cyclotomic field extension

$$K_\infty = K_{\mathrm{cyc}} \coloneqq \bigcup_{n\in\mathbb{N}} \mathbb{F}_{p^{\ell^n}} K,$$

which is everywhere unramified. If $\ell = p$, then K_{cyc}/K is the unique $\mathbb{Z}_\ell$-extension that is everywhere unramified, but there exist infinitely many $\mathbb{Z}_\ell$-extensions K_∞/K which are unramified outside any given non-empty set of places. There also exist $\mathbb{Z}_\ell$-extensions K_∞/K that ramify at infinitely many places [**GK88**].

Let L_∞ be a finite extension of K_∞ which is Galois over K. Write

$$\begin{aligned} G &\coloneqq \mathrm{Gal}(L_\infty/K),\\ H &\coloneqq \mathrm{Gal}(L_\infty/K_\infty),\\ \Gamma &\coloneqq \mathrm{Gal}(K_\infty/K) \end{aligned}$$

for the corresponding Galois groups. We fix a splitting $\Gamma \to G$ such that we may write G as the semi-direct product of H and Γ and let L be the fixed field of an ℓ-Sylow subgroup of G containing Γ. Write $L^{(\ell)}$ for the maximal Galois ℓ-extension of L inside a fixed separable closure $\overline{K}$ of K. Note that $L^{(\ell)} = L_\infty^{(\ell)}$ is the subfield of $\overline{K}$ fixed by the closed subgroup $\mathrm{Gal}_{L^{(\ell)}}$ generated by all q-Sylow subgroups of the absolute Galois group Gal_L for all primes $q \neq \ell$. Hence, $\mathrm{Gal}_{L^{(\ell)}} \subset \mathrm{Gal}_{L_\infty}$ is a characteristic subgroup and therefore, $L^{(\ell)}/K$ is a Galois extension. The following is an adaption of the proof of [**FK06**, Prop. 2.3.7].

PROPOSITION 2.1.2. *Set* $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(\ell)}/K)$. *Then*

$$\mathrm{H}^2(\mathcal{G}, X(\mathcal{G}_r)) = \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathcal{G}]]) = 0.$$

PROOF. Note that the projection $\mathcal{G} \to G$ induces a canonical isomorphism $X(\mathcal{G}_r) = X(H_r)$ and that $X(H_r)$ is of finite corank over $\mathbb{Z}_\ell$. We have

$$\mathrm{H}^s(\mathrm{Gal}(L^{(\ell)}/L), X(H_r)) = \mathrm{H}^s(\mathrm{Gal}_L, X(H_r))$$

for all s according to [**NSW00**, Cor. 10.1.4, Cor. 10.4.8] applied to the class of ℓ-groups and the set of all places of L. Moreover, $\mathrm{H}^2(\mathrm{Gal}_L, X(H_r)) = 0$ as a consequence of the fact that $\mathrm{H}^2(\mathrm{Gal}_F, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for any number field F [**FK06**, Prop. 2.3.7, Claim] and for any function field F [**Wit09**, Prop. 5.4].

Since $[L : K]$ is prime to ℓ, the restriction map

$$\mathrm{H}^2(\mathcal{G}, X(H_r)) \to \mathrm{H}^2(\mathrm{Gal}(L^{(\ell)}/L), X(H_r))$$

is split injective. In particular, $\mathrm{H}^2(\mathcal{G}, X(H_r)) = 0$ as claimed. □

Note that if K is a number field, L_∞/K is unramified in a real place of K, and $\ell \neq 2$, then $L^{(\ell)}/K$ is unramified in this real place as well. For the sake of completeness we also deal with the case $\ell = 2$ and consider for a set of real places Σ of K such that L_∞/K is unramified over Σ the maximal subfield $L^{(2)}_{\Sigma^c}$ of $L^{(2)}$ which is unramified over Σ. Note that $L^{(2)}_{\Sigma^c}/K$ is still Galois over K.

PROPOSITION 2.1.3. *Assume that K is a number field. Set $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(2)}_{\Sigma^c}/K)$. Then* $\mathrm{H}^2(\mathcal{G}, X(\mathcal{G}_r)) = \widehat{\mathrm{SK}}_1(\mathbb{Z}_2[[\mathcal{G}]]) = 0$.

PROOF. Let L' be the subfield fixed by the intersection of the centre of G with Γ and let $Y \coloneqq \mathrm{Map}(\mathrm{Gal}(L'/K), X(H_r))$ be the induced module. We obtain a canonical surjection $Y \to X(H_r)$ with kernel Z. For any discrete $\mathcal{G}$-module A we have

$$\mathrm{H}^3(\mathcal{G}, A) = \bigoplus_{v \in \Sigma^c_{\mathbb{R}}} \mathrm{H}^3(\mathrm{Gal}_{K_v}, A)$$

where v runs through set $\Sigma^c_{\mathbb{R}}$ of real places of K not in Σ and $\mathrm{Gal}_{K_v} = \mathbb{Z}/2\mathbb{Z}$ denotes the Galois group of the corresponding local field $K_v = \mathbb{R}$ [**NSW00**, Prop. 10.6.5]. By the proof of the ($\ell = 2$)-case in [**FK06**, Prop. 2.3.7, Claim] we have

$$\mathrm{H}^2(\mathrm{Gal}_{K_v}, X(H_r)) = 0$$

such that

$$\mathrm{H}^3(\mathcal{G}, Z) \to \mathrm{H}^3(\mathcal{G}, Y)$$

is injective and hence,

$$\mathrm{H}^2(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L'), X(H_r)) \cong \mathrm{H}^2(\mathcal{G}, Y) \to \mathrm{H}^2(\mathcal{G}, X(H_r))$$

is a surjection. Moreover, $\mathrm{Gal}_{L'}$ acts trivially on $X(H_r)$ such that it suffices to show that

$$\mathrm{H}^2(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L'), \mathbb{Q}_2/\mathbb{Z}_2) = 0.$$

By the proof of [**NSW00**, Thm. 10.6.1] we obtain an exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L')) \to \mathrm{H}^1(\mathrm{Gal}(L^{(2)}/L')) \to \bigoplus_{v \in \Sigma^c_{\mathbb{R}}(L')} \mathrm{H}^1(\mathrm{Gal}_{L'_v}) \to \mathrm{H}^2(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L')) \to \mathrm{H}^2(\mathrm{Gal}(L^{(2)}/L'))$$

where we have omitted the coefficients $\mathbb{Q}_2/\mathbb{Z}_2$ and $\Sigma^c_{\mathbb{R}}(L')$ denotes the real places of L' lying over $\Sigma^c_{\mathbb{R}}$. But

$$\mathrm{H}^2(\mathrm{Gal}(L^{(2)}/L')) = \mathrm{H}^2(\mathrm{Gal}_{L'}) = 0$$

by [**NSW00**, Cor. 10.4.8] and [**Sch79**, Satz 1.(ii)]. Moreover, L' is dense in the product of its real local fields such that for each real place v of L', we find an element a in L' which is negative with respect to v and positive with respect to all other real places. The element of $\mathrm{H}^1(\mathrm{Gal}(L^{(2)}/L'))$ corresponding via Kummer theory to a square root of a maps to the non-trivial element of $\mathrm{H}^1(\mathrm{Gal}_{L'_v}) = \mathbb{Z}/2\mathbb{Z}$ and to the trivial element for all other real places. This shows that

$$\mathrm{H}^1(\mathrm{Gal}(L^{(2)}_{\Sigma^c}/L')) \to \bigoplus_{v \in \Sigma^c_{\mathbb{R}}(L')} \mathrm{H}^1(\mathrm{Gal}_{L'_v})$$

must be surjective. □

COROLLARY 2.1.4. *Let K_∞/K be a $\mathbb{Z}_\ell$-extension of a global field K and L_∞/K_∞ be a finite extension such that L_∞/K is Galois with Galois group G. Assume further that L_∞/K is unramified in a (possibly empty) set of real places Σ of K. Then there exists a finite extension L'_∞/L_∞ such that*

(i) *$[L'_\infty : L_\infty]$ is a power of ℓ,*

(ii) *L'_∞/K is Galois with Galois group G',*
(iii) *L'_∞/K is unramified over Σ,*
(iv) *The canonical homomorphism $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G']]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]])$ is the zero map.*

In particular, L'_∞ may be chosen to be totally real if L_∞ is a totally real extension field of $\mathbb{Q}$.

Proof. With L as above, set $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(\ell)}/K)$ if $\ell \neq 2$ or K is a function field and $\mathcal{G} \coloneqq \mathrm{Gal}(L^{(2)}_{\Sigma^c}/K)$ if $\ell = 2$ and K is a number field. Further, set $\mathcal{H} \coloneqq \ker \mathcal{G} \to \mathrm{Gal}(K_\infty/K)$. According to Lemma 2.1.1, $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]])$ is finite and so, the image of

$$\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathcal{G}]]) = \varprojlim_{U \in \mathfrak{N}(\mathcal{G})} \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathcal{G}/U \cap \mathcal{H}]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]])$$

will be equal to the image of $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathcal{G}/U_0 \cap \mathcal{H}]])$ for some $U_0 \in \mathfrak{N}(\mathcal{G})$. We let L'_∞ be the fixed field of $U_0 \cap \mathcal{H}$. Then L'_∞ clearly satisfies (i), (ii), and (iii). Since $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathcal{G}]]) = 0$ by Proposition 2.1.2 and Proposition 2.1.3, it also satisfies (iv). ☐

Remark 2.1.5.

(1) If K is a number field, the extension L'_∞/K will be unramified outside a finite set of primes, but we cannot prescribe the ramification locus. However, assume L_∞/K is unramified outside the set S of places of K and that the Leopoldt conjecture holds for every finite extension F of K inside the maximal ℓ-extension $L_S^{(\ell)}$ which is unramified outside S, i. e. that
$$\mathrm{H}^2(\mathrm{Gal}(L_S^{(\ell)}/F), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0.$$
Then the same method of proof shows that we can additionally chose L'_∞ to lie in $L_S^{(\ell)}$.
(2) Assume that K is a function field and L_∞/K is unramified outside a non-empty set S of places of K. Then [**NSW00**, Thm. 8.3.17] implies
$$\mathrm{H}^2(\mathrm{Gal}(L_S^{(\ell)}/F), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$$
for every finite extension F of K inside the maximal ℓ-extension $L_S^{(\ell)}$ which is unramified outside S such that we can always chose L'_∞ to lie in $L_S^{(\ell)}$.

2.2. Waldhausen K-Theory

Classically, the first K-group of a ring R may be described as the quotient of the group
$$\mathrm{Gl}_\infty(R) \coloneqq \varinjlim_{d \in \mathbb{N}} \mathrm{Gl}_d(R)$$
by its commutator subgroup, but for the formulation of the main conjecture, it is more convenient to follow the constructions of higher K-theory. Among the many roads to higher K-theory, Waldhausen's S-construction [**Wal85**] turns out to be particularly well-suited for our purposes.

Recall that a *Waldhausen category* $\mathbf{W}$ is a category with a zero object 0 and two distinguished classes of morphisms, called *cofibrations* and *weak equivalences*, closed under composition and subject to the following set of axioms.

(1) Any isomorphism in $\mathbf{W}$ is both a cofibration and a weak equivalence.
(2) For every object A in $\mathbf{W}$, the unique map $0 \to A$ is a cofibration.
(3) If $A \to B$ is a cofibration and $A \to C$ is a map in $\mathbf{W}$, then the pushout $B \cup_A C$ exists and the canonical map $C \to B \cup_A C$ is a cofibration.

(4) If in the commutative diagram

$$\begin{tikzcd} B \arrow[d] & A \arrow[l] \arrow[r, "f"'] \arrow[d] & C \arrow[d] \\ B' & A' \arrow[l] \arrow[r, "g"'] & C' \end{tikzcd}$$

the morphisms f and g are cofibrations and the downwards pointing arrows are weak equivalences, then the natural map $B \cup_A C \to B' \cup_{A'} C'$ is a weak equivalence.

We usually denote cofibrations from A to B by $A \rightarrowtail B$ and weak equivalences by $A \xrightarrow{\sim} B$. If $C \cong B \cup_A 0$ is a cokernel of the cofibration $A \rightarrowtail B$, we denote the natural quotient map from B to C by $B \twoheadrightarrow C$. The sequence

$$A \rightarrowtail B \twoheadrightarrow C$$

is called *exact sequence* or *cofibre sequence*. A functor $F\colon \mathbf{W} \to \mathbf{W}'$ between Waldhausen categories is called *Waldhausen exact* if it preserves cofibrations, weak equivalences and pushouts along cofibrations.

For example, every exact category $\mathbf{E}$ in the sense of Quillen may be equipped with the structure of a Waldhausen category by choosing the cofibrations to be the injections that may be completed to admissible exact sequences and the weak equivalences to be the isomorphisms.

Waldhausen's S-construction then assigns to each Waldhausen category $\mathbf{W}$ a bisimplicial set $N.w\mathbf{S}.\mathbf{W}$. The n-th K-group $\mathrm{K}_n(\mathbf{W})$ of $\mathbf{W}$ is by definition the $n+1$-th homotopy group of the topological realisation of $N.w\mathbf{S}.\mathbf{W}$.

To construct the K-groups of R, one can simply apply the S-construction to the exact category of finitely generated, projective modules over R, but the true beauty of Waldhausen's construction is that we can choose among a multitude of different Waldhausen categories that all give rise to the same K-groups. Below, we will study a number of different Waldhausen categories whose K-theory agrees with that of R.

We recall that for any ring R, a complex $M^\bullet$ of R-modules is called *DG-flat* if every module M^n is flat and for every acyclic complex $N^\bullet$ of right R-modules, the total complex $(N \otimes_R M)^\bullet$ is acyclic. In particular, any bounded above complex of flat R-modules is DG-flat. The notion of DG-flatness can be used to define derived tensor products without this boundedness condition. Unbounded complexes will turn up naturally in our constructions. As usual, the complex $M^\bullet$ is called *strictly perfect* if M^n is finitely generated and projective for all n and $M^n = 0$ for almost all n. A complex of R-modules is a *perfect* complex if it is quasi-isomorphic to a strictly perfect complex.

Definition 2.2.1. For any ring R, we write $\mathbf{SP}(R)$ for the Waldhausen category of strictly perfect complexes, $\mathbf{PDG}(R)$ for the category of perfect DG-flat complexes, and $\mathbf{P}(R)$ for the Waldhausen category of perfect complexes of left R-modules. In both categories, the weak equivalences are given by quasi-isomorphisms. The cofibrations in $\mathbf{P}(R)$ are all injections, the cofibrations in $\mathbf{SP}(R)$ and $\mathbf{PDG}(R)$ are the injections with strictly perfect and DG-flat perfect cokernel, respectively.

It is a standard consequence of the Waldhausen approximation theorem [**TT90**, 1.9.1] that the inclusion functors $\mathbf{SP}(R) \to \mathbf{PDG}(R) \to \mathbf{P}(R)$ induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}(R)) \cong \mathrm{K}_n(\mathbf{PDG}(R)) \cong \mathrm{K}_n(\mathbf{P}(R))$$

between the Waldhausen K-groups of these categories. Moreover, they agree with the Quillen K-groups $\mathrm{K}_n(R)$ of R by the Gillet-Waldhausen theorem [**TT90**, Thm. 1.11.2, 1.11.7].

If S is another ring and $M^\bullet$ is a complex of S-R-bimodules which is strictly perfect as complex of S-modules, then the tensor product with $M^\bullet$ is a Waldhausen exact functor from $\mathbf{SP}(R)$ to $\mathbf{SP}(S)$ and from $\mathbf{PDG}(R)$ to $\mathbf{PDG}(S)$. Hence, it induces homomorphisms $\mathrm{K}_n(R) \to \mathrm{K}_n(S)$. Note, however, that the tensor product with $M^\bullet$ does not give a Waldhausen exact functor from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, as it does not preserve weak equivalences nor cofibrations. In the context of homological algebra, this problem can be solved by passing to the derived category, but there is no general recipe how to construct the K-groups of R on the basis of the derived category alone. As a consequence, in order to view certain homomorphisms between K-groups as being induced from a Waldhausen exact functor, one has to make a suitable choice of the underlying Waldhausen categories.

Thanks to a result of Muro and Tonks [**MT08**], the groups $\mathrm{K}_0(\mathbf{W})$ and $\mathrm{K}_1(\mathbf{W})$ of any Waldhausen category $\mathbf{W}$ can be described as the cokernel and kernel of a homomorphism

$$\partial\colon \mathcal{D}_1(\mathbf{W}) \to \mathcal{D}_0(\mathbf{W}) \tag{2.2.1}$$

between two nil-2-groups (i. e. $[a,[b,c]] = 1$ for any three group elements a,b,c) that are given by explicit generators and relations in terms of the structure of the underlying Waldhausen category. As additional structure, there exists a pairing

$$\mathcal{D}_0(\mathbf{W}) \times \mathcal{D}_0(\mathbf{W}) \to \mathcal{D}_1(\mathbf{W}), \qquad (A,B) \mapsto \langle A,B\rangle$$

satisfying

$$\begin{aligned}
\partial\,\langle A,B\rangle &= B^{-1}A^{-1}BA,\\
\langle \partial a,\partial b\rangle &= b^{-1}a^{-1}ba,\\
\langle A,B\rangle\,\langle B,A\rangle &= 1,\\
\langle A,BC\rangle &= \langle A,B\rangle\,\langle A,C\rangle\,.
\end{aligned}$$

In other words, $\mathcal{D}_\bullet(\mathbf{W})$ is a stable quadratic module in the sense of [**Bau91**]. In particular, $X \in \mathcal{D}_0(\mathbf{W})$ operates from the right on $a \in \mathcal{D}_1(\mathbf{W})$ via

$$a^X \coloneqq a\,\langle X,\partial a\rangle\,.$$

More explicitly, $\mathcal{D}_0(\mathbf{W})$ is the free nil-2-group generated by the objects of $\mathbf{W}$ different from the zero object, while $\mathcal{D}_1(\mathbf{W})$ is generated by all weak equivalences and exact sequences in $\mathbf{W}$ subject to the following list of relations:

(R1) $\partial[\alpha] = [B]^{-1}[A]$ for a weak equivalence $\alpha\colon A \xrightarrow{\sim} B$,
(R2) $\partial[\Delta] = [B]^{-1}[C][A]$ for an exact sequence $\Delta\colon A \rightarrowtail B \twoheadrightarrow C$.
(R3) $\langle [A],[B]\rangle = [B \rightarrowtail A\oplus B \twoheadrightarrow A]^{-1}[A \rightarrowtail A\oplus B \twoheadrightarrow B]$ for any pair of objects A,B.
(R4) $[0 \rightarrowtail 0 \twoheadrightarrow 0] = 1_{\mathcal{D}_1}$,
(R5) $[\beta\alpha] = [\beta][\alpha]$ for weak equivalences $\alpha\colon A \xrightarrow{\sim} B$, $\beta\colon B \xrightarrow{\sim} C$,
(R6) $[\Delta'][\alpha][\gamma]^{[A]} = [\beta][\Delta]$ for any commutative diagram

$$\begin{tikzcd}
\Delta\colon & A \arrow[r, tail] \arrow[d, "\alpha"', "\sim"] & B \arrow[r, two heads] \arrow[d, "\beta"', "\sim"] & C \arrow[d, "\gamma"', "\sim"] \\
\Delta'\colon & A' \arrow[r, tail] & B' \arrow[r, two heads] & C'
\end{tikzcd}$$

(R7) $[\Gamma_1][\Delta_1] = [\Delta_2][\Gamma_2]^{[A]}$ for any commutative diagram

$$\begin{tikzcd} & & \Gamma_1\colon & \Gamma_2\colon \\ \Delta_1\colon & A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & C \arrow[d, tail] \\ \Delta_2\colon & A \arrow[r, tail] \arrow[d, two heads] & D \arrow[r, two heads] \arrow[d, two heads] & E \arrow[d, two heads] \\ & 0 \arrow[r, tail] & F \arrow[r, equal] & F \end{tikzcd}$$

[**MT07**, Def. 1.2].

In particular, $\mathrm{K}_0(\mathbf{W})$ is the abelian group generated by the symbols $[P]$ with P in $\mathbf{W}$ modulo the relations

$$[P] = [Q] \qquad \text{if } P \text{ and } Q \text{ are weakly equivalent,}$$
$$[P_2] = [P_1] + [P_3] \qquad \text{if } P_1 \rightarrowtail P_2 \twoheadrightarrow P_3 \text{ is an exact sequence.}$$

If $f\colon P \xrightarrow{\sim} P$ is an endomorphism which is a weak equivalence in $\mathbf{W}$, we can assign to it a class $[f]$ in $\mathrm{K}_1(\mathbf{W})$. The relations that are satisfied by these classes can be read off from the above relations for $\mathcal{D}_1\mathbf{W}$. By the classical definition of the first K-group as factor group of the general linear group it is clear that these classes generate $\mathrm{K}_1(\mathbf{W})$ in the case that $\mathbf{W}$ is one of $\mathbf{SP}(R)$, $\mathbf{PDG}(R)$ and $\mathbf{P}(R)$ for a ring R.

REMARK 2.2.2. Some authors prefer the theory of determinant functors and Deligne's category of virtual objects [**Del87**] as an alternative model for the 1-type of the K-theory spectrum. We refer to [**MTW15**] for the precise connection of the two approaches.

2.3. Duality on the Level of K-Groups

Assume that $\mathbf{W}$ is a biWaldhausen category in the sense of [**TT90**, Def. 1.2.4]: $\mathbf{W}$ is a Waldhausen category, the class of quotient maps is closed under composition, the opposite category $\mathbf{W}^{\mathrm{op}}$ is a Waldhausen category with the same class of weak equivalences and with the classes of quotient maps and cofibrations mutually exchanged, and product and coproduct of any two objects in $\mathbf{W}$ are canonically isomorphic.

In particular, the opposite category $\mathbf{W}^{\mathrm{op}}$ is a biWaldhausen category as well and there are natural isomorphisms

$$I\colon \mathrm{K}_n(\mathbf{W}) \cong \mathrm{K}_n(\mathbf{W}^{\mathrm{op}}), \tag{2.3.1}$$

simply because the topological realisations of the bisimplicial sets $N.w\mathbf{S}.\mathbf{W}$ and $N.w\mathbf{S}.\mathbf{W}^{\mathrm{op}}$ resulting from Waldhausen's S-construction agree [**TT90**, §1.5.5]. However, the obvious identifications

$$N_m w\mathbf{S}_n\mathbf{W} \cong N_m w\mathbf{S}_n\mathbf{W}^{\mathrm{op}}$$

respect the boundary and degeneracy maps only up to reordering, so that we do not obtain an isomorphism of the bisimplicial sets themselves.

In order to understand the isomorphism (2.3.1) in terms of the presentation of $\mathrm{K}_1(\mathbf{W})$ given by (2.2.1), we will construct a canonical isomorphism

$$I\colon \mathcal{D}_\bullet(\mathbf{W}) \to \mathcal{D}_\bullet(\mathbf{W}^{\mathrm{op}}).$$

For any morphism $\alpha\colon A \to B$ in $\mathbf{W}$, write $\alpha^{\mathrm{op}}\colon B \to A$ for the corresponding morphism in the opposite category $\mathbf{W}^{\mathrm{op}}$. Further, note that by the definition of biWaldhausen categories, if $A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in $\mathbf{W}$, then the dual sequence $C \rightarrowtail B \twoheadrightarrow A$ is exact in $\mathbf{W}^{\mathrm{op}}$. We then set

$$\begin{aligned} I([A]) &= [A] && \text{for objects } A \text{ in } \mathbf{W}, \\ I([f\colon A \xrightarrow{\sim} B]) &= [f^{\mathrm{op}}\colon B \xrightarrow{\sim} A]^{-1} && \text{for weak equivalences } f, \\ I([A \rightarrowtail B \twoheadrightarrow C]) &= [C \rightarrowtail B \twoheadrightarrow A]\langle [A],[C]\rangle && \text{for exact sequences } A \rightarrowtail B \twoheadrightarrow C. \end{aligned}$$

PROPOSITION 2.3.1. *For any biWaldhausen category* $\mathbf{W}$*, the above assignment defines an isomorphism of stable quadratic modules*

$$I\colon \mathcal{D}_\bullet(\mathbf{W}) \to \mathcal{D}_\bullet(\mathbf{W}^{\mathrm{op}}).$$

PROOF. It is sufficient to check that I respects the relations (R1)–(R7) in the definition of $\mathcal{D}_\bullet(\mathbf{W})$. This is a straight-forward, but tedious exercise. □

Next, we investigate in how far I respects the boundary homomorphism of localisation sequences. For this, we consider the same situation as in [**Wit14**, Appendix], but with all Waldhausen categories replaced by biWaldhausen categories. Assume that $\mathbf{wW}$ is a biWaldhausen category with weak equivalences $\mathbf{w}$ that is saturated and extensional in the sense of [**TT90**, Def. 1.2.5, 1.2.6]. Let $\mathbf{vW}$ be a the same category with the same notion of fibrations and cofibrations, but with a coarser notion of weak equivalences $\mathbf{v} \subset \mathbf{w}$ and let $\mathbf{vW}^{\mathbf{w}}$ denote the full biWaldhausen subcategory of $\mathbf{vW}$ consisting of those objects which are weakly equivalent to the zero object in $\mathbf{wW}$. We assume that Cyl is a cylinder functor in the sense of [**Wit14**, Def. A.1] for both $\mathbf{wW}$ and $\mathbf{vW}$ and that it satisfies the cylinder axiom for $\mathbf{wW}$. We will write Cone and Σ for the associated cone and shift functors, i. e.

$$\begin{aligned} \operatorname{Cone}(\alpha) &\coloneqq \operatorname{Cyl}(\alpha)/A && \text{for any morphism } \alpha\colon A \to B, \\ \Sigma A &\coloneqq \operatorname{Cone}(A \to 0) && \text{for any object } A. \end{aligned}$$

Further, we assume that CoCyl is a cocylinder functor for both $\mathbf{wW}$ and $\mathbf{vW}$ in the sense that the opposite functor $\mathrm{CoCyl}^{\mathrm{op}}$ is a cylinder functor for $\mathbf{wW}^{\mathrm{op}}$ and $\mathbf{vW}^{\mathrm{op}}$. Again, we assume that $\mathrm{CoCyl}^{\mathrm{op}}$ satisfies the cylinder axiom for $\mathbf{wW}^{\mathrm{op}}$. We will write CoCone and CoΣ for the associated cocone and coshift functors.

Recall from [**Wit14**, Thm. A.5] that the assignment

(2.3.2)
$$\begin{aligned} d(\Delta) &= 0 && \text{for every exact sequence } \Delta \text{ in } \mathbf{wW}, \\ d(\alpha) &= -[\operatorname{Cone}(\alpha)] + [\operatorname{Cone}(\mathrm{id}_A)] && \text{for every weak equivalence } \alpha\colon A \xrightarrow{\sim} A' \text{ in } \mathbf{wW} \end{aligned}$$

defines a homomorphism $d\colon \mathcal{D}_1(\mathbf{wW}) \to K_0(\mathbf{vW}^{\mathbf{w}})$ such that the sequence

$$\mathrm{K}_1(\mathbf{vW}) \to \mathrm{K}_1(\mathbf{wW}) \xrightarrow{d} \mathrm{K}_0(\mathbf{vW}^{\mathbf{w}}) \to \mathrm{K}_0(\mathbf{vW}) \to \mathrm{K}_0(\mathbf{wW}) \to 0$$

is exact.

LEMMA 2.3.2. *For every weak equivalence* $\alpha\colon A \to B$ *in* $\mathbf{wW}$*,*

$$d(\alpha) = -[\operatorname{CoCone}(\mathrm{id}_B)] + [\operatorname{CoCone}(\alpha)]$$

in $\mathrm{K}_0(\mathbf{vW}^{\mathbf{w}})$.

PROOF. We first assume that A and B are objects of $\mathbf{vW}^{\mathbf{w}}$. Then

$$\begin{aligned} B &\rightarrowtail \operatorname{Cone}(\alpha) \twoheadrightarrow \Sigma A, \\ A &\rightarrowtail \operatorname{Cone}(\mathrm{id}_A) \twoheadrightarrow \Sigma A, \end{aligned}$$

are exact sequences in $\mathbf{vW}^{\mathbf{W}}$. Hence,

$$d(\alpha) = -[B] - [\Sigma A] + [\Sigma A] + [A] = -[B] + [A] \tag{2.3.3}$$

in $\mathrm{K}_0(\mathbf{vW}^{\mathbf{w}})$.

For a general weak equivalence $\alpha\colon A \to B$ in $\mathbf{wW}$, the natural morphism

$$\mathrm{Cone}(\alpha) \to 0$$

is a weak equivalence in $\mathbf{wW}$ by the cylinder axiom. The commutative square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "\alpha"', "\sim"] \\ A \arrow[r, "\sim", "\alpha"'] & B \end{tikzcd}$$

induces by the functoriality of the cone and shift functor a commutative diagram with exact rows

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\alpha"', "\sim"] & \mathrm{Cone}(\mathrm{id}_A) \arrow[r, two heads] \arrow[d, "\alpha_*"', "\sim"] & \Sigma A \arrow[d, equal] \\ B \arrow[r, tail] & \mathrm{Cone}(\alpha) \arrow[r, two heads] & \Sigma A \end{tikzcd}$$

where all downward pointing arrows are weak equivalences in $\mathbf{wW}$. Dually, we also obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Co}\Sigma B \arrow[r, tail] \arrow[d, equal] & \mathrm{CoCone}(\alpha) \arrow[r, two heads] \arrow[d, "\alpha^*"', "\sim"] & A \arrow[d, "\alpha"', "\sim"] \\ \mathrm{Co}\Sigma B \arrow[r, tail] & \mathrm{CoCone}(\mathrm{id}_B) \arrow[r, two heads] & B \end{tikzcd}$$

where all downward pointing arrows are weak equivalences in $\mathbf{wW}$.

From (R6) and (2.3.3) we conclude

$$\begin{aligned} -[\mathrm{Cone}(\alpha)] + [\mathrm{Cone}(\mathrm{id}_A)] &= d(\alpha_*) = d(\alpha) \\ &= d(\alpha^*) = -[\mathrm{CoCone}(\mathrm{id}_B)] + [\mathrm{CoCone}(\alpha)] \end{aligned}$$

as desired. □

REMARK 2.3.3. By basically the same argument, one also sees that d is independent of the choice of the particular cylinder functor.

PROPOSITION 2.3.4. *With the notation as above, the diagram*

$$\begin{tikzcd} \mathcal{D}_1(wW) \arrow[r, "I"] \arrow[d, "d"] & \mathcal{D}_1(wW^{\mathrm{op}}) \arrow[d, "d"] \\ \mathrm{K}_0(vW^w) \arrow[r, "I"] & \mathrm{K}_0((vW^w)^{\mathrm{op}}) \end{tikzcd}$$

commutes.

PROOF. This is a direct consequence of the definition of I and Lemma 2.3.2. □

If R is any ring and $P^\bullet$ is a strictly perfect complex of left R-modules, then

$$(P^\bullet)^{*_R} \coloneqq \mathrm{Hom}_R(P^{-\bullet}, R)$$

is a strictly perfect complex of left modules over the opposite ring R^{op} and

$$\mathbf{SP}(R)^{\mathrm{op}} \to \mathbf{SP}(R^{\mathrm{op}}) \qquad P^\bullet \mapsto (P^\bullet)^{*_R}$$

is a Waldhausen exact equivalence of categories. We omit the R from $*_R$ if it is clear from the context. By composing with the homomorphisms I, we obtain isomorphisms

$$\mathrm{K}_n(R) \cong \mathrm{K}_n(R^{\mathrm{op}}).$$

Note that the isomorphism $\mathrm{K}_1(R) \to \mathrm{K}_1(R^{\mathrm{op}})$ corresponds to the isomorphism induced by the group isomorphism

$$\mathrm{Gl}_\infty(R) \to \mathrm{Gl}_\infty(R^{\mathrm{op}}), \qquad A \mapsto (A^t)^{-1}$$

that maps a matrix A to the inverse of its transposed matrix.

If S is a second ring and $M^\bullet$ a complex of R-S-bimodules which is strictly perfect as complex of R-modules, then $(M^\bullet)^{*_R}$ is a complex of R^{op}-S^{op}-bimodules which is strictly perfect as complex of R^{op}-modules and there exists for any complex $P^\bullet$ in $\mathbf{SP}(S)$ a canonical isomorphism

$$(M^{*_R} \otimes_{S^{\mathrm{op}}} P^{*_S})^\bullet \cong ((M \otimes_S P)^\bullet)^{*_R}. \tag{2.3.4}$$

Hence, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_n(S) \arrow[r, "\cong"'] \arrow[d, "M^\bullet"'] & \mathrm{K}_n(S^{\mathrm{op}}) \arrow[d, "(M^\bullet)^{*_R}"] \\
\mathrm{K}_n(R) \arrow[r, "\cong"'] & \mathrm{K}_n(R^{\mathrm{op}})
\end{tikzcd}$$

2.4. K-Theory of Adic Rings

We are mainly interested in the first K-group of a certain class of rings introduced by Fukaya and Kato in [**FK06**]. It consists of those rings Λ such that for each $n \geq 1$ the n-th power of the Jacobson radical $\mathrm{Jac}(\Lambda)^n$ is of finite index in Λ and

$$\Lambda = \varprojlim_{n\geq 1} \Lambda/\mathrm{Jac}(\Lambda)^n.$$

In extension of the definition for commutative rings [**Gro60**, Ch. 0, Def. 7.1.9], these rings should be called *compact adic rings*. We will call these rings *adic rings* for short, as in [**Wit14**]. We do not intend to insinuate any relation to Huber's more recent concept of adic spaces with this denomination. By definition, an adic ring Λ carries a natural profinite topology. We will write $\mathfrak{I}_\Lambda$ for the set of open two-sided ideals of Λ, partially ordered by inclusion.

We mainly work with the following Waldhausen category taken from [**Wit14**]. Its main advantage is that it works well with our later definition of adic sheaves in Section 3.1 and that it allows a direct construction of most of the relevant Waldhausen exact functors.

DEFINITION 2.4.1. Let Λ be an adic ring. We denote by $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ are inverse system $(P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ satisfying the following conditions:

(1) for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ is a DG-flat perfect complex of Λ/I-modules,
(2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism of the system

$$\varphi_{IJ} : P_I^\bullet \to P_J^\bullet$$

induces an isomorphism of complexes

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

A morphism of inverse systems $(f_I : P_I^\bullet \to Q_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ is a weak equivalence if every f_I is a quasi-isomorphism. It is a cofibration if every f_I is injective and the system $(\operatorname{coker} f_I)$ is in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

DEFINITION 2.4.2. Let Λ' be another adic ring and $M^\bullet$ a complex of Λ'-Λ-bimodules which is strictly perfect as complex of Λ'-modules. We define $\Psi_{M^\bullet}$ to be the following Waldhausen exact functor

$$\Psi_{M^\bullet} : \mathbf{PDG}^{\mathrm{cont}}(\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'), \qquad P^\bullet \to (\varprojlim_{J\in\mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P_J)^\bullet)_{I\in\mathfrak{I}_{\Lambda'}}.$$

If $P^\bullet$ is a strictly perfect complex of Λ-modules, we may identify it with the system

$$(\Lambda/I \otimes_\Lambda P^\bullet)_{I\in\mathfrak{I}_\Lambda}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. By [**Wit14**, Prop. 3.7], the corresponding Waldhausen exact functor

$$\mathbf{SP}(\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$

induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}(\Lambda)) \cong \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$$

between the K-groups of the Waldhausen categories. Hence, $\mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(\Lambda))$ also coincide with the Quillen K-groups of the adic ring Λ and the homomorphism

$$\Psi_{M^\bullet}\colon \mathrm{K}_n(\Lambda) \to \mathrm{K}_n(\Lambda')$$

induced by the Waldhausen exact functor $\Psi_{M^\bullet}$ coincides with the homomorphism induced by

$$\mathbf{SP}(\Lambda) \to \mathbf{SP}(\Lambda'), \qquad P^\bullet \mapsto (M \otimes_\Lambda P)^\bullet.$$

The essential point in this observation is that $\mathfrak{I}_\Lambda$ is a countable set and that all the transition maps φ_{IJ} are surjective such that passing to the projective limit

$$\varprojlim_{I\in\mathfrak{I}_\Lambda} P_I^\bullet$$

is a Waldhausen exact functor from $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ to the Waldhausen category $\mathbf{P}(\Lambda)$ of perfect complexes of Λ-modules. We write

$$\mathrm{H}^s((P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}) \coloneqq \mathrm{H}^s(\varprojlim_{I\in\mathfrak{I}_\Lambda} P_I^\bullet)$$

for its cohomology groups and note that

$$\mathrm{H}^s((P_I^\bullet)_{I\in\mathfrak{I}_\Lambda}) = \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{H}^s(P_I^\bullet)$$

[**Wit08**, Prop. 5.2.3].

2.5. S-Torsion Complexes

Note that for any adic $\mathbb{Z}_\ell$-algebra Λ and any profinite group G such that G has an open pro-ℓ-subgroup which is topologically finitely generated, the profinite group algebra $\Lambda[[G]]$ is again an adic ring [**Wit14**, Prop. 3.2]. Assume further that $G = H \rtimes \Gamma$ is the semi-direct product of a closed normal subgroup H which is itself topologically finitely generated and a subgroup Γ which is isomorphic to $\mathbb{Z}_\ell$. We set

(2.5.1) $\quad S \coloneqq S_{\Lambda[[G]]} \coloneqq$

$$\{f \in \Lambda[[G]] \mid \Lambda[[G]] \xrightarrow{\cdot f} \Lambda[[G]] \text{ is perfect as complex of } \Lambda[[H]]\text{-modules}\}$$

and call it *Venjakob's canonical Ore set.* We may generalise the results of [**CFK⁺05**, §2] as follows.

Lemma 2.5.1. *Let*

$$P^\bullet\colon P^{-1} \xrightarrow{\partial} P^0$$

be a complex of length 2 *in* $\mathbf{SP}(\Lambda[[G]])$. *Then the following are equivalent:*

(1) $P^\bullet$ *is perfect as a complex of* $\Lambda[[H]]$*-modules.*
(2) P^{-1} *and* P^0 *are isomorphic as* $\Lambda[[G]]$*-modules and* $\mathrm{H}^0(P^\bullet)$ *is finitely generated as* $\Lambda[[H]]$*-module.*
(3) $\mathrm{H}^{-1}(P^\bullet) = 0$ *and* $\mathrm{H}^0(P^\bullet)$ *is finitely generated as* $\Lambda[[H]]$*-module.*

(4) $\mathrm{H}^{-1}(P^\bullet) = 0$ *and* $\mathrm{H}^0(P^\bullet)$ *is finitely generated and projective as* $\Lambda[[H]]$-*module.*

PROOF. Clearly, (4) implies (1).

We prove that (1) implies (2). Assume that $P^\bullet$ is perfect as complex of $\Lambda[[H]]$-modules. Then the class of $P^\bullet$ is trivial in $\mathrm{K}_0(\Lambda[[G]])$ by [**Wit13b**, Cor. 3.3]. As $\Lambda[[G]]$ is compact and semi-local, $\mathrm{K}_0(\Lambda[[G]])$ is the free abelian group over the isomorphism classes of indecomposable, projective $\Lambda[[G]]$-submodules of $\Lambda[[G]]$. Hence, P^{-1} and P^0 must be isomorphic. Moreover, as $P^\bullet$ is quasi-isomorphic to a strictly perfect complex of $\Lambda[[H]]$-modules, the highest non-vanishing cohomology group of $P^\bullet$ is a finitely presented $\Lambda[[H]]$-module.

We prove that (2) implies (3). It is sufficient to show that

$$\mathrm{H}^{-1}(\Lambda/I[[G/U]] \otimes_{\Lambda[[G]]} P^\bullet) = 0$$

for every open two-sided ideal I of Λ and every open subgroup U of H that is normal in G. Hence, we may assume that Λ and H are finite. Then ∂ is a homomorphism of the torsion $\mathbb{Z}_\ell[[\Gamma]]$-modules P^{-1} and P^0. As the two modules are isomorphic over $\mathbb{Z}_\ell[[\Gamma]]$ and $\operatorname{coker}\partial$ is finite, ∂ must be a pseudo-isomorphism. Hence, $\ker\partial$ ist finite, as well. But P^{-1} is finitely generated and projective as $\Lambda[[\Gamma]]$-module and therefore, it has no finite $\Lambda[[\Gamma]]$-submodules. We conclude that ∂ is injective.

We prove that (3) implies (4). Note that

$$\begin{aligned}\Lambda/I[[H/U]] \otimes_{\Lambda[[H]]} \mathrm{H}^0(P^\bullet) &\cong \Lambda/I[[G/U]] \otimes_{\Lambda[[G]]} \mathrm{H}^0(P^\bullet)\\ \cong \mathrm{H}^0(\Lambda/I[[G/U]] \otimes_{\Lambda[[G]]} P^\bullet) &\cong \mathrm{H}^0(\Lambda/I[[H/U]] \otimes_{\Lambda[[H]]} P^\bullet)\end{aligned}$$

for any $I \in \mathfrak{I}_\Lambda$ and any open subgroup $U \subset H$ which is normal in G. We conclude that $\mathrm{H}^0(P^\bullet)$ is finitely generated and projective as $\Lambda[[H]]$-module if and only if $\mathrm{H}^0(\Lambda/I[[H/U]] \otimes_{\Lambda[[H]]} P^\bullet)$ is finitely generated and projective as $\Lambda/I[[H/U]]$-module for every I and U. Hence, one may reduce to the case that Λ and H are finite. By replacing G by an appropriate open subgroup of G containing H, we may assume that Γ is central in G, such that we may identify $\Lambda[[G]]$ with the power series ring $\Lambda[[H]][[t]]$ over $\Lambda[[H]]$ in one indeterminate t. For any finite right $\Lambda[[H]]$-module N, the $\mathbb{Z}_\ell[[t]]$-module $N \otimes_{\Lambda[[H]]} P^{-1}$ cannot contain non-trivial finite $\mathbb{Z}_\ell[[t]]$-submodules. Moreover, P^{-1} and P^0 are flat $\Lambda[[H]]$-modules such that $P^\bullet$ is a flat resolution of $\mathrm{H}^0(P^\bullet)$ as $\Lambda[[H]]$-module. Hence, we have

$$\operatorname{Tor}_i^{\Lambda[[H]]}(N, \mathrm{H}^0(P^\bullet)) = 0$$

for $i > 1$ and

$$\operatorname{Tor}_1^{\Lambda[[H]]}(N, \mathrm{H}^0(P^\bullet)) \subset N \otimes_{\Lambda[[H]]} P^0$$

is a finite $\mathbb{Z}_\ell[[t]]$-submodule. Therefore,

$$\operatorname{Tor}_1^{\Lambda[[H]]}(N, \mathrm{H}^0(P^\bullet)) = 0$$

and $\mathrm{H}^0(P^\bullet)$ is finite and projective. □

LEMMA 2.5.2. *If* Λ *and* H *are finite and* $\gamma \in \Gamma$ *is a topological generator of* Γ, *then*

$$T \coloneqq \{(\gamma - 1)^n \mid n \in \mathbb{N}\}$$

is a left and right denominator set in $\Lambda[[G]]$ *consisting of left and right non-zero divisors in the sense of* [**GW04**, Ch. 10] *such that the left and right localisation*

$$\Lambda[[G]]_T.$$

exists. Moreover, S *is equal to the set of elements of* $\Lambda[[G]]$ *that become units in* $\Lambda[[G]]_T$. *In particular,* S *is also a left and right denominator set and*

$$\Lambda[[G]]_S = \Lambda[[G]]_T.$$

PROOF. Set $t = \gamma - 1$. Viewing $\Lambda[[G]]$ as a skew power series ring over $\Lambda[[H]]$ in t, it is clear that left and right multiplication with t^n on $\Lambda[[G]]$ is injective with finite cokernel.

According to Lemma 2.5.1 we have $s \in S$ if and only if $\Lambda[[G]]/\Lambda[[G]]s$ is finite. In particular, we have $T \subset S$. Considering $\Lambda[[G]]/\Lambda[[G]]s$ as a finite $\mathbb{Z}_\ell[[t]]$-module we see that it is annihilated by a power of t. We conclude that there exists an integer $n \geq 0$ such that for any $a \in \Lambda[[G]]$ there exists a $b \in \Lambda[[G]]$ such that

$$t^n a = bs.$$

Applying this to elements of $T \subset S$, we see that T and S are left denominator set consisting of left and right non-zero divisors such that all elements of S are units in $\Lambda[[G]]_T = \Lambda[[G]]_S$.

Applying the same arguments to $s \in \Lambda[[G]]$ with $\Lambda[[G]]/s\Lambda[[G]]$ finite, we see that T is also a right denominator set.

Assume that $s \in \Lambda[[G]]$ becomes a unit in $\Lambda[[G]]_T$. Then kernel and cokernel of

$$\Lambda[[G]] \xrightarrow{\cdot s} \Lambda[[G]]$$

are annihilated by powers of t. Considering $\Lambda[[G]]$ as a finitely generated $\mathbb{Z}_\ell[[t]]$-module annihilated by a power of ℓ, we conclude that the cokernel is finite, which implies that $s \in S$. Since T is a right denominator set, the same is then true for

$$S = \Lambda[[G]] \cap \Lambda[[G]]_T^\times.$$

□

LEMMA 2.5.3. *Assume that* $\Lambda[[H]]$ *is noetherian. Then:*

(1) $S = \{f \in \Lambda[[G]] \mid \Lambda[[G]]/\Lambda[[G]]f$ *is a f. g. left* $\Lambda[[H]]$*-module*$\}$.
(2) $S = \{f \in \Lambda[[G]] \mid \Lambda[[G]]/f\Lambda[[G]]$ *is a f. g. right* $\Lambda[[H]]$*-module*$\}$.
(3) S *is a left and right denominator set consisting of left and right non-zero divisors.*
(4) *A perfect complex of left* $\Lambda[[G]]$*-modules is perfect as complex of* $\Lambda[[H]]$*-modules if and only its cohomology groups are* S*-torsion.*

PROOF. Lemma 2.5.1 implies that the elements of S are right non-zero divisors and that (1) holds. Under the assumption that $\Lambda[[H]]$ is noetherian, we know by [**Wit13b**, Cor. 2.21] that S is a left denominator set. Assertion (4) follows from [**Wit13b**, Thm. 2.18]. Write $(\Lambda[[G]])^{\mathrm{op}}$ and Λ^{op} for the opposite rings of $\Lambda[[G]]$ and Λ, respectively. Consider the ring isomorphism

$$\sharp\colon (\Lambda[[G]])^{\mathrm{op}} \to \Lambda^{\mathrm{op}}[[G]]$$

that maps $g \in G$ to g^{-1}. To prove the remaining assertions, it is sufficient to show that $\sharp$ maps $S_{\Lambda[[G]]} \subset (\Lambda[[G]])^{\mathrm{op}}$ to $S_{\Lambda^{\mathrm{op}}[[G]]}$.

If Λ and H are finite and $\gamma \in \Gamma$ is a topological generator, then $\sharp$ maps $t \coloneqq \gamma - 1$ to $t' \coloneqq \gamma^{-1} - 1$ and hence, it maps $T = \{t^n \mid n \in \mathbb{N}\}$ to $T' = \{t'^n \mid n \in \mathbb{N}\}$. Using Lemma 2.5.2 for T and T', we conclude that $\sharp(S_{\Lambda[[G]]}) = S_{\Lambda^{\mathrm{op}}[[G]]}$.

In the general case, write

$$\Lambda^{\mathrm{op}}[[G]] = \varprojlim_{U,I} \Lambda^{\mathrm{op}}/I[[G/H \cap U]]$$

where the limit runs over all open two-sided ideals I of Λ and all open normal subgroups U of G and note that $\Lambda^{\mathrm{op}}[[G]] \xrightarrow{\cdot s^\sharp} \Lambda^{\mathrm{op}}[[G]]$ is perfect over $\Lambda^{\mathrm{op}}[[H]]$ if and only if $(\Lambda/I)^{\mathrm{op}}[[G/H \cap U]] \xrightarrow{\cdot s^\sharp} (\Lambda/I)^{\mathrm{op}}[[G/H \cap U]]$ is perfect over the finite ring $(\Lambda/I)^{\mathrm{op}}[[H/H \cap U]]$ for all I and U. □

For general Λ and H, the set S is no longer a left or right denominator set, as the following example shows.

EXAMPLE 2.5.4. Assume that either $\Lambda = \mathbb{F}_\ell$ is the finite field with ℓ elements and H is the free pro-ℓ group on two topological generators with trivial action of Γ or $\Lambda = \mathbb{F}_\ell\langle\langle x, y\rangle\rangle$ is the power series ring in two non-commuting indeterminates x, y and H is trivial. In both cases, $\Lambda[[G]] = \mathbb{F}_\ell\langle\langle x, y\rangle\rangle[[t]]$ is the power series ring over $\mathbb{F}_\ell\langle\langle x, y\rangle\rangle$ with t commuting with x and y and the set S is the set of those power series $f(x, y, t)$ with $f(0, 0, t) \neq 0$. Set $s \coloneqq x - t \in S$. If S were a left denominator set, we could find

$$a \coloneqq \sum_{i=0}^{\infty} a_i t^i \in \mathbb{F}_\ell\langle\langle x, y\rangle\rangle[[t]], \quad b \coloneqq \sum_{i=0}^{\infty} b_i t^i \in S$$

such that $as = by$, i. e.

$$a_0 x = b_0 y, \qquad a_i x - a_{i-1} = b_i y \quad \text{for } i > 0.$$

The only solution for this equation is $a = b = 0$, which contradicts the assumption $b \in S$.

Nevertheless, using Waldhausen K-theory, we can still give a sensible definition of $\mathrm{K}_1(\Lambda[[G]]_S)$ even if $\Lambda[[G]]_S$ does not exist.

DEFINITION 2.5.5. We write $\mathbf{SP}^{w_H}(\Lambda[[G]])$ for the full Waldhausen subcategory of $\mathbf{SP}(\Lambda[[G]])$ of strictly perfect complexes of $\Lambda[[G]]$-modules which are perfect as complexes of $\Lambda[[H]]$-modules.

We write $w_H\mathbf{SP}(\Lambda[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{SP}(\Lambda[[G]])$, but with a new set of weak equivalences given by those morphisms whose cones are objects of the category $\mathbf{SP}^{w_H}(\Lambda[[G]])$.

The same construction also works for $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$:

DEFINITION 2.5.6. We write $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ for the full Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ of objects $(P_J^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$ such that

$$\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} P_J^\bullet$$

is a perfect complex of $\Lambda[[H]]$-modules.

We write $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$, but with a new set of weak equivalences given by those morphisms whose cones are objects of the category $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$.

From the Waldhausen approximation theorem [**TT90**, 1.9.1] and from [**Wit14**, Prop. 3.7] we conclude that

$$\mathrm{K}_n(\mathbf{SP}^{w_H}(\Lambda[[G]])) \cong \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])),$$
$$\mathrm{K}_n(w_H\mathbf{SP}(\Lambda[[G]])) \cong \mathrm{K}_n(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$$

We may then set for all $n \geq 0$

$$\mathrm{K}_n(\Lambda[[G]], S) \coloneqq \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])),$$
$$\mathrm{K}_{n+1}(\Lambda[[G]]_S) \coloneqq \mathrm{K}_{n+1}(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])).$$

If $\Lambda[[H]]$ is noetherian, these groups agree with their usual definition [**Wit14**, § 4].

A closely related variant of $\mathbf{SP}^{w_H}(\Lambda[[G]])$ is the following Waldhausen category.

DEFINITION 2.5.7. Let $\mathbf{SP}(\Lambda[[H]],G)$ be the Waldhausen category of complexes of $\Lambda[[G]]$-modules which are strictly perfect as complexes of $\Lambda[[H]]$-modules. Cofibrations are the injective morphisms with cokernel in $\mathbf{SP}(\Lambda[[H]],G)$; the weak equivalences are given by the quasi-isomorphisms.

In other words, $\mathbf{SP}(\Lambda[[H]],G)$ is the Waldhausen category of bounded complexes over the exact category of $\Lambda[[G]]$-modules which are finitely generated and projective as $\Lambda[[H]]$-modules and hence, the groups $\mathrm{K}_n(\mathbf{SP}(\Lambda[[H]],G))$ agree with the Quillen K-groups of this exact category. Unfortunately, we cannot prove in general that $\mathrm{K}_n(\mathbf{SP}(\Lambda[[H]],G))$ agrees with $\mathrm{K}_n(\Lambda[[G]],S)$. However, we shall see below that we always have a surjection

$$\mathrm{K}_0(\mathbf{SP}(\Lambda[[H]],G)) \to \mathrm{K}_0(\Lambda[[G]],S).$$

This is sufficient for our applications.

LEMMA 2.5.8. *Let $P^\bullet$ be a complex of projective compact $\Lambda[[G]]$-modules that is bounded above. Assume that there exists a bounded above complex $K^\bullet$ of finitely generated, projective $\Lambda[[H]]$-modules that is quasi-isomorphic to $P^\bullet$ as complex of $\Lambda[[H]]$-modules. Then there exists in the category of complexes of $\Lambda[[G]]$-modules an injective endomorphism*

$$\psi\colon \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet \to \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet$$

and a quasi-isomorphism

$$f\colon P^\bullet \to \operatorname{coker}\psi.$$

such that $\operatorname{coker}\psi$ *is a bounded above complex of $\Lambda[[G]]$-modules which are finitely generated and projective as $\Lambda[[H]]$-modules.*

In particular, if $P^\bullet$ is perfect as complex of $\Lambda[[H]]$-modules, then $P^\bullet$ is perfect as complex of $\Lambda[[G]]$-modules and $\operatorname{coker}\psi$ *is in* $\mathbf{SP}(\Lambda[[H]],G)$.

PROOF. Since $K^\bullet$ is a bounded above complex of finitely generated projective $\Lambda[[H]]$-modules, there exists a quasi-isomorphism $\alpha\colon K^\bullet \to P^\bullet$ of complexes of $\Lambda[[H]]$-modules, which is automatically continuous for the compact topologies on $K^\bullet$ and $P^\bullet$. Every projective compact $\Lambda[[G]]$-module is also projective in the category of compact $\Lambda[[H]]$-modules. Hence, there exists a weak equivalence $\beta\colon P^\bullet \to K^\bullet$ in the category of complexes of compact $\Lambda[[H]]$-modules such that $\alpha\circ\beta$ and $\beta\circ\alpha$ are homotopic to the identity. Fix a topological generator $\gamma\in\Gamma$ and set

$$g\colon K^\bullet \to K^\bullet, \qquad x \mapsto \beta(\gamma\alpha(x)),$$
$$\psi\colon \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet \to \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet, \qquad \lambda\otimes x \mapsto \lambda\otimes x - \lambda\gamma^{-1}\otimes g(x).$$

Then ψ is a $\Lambda[[G]]$-linear complex morphism. Moreover, $\operatorname{coker}\psi$ is finitely generated over $\Lambda[[H]]$ in each degree. Indeed, if we set $t=\gamma-1$ and let $(e_1,\dots,e_m)$ denote a generating system of the $\Lambda[[H]]$-module K^n in degree n, then $(t^k\otimes e_i)_{k\in\mathbb{N}_0, i=1,\dots,m}$ is a topological generating system of $\Lambda[[G]]\otimes_{\Lambda[[H]]} K^n$ over $\Lambda[[H]]$. But

$$t^k\otimes v = t^{k-1}\otimes(g(v)-v) + \psi(\gamma t^{k-1}\otimes v)$$

for all $v\in K^n$, such that $\operatorname{coker}\psi$ is already generated by the images of $1\otimes e_1,\dots,1\otimes e_m$.

From Lemma 2.5.1 we conclude that ψ is injective and that $\operatorname{coker}\psi$ is finitely generated and projective over $\Lambda[[H]]$ in each degree. Set $Q^\bullet \coloneqq \operatorname{coker}\psi$. Since $P^\bullet$ is a bounded above complex of projective compact $\Lambda[[G]]$-modules, there exists a quasi-isomorphism f completing the homotopy-commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet \arrow[r, "\lambda\hat{\otimes}x\mapsto\lambda\hat{\otimes}x-\lambda\gamma^{-1}\hat{\otimes}\gamma x"] \arrow[d, "\mathrm{id}\hat{\otimes}\beta", "\sim"'] & \Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet \arrow[r, "\lambda\hat{\otimes}x\mapsto\lambda x"] \arrow[d, "\mathrm{id}\hat{\otimes}\beta", "\sim"'] & P^\bullet \arrow[r] \arrow[d, dotted, "f", "\sim"'] & 0 \\ 0 \arrow[r] & \Lambda[[G]]\otimes_{\Lambda[[H]]}K^\bullet \arrow[r, "\psi"] & \Lambda[[G]]\otimes_{\Lambda[[H]]}K^\bullet \arrow[r] & Q^\bullet \arrow[r] & 0 \end{tikzcd}$$

in the category of complexes of compact $\Lambda[[G]]$-modules. Here, $\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet$ denotes the completed tensor product. The exactness of the first row follows from [**Wit13b**, Prop. 2.4]. If we can choose $K^\bullet$ to be a strictly perfect complex of $\Lambda[[H]]$-modules, then $P^\bullet$ is also quasi-isomorphic to the cone of ψ, which is strictly perfect as complex of $\Lambda[[G]]$-modules. Moreover, coker ψ is a bounded complex and hence, an object of $\mathbf{SP}(\Lambda[[H]], G)$. □

PROPOSITION 2.5.9. *Let $\gamma \in \Gamma$ be a topological generator. The functor*

$$C_\gamma : \mathbf{SP}(\Lambda[[H]], G) \to \mathbf{SP}^{w_H}(\Lambda[[G]]),$$

$$P^\bullet \mapsto \mathrm{Cone}(\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\lambda\otimes p\mapsto\lambda\otimes p-\lambda\gamma^{-1}\otimes\gamma p} \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet)$$

is well defined and Waldhausen exact. It induces a surjection

$$C_\gamma : \mathrm{K}_0(\mathbf{SP}(\Lambda[[H]], G)) \to \mathrm{K}_0(\mathbf{SP}^{w_H}(\Lambda[[G]]))$$

which is independent of the choice of γ.

PROOF. From [**Wit13b**, Prop. 2.4] we conclude that

$$0 \to \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\mathrm{id}-(\cdot\gamma^{-1}\otimes\gamma\cdot)} \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\lambda\otimes p\mapsto\lambda p} P^\bullet \to 0$$

is an exact sequence of complexes of $\Lambda[[G]]$-modules for any $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$. In particular, the strictly perfect complex of $\Lambda[[G]]$-modules $C_\gamma(P^\bullet)$ is quasi-isomorphic to $P^\bullet$ in the category of complexes of $\Lambda[[G]]$-modules and therefore perfect as complex of $\Lambda[[H]]$-modules. Thus, $C_\gamma(P^\bullet)$ is an object of $\mathbf{SP}^{w_H}(\Lambda[[G]])$. The Waldhausen exactness of the functor C_γ follows easily from the Waldhausen exactness of the cone construction.

Consider the Waldhausen category $\mathbf{P}^{w_H}(\Lambda[[G]])$ of those perfect complexes of $\Lambda[[G]]$-modules which are also perfect as complexes of $\Lambda[[H]]$-modules. The Waldhausen approximation theorem [**TT90**, 1.9.1] implies that the inclusion

$$\iota : \mathbf{SP}^{w_H}(\Lambda[[G]]) \to \mathbf{P}^{w_H}(\Lambda[[G]])$$

induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}^{w_H}(\Lambda[[G]])) \cong \mathrm{K}_n(\mathbf{P}^{w_H}(\Lambda[[G]]))$$

for all n. The functorial quasi-isomorphism $C_\gamma(P^\bullet) \xrightarrow{\sim} P^\bullet$ in $\mathbf{P}^{w_H}(\Lambda[[G]])$ implies that the homomorphism of K-groups induced by $\iota \circ C_\gamma$ agrees with the homomorphism induced by the inclusion $\iota' : \mathbf{SP}(\Lambda[[H]], G) \to \mathbf{P}^{w_H}(\Lambda[[G]])$. Since $\mathrm{K}_0(\mathbf{P}^{w_H}(\Lambda[[G]]))$ is generated by the quasi-isomorphism classes of complexes in $\mathbf{P}^{w_H}(\Lambda[[G]])$, we deduce from Lemma 2.5.8 that ι' induces a surjection

$$\mathrm{K}_0(\mathbf{SP}(\Lambda[[H]], G)) \to \mathrm{K}_0(\mathbf{P}^{w_H}(\Lambda[[G]])).$$

□

REMARK 2.5.10. In order to deduce from the Waldhausen approximation theorem (applied to the opposite categories) that C_γ induces isomorphisms

$$\mathrm{K}_n(\mathbf{SP}(\Lambda[[H]], G)) \cong \mathrm{K}_n(\mathbf{SP}^{w_H}(\Lambda[[G]]))$$

for all n, it would suffice to verify that for every complex $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$ and every morphism $f : K^\bullet \to P^\bullet$ in $\mathbf{P}^{w_H}(\Lambda[[G]])$, there exists a morphism $f' : Q^\bullet \to P^\bullet$

in $\mathbf{SP}(\Lambda[[H]], G)$ and a quasi-isomorphism $w: K^\bullet \xrightarrow{\sim} Q^\bullet$ in $\mathbf{P}^{w_H}(\Lambda[[G]])$ such that $f = f' \circ w$.

Remark 2.5.11.

(1) In the light of Proposition 2.5.9, we will write

$$[P^\bullet] \coloneqq [C_\gamma(P^\bullet)] \in \mathrm{K}_0(\Lambda[[G]], S)$$

for any $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$.

(2) More generally, let M be a $\Lambda[[G]]$-module which has a resolution by a strictly perfect complex of $\Lambda[[H]]$-modules $Q^\bullet$. By Lemma 2.5.8, M then also has a resolution by a complex $P^\bullet$ in $\mathbf{SP}(\Lambda[[H]], G)$. We set

$$[M] \coloneqq [P^\bullet] \in \mathrm{K}_0(\Lambda[[G]], S).$$

Note that $[M]$ does not depend on the particular choice of the resolutions $P^\bullet$ or $Q^\bullet$.

2.6. Base Change with Bimodules

Let Λ and Λ' be two adic $\mathbb{Z}_\ell$-algebras and $G = H \rtimes \Gamma$, $G = H' \rtimes \Gamma'$ be profinite groups, such that H and H' contain open, topologically finitely generated pro-ℓ subgroups and $\Gamma \cong \mathbb{Z}_\ell \cong \Gamma'$. Suppose that $K^\bullet$ is a complex of $\Lambda'[[G']]$-$\Lambda[[G]]$-bimodules, strictly perfect as complex of $\Lambda'[[G']]$-modules and assume that there exists a complex $L^\bullet$ of $\Lambda'[[H']]$-$\Lambda[[H]]$-bimodules, strictly perfect as complex of $\Lambda'[[H']]$-modules, and a quasi-isomorphism of complexes of $\Lambda'[[H']]$-$\Lambda[[G]]$-bimodules

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet.$$

Here,

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \coloneqq \varprojlim_{I \in \mathfrak{I}_{\Lambda'[[G']]}} \varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} L/IL^\bullet \otimes_{\Lambda[[H]]} \Lambda[[G]]/J$$

denotes the *completed tensor product.*

In the above situation, the Waldhausen exact functor

$$\Psi_{K^\bullet}: \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']]) \tag{2.6.1}$$

takes objects of the category $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ to objects of the category $\mathbf{PDG}^{\mathrm{cont}, w_{H'}}(\Lambda'[[G']])$ and weak equivalences of $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ to weak equivalences of $w_{H'}\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G']])$ [**Wit14**, Prop. 4.6]. Hence, it also induces homomorphisms between the corresponding K-groups. In particular, this applies to the following examples:

Example 2.6.1. [**Wit14**, Prop. 4.7]

(1) Assume $G = G'$, $H = H'$. For any complex $P^\bullet$ of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules, let $K^\bullet$ be the complex

$$P[[G]]^{\delta^\bullet} \coloneqq \Lambda'[[G]] \otimes_{\Lambda'} P^\bullet$$

of $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodules with the right G-operation given by the diagonal action on both factors. This applies in particular for any complex $P^\bullet$ of Λ'-Λ-bimodules, strictly perfect as complex of Λ'-modules and equipped with the trivial G-operation.

(2) Assume $\Lambda = \Lambda'$. Let $\alpha: G \to G'$ be a continuous homomorphism such that α maps H to H' and induces a bijection of G/H and G'/H'. Let $K^\bullet$ be the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G']]$.

(3) Assume that G' is an open subgroup of G and set $H' \coloneqq H \cap G'$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the complex concentrated in degree 0 given by the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G]]$.

EXAMPLE 2.6.2. The assumptions in Example 2.6.1.(2) are in fact stronger than necessary. We may combine it with the following result. Assume that G is an open subgroup of G' such that $H \coloneqq H' \cap G = H'$ and $\Gamma = (\Gamma')^{\ell^n}$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule $\Lambda[[G']]$. Fix a topological generator $\gamma' \in \Gamma'$ and let $L^\bullet$ be the $\Lambda[[H]]$-$\Lambda[[H]]$-sub-bimodule of $\Lambda[[G']]$ generated as left $\Lambda[[H]]$-module by $1, \gamma', (\gamma')^2, \dots, (\gamma')^{\ell^n - 1}$. Then $L^\bullet$ is a strictly perfect complex of $\Lambda[[H]]$-modules concentrated in degree 0 and the canonical map

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet, \qquad \ell \hat{\otimes} \lambda \mapsto \ell\lambda$$

is an isomorphism of $\Lambda'[[H']]$-$\Lambda[[G]]$-bimodules such that [**Wit14**, Prop. 4.6] applies. In combination with Example 2.6.1.(2) this implies that any continuous group homomorphism $\alpha \colon G \to G'$ such that $\alpha(G) \notin H'$ induces Waldhausen exact functors between all three variants of the above Waldhausen categories.

EXAMPLE 2.6.3. As a special case of Example 2.6.1.(1), assume that $\Lambda = \mathbb{Z}_\ell$ and that ρ is some continuous representation of G on a finitely generated and projective Λ'-module. Let $\rho^\sharp$ be the Λ'-$\mathbb{Z}_\ell[[G]]$-bimodule which agrees with ρ as Λ'-module, but on which $g \in G$ acts from the right by the left operation of g^{-1} on ρ. We thus obtain Waldhausen exact functors

$$\Phi_\rho \coloneqq \Psi_{\Lambda'[[\Gamma]]} \circ \Psi_{\rho^\sharp[[G]]^\delta} \tag{2.6.2}$$

from all three variants of the Waldhausen category $\mathbf{PDG}^{\mathrm{cont}}(\mathbb{Z}_\ell[[G]])$ to the corresponding variant of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[\Gamma]])$. If Λ' is a commutative adic $\mathbb{Z}_\ell$-algebra, then the image of

$$\left[\mathbb{Z}_\ell[[G]] \xrightarrow{\cdot g} \mathbb{Z}_\ell[[G]]\right] \in \mathrm{K}_1(\mathbb{Z}_\ell[[G]]), \quad g \in G,$$

under the composition of Φ_ρ with

$$\det \colon \mathrm{K}_1(\Lambda'[[\Gamma]]) \xrightarrow{\cong} \Lambda'[[\Gamma]]^\times$$

is $\bar{g}\det(\rho(g))^{-1}$, where $\bar{g}$ denotes the image of g under the projection $G \to \Gamma$. Note that this differs from [**CFK$^+$05**, (22)] by a sign. So, our evaluation at ρ corresponds to the evaluation at the representation dual to ρ in terms of the cited article.

2.7. Duality for S-Torsion Complexes

As before, Λ is an adic ring and $G = H \rtimes \Gamma$ is a profinite group such that H contains an open, topologically finitely generated pro-ℓ subgroup and $\Gamma \cong \mathbb{Z}_\ell$.

We define

$$\sharp \colon \Lambda[[G]]^{\mathrm{op}} \to \Lambda^{\mathrm{op}}[[G]], \qquad a \mapsto a^\sharp,$$

to be the ring homomorphism that is the identity on the coefficients and maps $g \in G$ to g^{-1} and write $\Lambda^{\mathrm{op}}[[G]]^\sharp$ for $\Lambda^{\mathrm{op}}[[G]]$ considered as $\Lambda^{\mathrm{op}}[[G]]$-$\Lambda[[G]]^{\mathrm{op}}$-bimodule via $\sharp$. If $P^\bullet$ is a bounded above complex of finitely generated, projective $\Lambda[[G]]$-modules, we set

$$^\sharp(P^\bullet)^{*_{\Lambda[[G]]}} \coloneqq \Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda[[G]]^{\mathrm{op}}} (P^\bullet)^{*_{\Lambda[[G]]}},$$

i. e. the complex $(P^\bullet)^{*_{\Lambda[[G]]}}$ of finitely generated, projective $\Lambda[[G]]^{\mathrm{op}}$-modules is turned into a complex of finitely generated, projective $\Lambda^{\mathrm{op}}[[G]]$-modules by letting $g \in G$ act as g^{-1}.

In particular, we obtain Waldhausen exact equivalence of categories

$$\mathbf{SP}(\Lambda[[G]])^{\mathrm{op}} \to \mathbf{SP}(\Lambda^{\mathrm{op}}[[G]]), \qquad P^\bullet \mapsto {}^\sharp(P^\bullet)^{*_{\Lambda[[G]]}}.$$

DEFINITION 2.7.1. We write

$$\circledast\colon \mathrm{K}_n(\Lambda[[G]]) \to \mathrm{K}_n(\Lambda^{\mathrm{op}}[[G]])$$

for the homomorphisms obtained by composing I with the homomorphism

$$\mathrm{K}_n(\mathbf{SP}(\Lambda[[G]])^{\mathrm{op}}) \to \mathrm{K}_n(\mathbf{SP}(\Lambda^{\mathrm{op}}[[G]]))$$

induced by the Waldhausen exact functor $P^\bullet \mapsto {}^\sharp(P^\bullet)^{*_{\Lambda[[G]]}}$.

REMARK 2.7.2. The author does not know wether it is possible to produce an extension of $P^\bullet \mapsto {}^\sharp(P^\bullet)^{*_{\Lambda[[G]]}}$ to a Waldhausen exact functor

$$\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])^{\mathrm{op}} \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}}[[G]])$$

inducing the same homomorphisms on K-theory. This would avoid some technicalities that we need to deal with later on.

LEMMA 2.7.3. *Assume that $K^\bullet$ is in $\mathbf{SP}(\Lambda[[G]])$.*

(1) *Let Λ' be another adic $\mathbb{Z}_\ell$-algebra. For any complex $P^\bullet$ of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules, set*

$$(P^\bullet)^{*_\Lambda,\sharp} \coloneqq (P^\bullet)^{*_\Lambda} \otimes_{\Lambda[[G]]^{\mathrm{op}}} (\Lambda[[G]]^{\mathrm{op}})^\sharp$$

such that $(P^\bullet)^{*_\Lambda,\sharp}$ is a complex of Λ'^{op}-$\Lambda^{\mathrm{op}}[[G]]$-bimodules, with $g \in G$ acting by $(g^{-1})^*$. With $P[[G]]^{\delta^\bullet}$ as in Example 2.6.1,

$${}^\sharp(\Psi_{P[[G]]^{\delta^\bullet}}(K^\bullet))^{*_{\Lambda[[G]]}} \cong \Psi_{(P^\bullet)^{*_\Lambda,\sharp}[[G]]^\delta}({}^\sharp(K^\bullet)^{*_{\Lambda[[G]]}}).$$

(2) *Let $G' = H' \rtimes \Gamma'$ be another profinite group such that H' contains an open, topologically finitely generated pro-ℓ subgroup and $\Gamma' \cong \mathbb{Z}_\ell$. Let $\alpha\colon G \to G'$ be a continuous homomorphism such that $\alpha(G) \not\subset H'$. Consider $\Lambda[[G']]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then*

$${}^\sharp(\Psi_{\Lambda[[G']]}(K^\bullet))^{*_{\Lambda[[G]]}} \cong \Psi_{\Lambda^{\mathrm{op}}[[G']]}({}^\sharp(K^\bullet)^{*_{\Lambda[[G]]}}).$$

(3) *Assume that G' is an open subgroup of G and set $H' \coloneqq H \cap G'$. Consider $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then*

$${}^\sharp(\Psi_{\Lambda[[G]]}(K^\bullet))^{*_{\Lambda[[G]]}} \cong \Psi_{\Lambda^{\mathrm{op}}[[G]]}({}^\sharp(K^\bullet)^{*_{\Lambda[[G]]}}).$$

PROOF. Using the canonical isomorphism (2.3.4), it remains to notice that

$$\Lambda'^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda'[[G]]^{\mathrm{op}}} (P[[G]]^{\delta^\bullet})^{*_{\Lambda'[[G]]}} \cong (P^\bullet)^{*,\sharp}[[G]]^\delta \otimes_{\Lambda^{\mathrm{op}}[[G]]} \Lambda^{\mathrm{op}}[[G]]^\sharp$$

as complexes of $\Lambda'^{\mathrm{op}}[[G]]$-$\Lambda[[G]]^{\mathrm{op}}$-bimodules to prove (1). The other two parts are straightforward. □

PROPOSITION 2.7.4. *The functor $P^\bullet \mapsto {}^\sharp(P^\bullet)^{*_{\Lambda[[G]]}}$ extends to Waldhausen exact equivalences*

$$(w_H\mathbf{SP}(\Lambda[[G]]))^{\mathrm{op}} \to w_H\mathbf{SP}(\Lambda^{\mathrm{op}}[[G]]),$$
$$(\mathbf{SP}^{w_H}(\Lambda[[G]]))^{\mathrm{op}} \to \mathbf{SP}^{w_H}(\Lambda^{\mathrm{op}}[[G]])$$

and hence, it induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_1(\Lambda[[G]]) \arrow[r] \arrow[d, "\circledast", "\cong"'] & \mathrm{K}_1(\Lambda[[G]]_S) \arrow[r, "d"] \arrow[d, "\circledast", "\cong"'] & \mathrm{K}_0(\Lambda[[G]], S) \arrow[r] \arrow[d, "\circledast", "\cong"'] & 0 \\
0 \arrow[r] & \mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]) \arrow[r] & \mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]_S) \arrow[r, "d"] & \mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]], S) \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

PROOF. The exactness of the rows follows from [**Wit13b**, Cor. 3.3]. To extend $\circledast$, it suffices to show that for any strictly perfect complex $P^\bullet$ of $\Lambda[[G]]$-modules which is also perfect as complex of $\Lambda[[H]]$-modules, the complex ${}^\sharp(P^\bullet)^{*_{\Lambda[[G]]}}$ is perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules. By [**Wit14**, Prop. 4.8] we may check this after tensoring with $(\Lambda/\operatorname{Jac}(\Lambda))^{\mathrm{op}}[[G/V]]$ with $V \subset G$ a closed normal pro-ℓ-subgroup which is open in H. Using Lemma 2.7.3, we may therefore assume that Λ and H are finite.

By Lemma 2.5.3, $S = S_{\Lambda[[G]]} \subset \Lambda[[G]]$ is a left and right denominator set and $\sharp$ maps $S_{\Lambda[[G]]}$ to the set $S_{\Lambda^{\mathrm{op}}[[G]]} \subset \Lambda^{\mathrm{op}}[[G]]$. Moreover ${}^\sharp(P^\bullet)^{*_{\Lambda[[G]]}}$ is perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules if and only if its cohomology is $S_{\Lambda^{\mathrm{op}}[[G]]}$-torsion.

As $P^\bullet$ has $S_{\Lambda[[G]]}$-torsion cohomology and as

$$(\Lambda[[G]]_S)^{\mathrm{op}} \otimes_{\Lambda[[G]]^{\mathrm{op}}} (P^\bullet)^{*_{\Lambda[[G]]}} \cong (\Lambda[[G]]_S \otimes_{\Lambda[[G]]} P^\bullet)^{*_{\Lambda[[G]]_S}},$$

we conclude that ${}^\sharp(P^\bullet)^{*_{\Lambda[[G]]}}$ is indeed perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules. $\square$

We may extend $\circledast$ to the Waldhausen category $\mathbf{SP}(\Lambda[[H]], G)$ from Definition 2.5.7. More generally, we can also explicitly describe the class $[M]^\circledast$ for any $\Lambda[[G]]$-module M that has a strictly perfect resolution as a $\Lambda[[H]]$-module.

Assume that P is a $\Lambda[[G]]$-module that is finitely generated and projective as $\Lambda[[H]]$-module. We may let $g \in G$ act on $\phi \in P^{*_{\Lambda[[H]]}}$ by setting

$$g\phi\colon P \to \Lambda[[H]], \qquad p \mapsto g\phi(g^{-1}p)g^{-1}.$$

We write ${}^\sharp P^{*_{\Lambda[[H]]}}$ for the resulting $\Lambda^{\mathrm{op}}[[G]]$-module.

LEMMA 2.7.5. *Let $\gamma \in \Gamma$ be a topological generator. Then for any bounded above complex $P^\bullet$ of $\Lambda[[G]]$-modules which are finitely generated and projective as $\Lambda[[H]]$-modules, we have a commutative diagram*

$$\begin{tikzcd}
\Lambda^{\mathrm{op}}[[G]] \otimes_{\Lambda^{\mathrm{op}}[[H]]} {}^\sharp(P^\bullet)^{*_{\Lambda[[H]]}} \arrow[r, "\mathrm{id} - \cdot\gamma^{-1} \otimes \gamma"] \arrow[d, "\alpha", "\cong"'] & \Lambda^{\mathrm{op}}[[G]] \otimes_{\Lambda^{\mathrm{op}}[[H]]} {}^\sharp(P^\bullet)^{*_{\Lambda[[H]]}} \arrow[d, "\alpha", "\cong"'] \\
{}^\sharp(\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet)^{*_{\Lambda[[G]]}} \arrow[r, "\mathrm{id} - (\cdot\gamma \otimes \gamma^{-1}\cdot)^*"] & {}^\sharp(\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet)^{*_{\Lambda[[G]]}}
\end{tikzcd}$$

of complexes of $\Lambda[[G]]$-modules.

PROOF. For any degree n and any $f \in (P^n)^{*_{\Lambda[[H]]}}$, we write

$$\tilde{f}\colon \Lambda[[G]] \otimes_{\Lambda[[H]]} P^n \to \Lambda[[G]], \qquad \lambda \otimes p \mapsto \lambda f(p).$$

We then set

$$\begin{tikzcd}
\Lambda^{\mathrm{op}}[[G]] \otimes_{\Lambda^{\mathrm{op}}[[H]]} {}^\sharp(P^n)^{*_{\Lambda[[H]]}} \arrow[d, "\alpha"', "\cong"] & \lambda \otimes f \arrow[d, mapsto] \\
{}^\sharp(\Lambda[[G]] \otimes_{\Lambda[[H]]} P^n)^{*_{\Lambda[[G]]}} & \lambda\tilde{f}
\end{tikzcd}$$

It is then straightforward to check that the above diagram commutes. $\square$

COROLLARY 2.7.6. *Assume that $P^\bullet$ is in $\mathbf{SP}(\Lambda[[H]], G)$ and let γ be a topological generator of Γ. Then there exists a canonical isomorphism*

$${}^\sharp C_{\gamma^{-1}}(P^\bullet)^{*_{\Lambda[[G]]}} \cong C_\gamma({}^\sharp(P^\bullet)^{*_{\Lambda[[H]]}})[1].$$

In particular, we have

$$[P^\bullet]^\circledast = -[{}^\sharp(P^\bullet)^{*_{\Lambda[[H]]}}]$$

in $\mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]], S)$.

PROOF. Consider the diagram of Lemma 2.7.5. The cone of the first row is $C_\gamma({}^\sharp(P^\bullet)^{*_{\Lambda[[H]]}})$, the cone of the second row is the $\Lambda[[G]]$-dual of the cocone of

$$\Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet \xrightarrow{\mathrm{id}-(\cdot\gamma\otimes\gamma^{-1}\cdot)} \Lambda[[G]] \otimes_{\Lambda[[H]]} P^\bullet$$

in $\mathbf{SP}^{w_H}(\Lambda[[G]])$, which is in turn the same as $C_{\gamma^{-1}}(P^\bullet)[-1]$. Finally, recall from Proposition 2.5.9 that the class of $C_\gamma(P^\bullet)$ in $\mathrm{K}_0(\Lambda[[G]],S)$ is independent of the choice of the topological generator γ. Hence,

$$[P^\bullet]^\circledast = [{}^\sharp C_\gamma(P^\bullet)^{*_{\Lambda[[G]]}}] = [{}^\sharp C_{\gamma^{-1}}(P^\bullet)^{*_{\Lambda[[G]]}}] = -[C_\gamma({}^\sharp(P^\bullet)^{*_{\Lambda[[H]]}})] = -[{}^\sharp(P^\bullet)^{*_{\Lambda[[H]]}}].$$

□

The following lemma is a minor improvement on [**NSW00**, Prop. 5.4.17].

LEMMA 2.7.7. *Let $H' \subset H$ be an open subgroup and assume M is a $\Lambda[[H]]$-module which has a resolution by finitely generated, free $\Lambda[[H']]$-modules. Then there exist an isomorphism*

$$\alpha\colon \mathrm{R}\operatorname{Hom}_{\Lambda[[H']]}(M, \Lambda[[H']]) \xrightarrow{\cong} \mathrm{R}\operatorname{Hom}_{\Lambda[[H]]}(M, \Lambda[[H]])$$

in the derived category of complexes of $\Lambda[[H']]^{\mathrm{op}}$-modules.

PROOF. Choose a system $g_1, \dots, g_d$ of right coset representatives of $H' \backslash H$. For any finitely generated, free $\Lambda[[H]]$-module P,

$$\alpha\colon \operatorname{Hom}_{\Lambda[[H']]}(P, \Lambda[[H']]) \to \operatorname{Hom}_{\Lambda[[H]]}(P, \Lambda[[H]]),$$

$$\alpha(\phi)(p) \coloneqq \sum_{i=1}^{d} g_i \phi(g_i^{-1} p) \qquad \text{for } p \in P,\ \phi \in \operatorname{Hom}_{\Lambda[[H']]}(P, \Lambda[[H']]),$$

is an isomorphism of $\Lambda[[H']]^{\mathrm{op}}$-modules and does not depend on the choice of $g_1, \dots, g_d$. By Lemma 2.7.8 below, M also has a resolution $P^\bullet$ by finitely generated, free $\Lambda[[H]]$-modules. Moreover, any finitely generated, free $\Lambda[[H]]$-module is also finitely generated and free as $\Lambda[[H']]$-module, so that we can use the same $P^\bullet$ to compute the total derived functor of $\operatorname{Hom}_{\Lambda[[H]]}(M, \Lambda[[H]])$ in the categories of complexes of $\Lambda[[H]]$-modules or of $\Lambda[[H']]$-modules. □

LEMMA 2.7.8. *Let A be a subring of a ring B and assume that B has a resolution by finitely generated, free A-modules as a left A-module. Then a B-module M has a resolution by finitely generated, free B-modules if and only if it has a resolution by finitely generated, free A-modules.*

PROOF. If M has a resolution $P^\bullet$ by finitely generated, free B-modules, then we may find a resolution of P^{-n} by finitely generated, free A-modules for each $n \geq 0$. We obtain a resolution of M by finitely generated, free A-modules by taking the total complex of the resulting double complex.

To prove the converse, we proceed by induction. For any ring R and any R-module N, recall that a finite free presentation of length μ is an exact sequence

$$P^{-\mu} \to P^{1-\mu} \to \cdots \to P^0 \to N \to 0$$

with finitely generated, free R-modules P^k. Set $\lambda_R(N) \coloneqq -1$ if N is not finitely generated and

$$\lambda_R(N) \coloneqq \sup\{\mu \mid \text{there exists a finite free presentation of length } \mu\}$$

else. Clearly, for all B-modules N, if $\lambda_A(N) \geq 0$, then also $\lambda_B(N) \geq 0$. Assume that we know for some $n \geq 0$ that $\lambda_A(N) \geq n$ implies $\lambda_B(N) \geq n$ for B-modules N. Let N' be a B-module with $\lambda_A(N') \geq n+1$. Then there exists an exact sequence

$$0 \to Q \to P \to N' \to 0$$

of B-modules with P finitely generated and free. By [**Bou89**, §II.2, Ex. 6.(d)],

$$\lambda_A(Q) \geq \inf\{\lambda_A(P), \lambda_A(N') - 1\} \geq n.$$

Hence, $\lambda_B(Q) \geq n$ by the induction assumption. By [**Bou89**, §II.2, Ex. 6.(c)],

$$\lambda_B(N') \geq \inf\{\lambda_B(P), \lambda_B(Q) + 1\} \geq n + 1$$

In particular, we conclude that $\lambda_A(M) = \infty$ implies $\lambda_B(M) = \infty$. $\square$

The following Lemma is a variant of [**SV06**, Prop. 3.1].

LEMMA 2.7.9. *Assume that M is a $\Lambda[[G]]$-module that has a resolution by finitely generated, projective $\Lambda[[H]]$-modules. Then M also has a resolution by finitely generated, projective $\Lambda[[G]]$-modules and there exists an isomorphism*

$$\beta \colon \mathrm{R}\operatorname{Hom}_{\Lambda[[H]]}(M, \Lambda[[H]]) \xrightarrow{\cong} \mathrm{R}\operatorname{Hom}_{\Lambda[[G]]}(M, \Lambda[[G]])[1]$$

in the derived category of complexes of $\Lambda[[H]]^{\mathrm{op}}$-modules.

PROOF. By Lemma 2.5.8 we may find a resolution $K^\bullet$ of M by $\Lambda[[G]]$-modules which are finitely generated and projective as $\Lambda[[H]]$-modules. Choose a topological generator $\gamma \in \Gamma$. We then obtain an exact sequence of complexes of $\Lambda[[G]]$-modules

$$0 \to \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet \xrightarrow{\mathrm{id} - (\cdot\gamma^{-1} \otimes \gamma\cdot)} \Lambda[[G]] \otimes_{\Lambda[[H]]} K^\bullet \to K^\bullet \to 0$$

from [**Wit13b**, Prop. 2.4]. The cone of $\mathrm{id} - (\cdot\gamma^{-1} \otimes \gamma\cdot)$ is a resolution of M by finitely generated, projective $\Lambda[[G]]$-modules. One then uses Lemma 2.7.5. $\square$

Assume that M is a $\Lambda[[G]]$-module that has a resolution by finitely generated, projective $\Lambda[[H']]$-modules for some open subgroup H' of H and that

$$\operatorname{Ext}^n_{\Lambda[[H']]}(M, \Lambda[[H']]) = \operatorname{Ext}^{n+1}_{\Lambda[[G]]}(M, \Lambda[[G]]) = 0$$

for all $n \neq 0$. By Lemma 2.7.7 and Lemma 2.7.9,

$$\operatorname{Ext}^n_{\Lambda[[H'']]}(M, \Lambda[[H'']]) = \operatorname{Ext}^{n+1}_{\Lambda[[G]]}(M, \Lambda[[G]]) = 0$$

for all open subgroups H'' of H and all $n \neq 0$. We may then extend the notation introduced above as follows: Let ${}^\sharp\operatorname{Ext}^1_{\Lambda[[G]]}(M, \Lambda[[G]])$ denote the $\Lambda[[G]]^{\mathrm{op}}$-module $\operatorname{Ext}^1_{\Lambda[[G]]}(M, \Lambda[[G]])$ considered as $\Lambda^{\mathrm{op}}[[G]]$-module.

DEFINITION 2.7.10. We write ${}^\sharp M^{*_{\Lambda[[H'']]}}$ for $\operatorname{Hom}_{\Lambda[[H'']]}(M, \Lambda[[H']])$ considered as $\Lambda^{\mathrm{op}}[[G]]$-module via the isomorphism

$$ {}^\sharp M^{*_{\Lambda[[H'']]}} \underset{\cong}{\xrightarrow{\alpha}} {}^\sharp M^{*_{\Lambda[[H]]}} \underset{\cong}{\xrightarrow{\beta}} {}^\sharp\operatorname{Ext}^1_{\Lambda[[G]]}(M, \Lambda[[G]]).$$

If H'' is normal in G, $g \in G$ acts on $\phi \in {}^\sharp M^{*_{\Lambda[[H'']]}}$ via

$$g\phi \colon M \to \Lambda[[H'']], \qquad m \mapsto g\phi(g^{-1}m)g^{-1}.$$

In the case that H'' is not normal in G, it is more difficult to give an explicit description of the G-operation.

We conclude that if M has a resolution by a strictly perfect complex of $\Lambda[[H]]$-modules, then

$$[M]^{\circledast} = -[{}^\sharp M^{*_{\Lambda[[H']]}}] \tag{2.7.1}$$

in $\mathrm{K}_0(\Lambda[[G]], S)$ for every open subgroup H' of H.

2.8. Another Property of S-Torsion Complexes

In this section, we prove Proposition 2.8.1, which is an abstract generalisation of [**Wit13a**, Prop. 2.1]. We will apply this proposition later in Section 3.4.

With the notation of the previous section, fix a topological generator $\gamma \in \Gamma$ and set $t \coloneqq \gamma - 1$. Assume for the moment that Λ is a finite $\mathbb{Z}_\ell$-algebra and that H is a finite group. By Lemma 2.5.2, we have

$$\Lambda[[G]]_S = \varinjlim_{n\geq 0} \Lambda[[G]]t^{-n}$$

as $\Lambda[[G]]$-modules.

Assume that $\ell^{i+1} = 0$ in Λ. Then

$$\binom{\ell^{n+i}}{k} = 0$$

in Λ whenever $\ell^n \nmid k$. Hence,

$$\begin{aligned}
\gamma^{\ell^{n+i}} - 1 = (t+1)^{\ell^{n+i}} - 1 &= t^{\ell^n} \sum_{k=1}^{\ell^i} \binom{\ell^{n+i}}{k\ell^n} t^{\ell^n(k-1)}, \\
t^{\ell^{n+i}} = (\gamma-1)^{\ell^{n+i}} - (1-1)^{\ell^{n+i}} &= \sum_{k=1}^{\ell^i} \binom{\ell^{n+i}}{k\ell^n} (\gamma^{k\ell^n} - 1)(-1)^{\ell^n(\ell^i - k)} \\
&= (\gamma^{\ell^n} - 1) \sum_{k=1}^{\ell^i} \binom{\ell^{n+i}}{k\ell^n} (-1)^{\ell^n(\ell^i-k)} \sum_{v=0}^{k-1} \gamma^{v\ell^n}
\end{aligned}$$

and therefore,

$$\Lambda[[G]]_S = \varinjlim_{n\geq 0} \Lambda[[G]](\gamma^{\ell^n} - 1)^{-1}.$$

Since H was assumed to be finite, the same is true for the automorphism group of H. We conclude that γ^{ℓ^n} is a central element of G and $\Gamma^{\ell^n} \subset G$ a central subgroup for all $n \geq n_0$ and n_0 large enough. Set

$$N_n \coloneqq \sum_{k=0}^{\ell-1} \gamma^{\ell^n k}.$$

The homomorphism

$$\Lambda[[G]](\gamma^{\ell^n} - 1)^{-1} \to \Lambda[[G/\Gamma^{\ell^n}]], \qquad \lambda(\gamma^{\ell^n} - 1)^{-1} \mapsto \lambda + \Lambda[[G]](\gamma^{\ell^n} - 1)$$

induces an isomorphism $\Lambda[[G]](\gamma^{\ell^n} - 1)^{-1}/\Lambda[[G]] \cong \Lambda[[G/\Gamma^{\ell^n}]]$ such that the diagram

$$\begin{tikzcd}
\Lambda[[G]](\gamma^{\ell^n} - 1)^{-1}/\Lambda[[G]] \arrow[r, "\subset"] \arrow[d, "\cong"] & \Lambda[[G]](\gamma^{\ell^{n+1}} - 1)^{-1}/\Lambda[[G]] \arrow[d, "\cong"] \\
\Lambda[[G/\Gamma^{\ell^n}]] \arrow[r, "\cdot N_n"] & \Lambda[[G/\Gamma^{\ell^{n+1}}]]
\end{tikzcd}$$

commutes. Hence, we obtain an isomorphism of (left and right) $\Lambda[[G]]$-modules

$$\Lambda[[G]]_S/\Lambda[[G]] \cong \varinjlim_{n} \Lambda[[G/\Gamma^{\ell^n}]].$$

We note that this isomorphism may depend on the choice of the topological generator γ.

For any strictly perfect complex $P^\bullet$ of $\Lambda[[G]]$-modules, we thus obtain an exact sequence

$$0 \to P^\bullet \to \Lambda[[G]]_S \otimes_{\Lambda[[G]]} P^\bullet \to \varinjlim_{n} \Lambda[[G/\Gamma^{\ell^n}]] \otimes_{\Lambda[[G]]} P^\bullet \to 0.$$

If $P^\bullet$ is also perfect as a complex of $\Lambda[[H]]$-modules such that the cohomology of $P^\bullet$ is S-torsion by Lemma 2.5.3, then we conclude that there exists an isomorphism

$$P^\bullet[1] \cong \varinjlim_n \Lambda[[G/\Gamma^{\ell^n}]] \otimes_{\Lambda[[G]]} P^\bullet$$

in the derived category of complexes of $\Lambda[[G]]$-modules. In particular, the right-hand complex is perfect as complex of $\Lambda[[G]]$-modules and of $\Lambda[[H]]$-modules. This signifies that its cohomology modules

$$\mathrm{H}^s(\varinjlim_n \Lambda[[G/\Gamma^{\ell^n}]] \otimes_{\Lambda[[G]]} P^\bullet) \cong \varinjlim_n \mathrm{H}^s(\Lambda[[G/\Gamma^{\ell^n}]] \otimes_{\Lambda[[G]]} P^\bullet) \cong \mathrm{H}^{s+1}(P^\bullet)$$

are finite as abelian groups.

We now drop the assumption that Λ and H are finite. Let $I \subset J$ be two open ideals of Λ and $U \subset V$ be the intersections of two open normal subgroups of G with H. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda/I[[G/U]] \arrow[r] \arrow[d] & \Lambda/I[[G/U]]_S \arrow[r] \arrow[d] & \varinjlim_n \Lambda/I[[G/U\Gamma^{\ell^n}]] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda/J[[G/V]] \arrow[r] & \Lambda/J[[G/V]]_S \arrow[r] & \varinjlim_n \Lambda/J[[G/V\Gamma^{\ell^n}]] \arrow[r] & 0
\end{tikzcd}$$

commutes and the downward pointing arrows are surjections. Tensoring with $P^\bullet$ and passing to the inverse limit we obtain the exact sequence

$$0 \to P^\bullet \to \varprojlim_{I,U} \Lambda/I[[G/U]]_S \otimes_{\Lambda[[G]]} P^\bullet \to \varprojlim_{I,U} \varinjlim_n \Lambda/I[[G/U\Gamma^{\ell^n}]] \otimes_{\Lambda[[G]]} P^\bullet \to 0.$$

If $P^\bullet$ is also perfect as a complex of $\Lambda[[H]]$-modules, then complex in the middle is acyclic and we obtain again an isomorphism

$$P^\bullet[1] \cong \varprojlim_{I,U} \varinjlim_n \Lambda/I[[G/U\Gamma^{\ell^n}]] \otimes_{\Lambda[[G]]} P^\bullet$$

in the derived category of complexes of $\Lambda[[G]]$-modules and hence, isomorphisms of $\Lambda[[G]]$-modules

$$\mathrm{H}^{s+1}(P^\bullet) \cong \varprojlim_{I,U} \varinjlim_n \mathrm{H}^s(\Lambda/I[[G/U\Gamma^{\ell^n}]] \otimes_{\Lambda[[G]]} P^\bullet).$$

Here, we use that the modules in the projective system on the righthand side are finite and thus $\varprojlim$-acyclic.

Finally, assume that $(Q^\bullet_J)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$ is a complex in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. Then we can find a strictly perfect complex of $\Lambda[[G]]$-modules $P^\bullet$ and a weak equivalence

$$f\colon (\Lambda[[G]]/J \otimes_{\Lambda[[G]]} P^\bullet)_{J\in\mathfrak{I}_{\Lambda[[G]]}} \to (Q^\bullet_J)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$$

in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda)$ [**Wit08**, Cor. 5.2.6]. Moreover, this complex $P^\bullet$ will also be perfect as a complex of $\Lambda[[H]]$-modules. For $I \in \mathfrak{I}_\Lambda$, U the intersection of an open normal subgroup of G with H and a positive integer n such that Γ^{ℓ^n} is central in G/U we set

$$J_{I,U,n} \coloneqq \ker \Lambda[[G]] \to \Lambda/I[[G/U\Gamma^{\ell^n}]],$$

such that the $J_{I,U,n}$ form a final subsystem in $\mathfrak{I}_{\Lambda[[G]]}$. We conclude:

Proposition 2.8.1. *For $(Q^\bullet_J)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$ in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ there exists an isomorphism*

$$\mathrm{R}\varprojlim_{J\in\mathfrak{I}_{\Lambda[[G]]}} Q^\bullet_J[1] \cong \mathrm{R}\varprojlim_{I,U} \varinjlim_n Q^\bullet_{J_{I,U,n}}$$

in the derived category of $\Lambda[[G]]$*-modules and isomorphisms of* $\Lambda[[G]]$*-modules*

$$\varprojlim_{J\in\mathfrak{J}_{\Lambda[[G]]}} \mathrm{H}^{s+1}(Q_J^\bullet) \cong \varprojlim_{I,U} \varinjlim_{n} \mathrm{H}^s(Q^\bullet_{J_{I,U,n}}).$$

REMARK 2.8.2. For any $(Q_J^\bullet)_{J\in\mathfrak{J}_{\Lambda[[G]]}}$ in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ we obtain in the same way a distinguished triangle

$$\mathrm{R}\varprojlim_{J\in\mathfrak{J}_{\Lambda[[G]]}} Q_J^\bullet \to \mathrm{R}\varprojlim_{I,U}\left(\Lambda/I[[G/U]]_S \otimes^{\mathbb{L}}_{\Lambda/I[[G/U]]} \mathrm{R}\varprojlim_{n} Q^\bullet_{J_{I,U,n}}\right) \to \mathrm{R}\varprojlim_{I,U}\varinjlim_{n} Q^\bullet_{J_{I,U,n}}$$

in the derived category of complexes of $\Lambda[[G]]$-modules.

2.9. Non-Commutative Algebraic L-Functions

Let $G = H \rtimes \Gamma$ as before. Recall the split exact sequence

$$0 \to \mathrm{K}_1(\Lambda[[G]]) \to \mathrm{K}_1(\Lambda[[G]]_S) \xrightarrow{d} \mathrm{K}_0(\Lambda[[G]], S) \to 0.$$

[**Wit13b**, Cor. 3.4], which is central for the formulation of the non-commutative main conjecture: The map $\mathrm{K}_1(\Lambda[[G]]) \to \mathrm{K}_1(\Lambda[[G]]_S)$ is the obvious one; the boundary map

$$d\colon \mathrm{K}_1(\Lambda[[G]]_S) \to \mathrm{K}_0(\Lambda[[G]], S)$$

on the class $[f]$ of an endomorphism f which is a weak equivalence in the Waldhausen category $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ is given by

$$d[f] = -[\mathrm{Cone}(f)^\bullet]$$

where $\mathrm{Cone}(f)^\bullet$ denotes the cone of f [**Wit14**, Thm. A.5]. (Note that other authors use $-d$ instead.) For a fixed choice of a topological generator $\gamma \in \Gamma$, a splitting s_γ of d is given by

$$s_\gamma([P^\bullet]) \coloneqq [\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet \xrightarrow{x\hat{\otimes}y\mapsto x\hat{\otimes}y - x\gamma^{-1}\hat{\otimes}\gamma y} \Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet]^{-1} \tag{2.9.1}$$

for any $P^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$, where the precise definition of $\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet$ as an object of the Waldhausen category $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ is

$$\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}P^\bullet = (\varprojlim_{J\in\mathfrak{J}_{\Lambda[[G]]}} \Lambda[[G]]/I \otimes_{\Lambda[[H]]} P_J^\bullet)_{I\in\mathfrak{J}_{\Lambda[[G]]}}$$

[**Wit13b**, Def. 2.12]. A short inspection of the definition shows that s_γ only depends on the image of γ in G/H. Following [**Bur09**], we may call $s_\gamma(-A)$ the *non-commutative algebraic L-function* of $A \in \mathrm{K}_0(\Lambda[[G]], S)$.

PROPOSITION 2.9.1. *Consider an element* $A \in \mathrm{K}_0(\Lambda[[G]], S)$.

(1) *Let* Λ' *be another adic* $\mathbb{Z}_\ell$*-algebra. For any complex* $P^\bullet$ *of* Λ'-$\Lambda[[G]]$*-bimodules which is strictly perfect as complex of* Λ'*-modules we have*

$$\Psi_{P[[G]]^{\delta\bullet}}(s_\gamma(A)) = s_\gamma(\Psi_{P[[G]]^{\delta\bullet}}(A))$$

in $\mathrm{K}_1(\Lambda'[[G]]_S)$.

(2) *Let* $G' = H' \rtimes \Gamma'$ *such that* H' *has an open, topologically finitely generated pro-*ℓ*-subgroup and* $\Gamma' \cong \mathbb{Z}_\ell$*. Assume that* $\alpha\colon G \to G'$ *is a continuous homomorphism such that* $\alpha(G) \not\subset H'$*. Set* $r \coloneqq [G' : \alpha(G)H']$*. Let* $\gamma' \in \Gamma'$ *be a topological generator such that* $\alpha(\gamma) = (\gamma')^r$ *in* G'/H'*. Then*

$$\Psi_{\Lambda[[G']]}(s_\gamma(A)) = s_{\gamma'}(\Psi_{\Lambda[[G']]}(A))$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$.

(3) *Assume that G' is an open subgroup of G and set $H' \coloneqq H \cap G'$, $r \coloneqq [G : G'H]$. Consider $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then γ^r generates $G'/H' \subset G/H$ and*

$$\Psi_{\Lambda[[G]]}(s_\gamma(A)) = s_{\gamma^r}(\Psi_{\Lambda[[G]]}(A))$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$.

Proof. For (1), we first note that by applying the Waldhausen additivity theorem [**Wal85**, Prop. 1.3.2] to the short exact sequences resulting from stupid truncation, we have

$$\Psi_{P[[G]]^{\delta\bullet}} = \sum_{i\in\mathbb{Z}} (-1)^i \Psi_{P^i[[G]]^\delta}$$

as homomorphisms between the K-groups. Hence we may assume that $P = P^\bullet$ is concentrated in degree 0. We now apply [**Wit13b**, Prop 2.14.1] to the $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodule $M \coloneqq P[[G]]^\delta$ and its $\Lambda'[[H]]$-$\Lambda[[H]]$-sub-bimodule

$$N \coloneqq \Lambda'[[H]] \otimes_{\Lambda'} P$$

(with the diagonal right action of H) and $t_1 \coloneqq t_2 \coloneqq \gamma - 1$, $\gamma_1 \coloneqq \gamma_2 \coloneqq \gamma$.

For (2), we first assume that α induces an isomorphism $G/H \cong G/H'$ and that $\gamma' = \alpha(\gamma)$. We then apply [**Wit13b**, Prop 2.14.1] to $M \coloneqq \Lambda[[G']]$, $N \coloneqq \Lambda[[H']]$, and $t_1 \coloneqq \gamma - 1$, $t_2 \coloneqq \alpha(\gamma) - 1$, $\gamma_1 \coloneqq \gamma$, $\gamma_2 \coloneqq \alpha(\gamma)$.

Next, we assume that $G \subset G'$, $H = H'$, and $\gamma = (\gamma')^r$. This case is not covered by [**Wit13b**, Prop 2.14] and therefore, we will give more details. Consider the isomorphism of $\Lambda[[G']]$-$\Lambda[[G]]$-bimodules

$$\kappa\colon \Lambda[[G']]\hat{\otimes}_{\Lambda[[H]]}\Lambda[[G]]^r \to \Lambda[[G']]\hat{\otimes}_{\Lambda[[H]]}\Lambda[[G']],$$
$$\mu\hat{\otimes}\begin{pmatrix}\lambda_0\\ \vdots \\ \lambda_{r-1}\end{pmatrix} \mapsto \sum_{i=0}^{r-1} \mu(\gamma')^{-i}\hat{\otimes}(\gamma')^i(\lambda_i).$$

Then the map $\mu\hat{\otimes}\lambda \mapsto \mu\hat{\otimes}\lambda - \mu(\gamma')^{-1}\hat{\otimes}\gamma'\lambda$ on the righthand side corresponds to left multiplication with the matrix

$$A \coloneqq \begin{pmatrix} \mathrm{id} & 0 & \cdots & 0 & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ -\mathrm{id} & \mathrm{id} & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \mathrm{id} & 0 \\ 0 & \cdots & 0 & -\mathrm{id} & \mathrm{id} \end{pmatrix}$$

on the left-hand side. Let $P^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. Then κ induces an isomorphism

$$\kappa\colon \Psi_{\Lambda[[G']]}(\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}(P^\bullet)^r) \to \Lambda[[G']]\hat{\otimes}_{\Lambda[[H]]}\Psi_{\Lambda[[G']]}(P^\bullet)$$

in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G']])$ while $A \circlearrowleft \Psi_{\Lambda[[G']]}(\Lambda[[G]]\hat{\otimes}_{\Lambda[[H]]}(P^\bullet)^r)$ is a weak equivalence. Hence,

$$[A]^{-1} = s_{\gamma'}([\Psi_{\Lambda[[G']]}(P^\bullet)])$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$. Moreover,

$$\begin{pmatrix} \mathrm{id} & 0 & \cdots & \cdots & 0 \\ \mathrm{id} & \mathrm{id} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathrm{id} & 0 \\ \mathrm{id} & \cdots & \cdots & \mathrm{id} & \mathrm{id} \end{pmatrix} A = \begin{pmatrix} \mathrm{id} & 0 & \cdots & 0 & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ 0 & \mathrm{id} & \ddots & \vdots & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \mathrm{id} & -(\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \\ 0 & \cdots & \cdots & 0 & \mathrm{id} - (\cdot\gamma^{-1})\hat{\otimes}(\gamma\cdot) \end{pmatrix}.$$

The relations (R1)–(R7) in the definition of $\mathcal{D}_\bullet(\mathbf{W})$ imply that the class of a triangular matrix is the product of the classes of its diagonal entries in $\mathrm{K}_1(\Lambda[[G']]_S)$. Hence, $[A]^{-1} = \Psi_{\Lambda[[G']]}(s_\gamma[P^\bullet])$, as desired.

In the general case, we note that the image of α is contained in the subgroup G'' of G' topologically generated by $(\gamma')^r$ and H' and recall that s_γ only depends on the image of γ in G/H. We are then reduced to the two cases already treated above.

For (3), we first treat the case $r = 1$, i. e. $G' \to G/H$ is a surjection. Hence, we may assume $\gamma \in G'$. We then apply [**Wit13b**, Prop 2.14.1] to $M \coloneqq \Lambda[[G]]$, $N \coloneqq \Lambda[[H]]$, and $t_1 \coloneqq t_2 \coloneqq \gamma - 1$, $\gamma_1 \coloneqq \gamma_2 \coloneqq \gamma$ as above. If $r > 1$ we can thus reduce to the case that G' is topologically generated by H and γ^r and apply [**Wit13b**, Prop 2.14.2].

In [**Wit13b**], we use a slightly different Waldhausen category for the construction of the K-theory of $\Lambda[[G]]$, but the proof of [**Wit13b**, Prop 2.14] goes through without changes. $\square$

Example 2.9.2.

(1) Assume that M is a $\Lambda[[G]]$-module which is finitely generated and projective as a $\Lambda[[H]]$-module. Then the complex

$$C_\gamma(M)\colon \qquad \underbrace{\Lambda[[G]] \otimes_{\Lambda[[H]]} M}_{\text{degree } -1} \xrightarrow{\mathrm{id}-(\cdot\gamma^{-1}\otimes\gamma\cdot)} \underbrace{\Lambda[[G]] \otimes_{\Lambda[[H]]} M}_{\text{degree } 0}$$

is an object of $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ whose cohomology is M in degree 0 and zero otherwise. Moreover,

$$s_\gamma([M]) = s_\gamma([C_\gamma(M)]) = [\mathrm{id} - (\cdot\gamma^{-1} \otimes \gamma\cdot) \circlearrowright \Lambda[[G]] \otimes_{\Lambda[[H]]} M]^{-1}$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$. If $\Lambda[[G]]$ is commutative, then the image of the element $s_\gamma([M])^{-1}$ under

$$\det\colon \mathrm{K}_1(\Lambda[[G]]_S) \to \Lambda[[G]]_S^\times$$

is precisely the reverse characteristic polynomial

$$\det_{\Lambda[[H]][t]}(\mathrm{id} - (\cdot t \otimes \gamma\cdot) \circlearrowright \Lambda[[H]][t] \otimes_{\Lambda[[H]]} M)$$

evaluated at $t = \gamma^{-1} \in \Gamma$. In fact, one may extend this to non-commutative $\Lambda[[H]]$ and $G = H \times \Gamma$ as well, using the results of the appendix.

(2) If $M = \Lambda[[G]]/\Lambda[[G]]f$ with

$$f \coloneqq t^n + \sum_{i=0}^{n-1} \lambda_i t^i \in \Lambda[[G]]$$

a polynomial of degree n in $t \coloneqq \gamma - 1$ with $\lambda_i \in \mathrm{Jac}(\Lambda[[H]])$, then M is finitely generated and free as $\Lambda[[H]]$-module. A $\Lambda[[H]]$-basis is given by the residue classes of $1, t, \dots, t^{n-1} \in \Lambda[[G]]$. If we use this basis to identify $\Lambda[[G]] \otimes_{\Lambda[[H]]} M$ with $\Lambda[[G]]^n$, then the $\Lambda[[G]]$-linear endomorphism $\mathrm{id} - (\cdot\gamma^{-1} \otimes \gamma\cdot)$ is given by right multiplication with the matrix

$$A \coloneqq \begin{pmatrix} \gamma^{-1}t & -\gamma^{-1} & 0 & \cdots & 0 \\ 0 & \gamma^{-1}t & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma^{-1}t & -\gamma^{-1} \\ \gamma^{-1}\lambda_0 & \gamma^{-1}\lambda_1 & \cdots & \gamma^{-1}\lambda_{n-2} & \gamma^{-1}(t+\lambda_{n-1}) \end{pmatrix}.$$

By right multiplication with

$$E \coloneqq \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ t & 1 & \ddots & & \vdots \\ t^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ t^{n-1} & \cdots & t^2 & t & 1 \end{pmatrix}$$

one can transform A into

$$A' \coloneqq \begin{pmatrix} 0 & -\gamma^{-1} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -\gamma^{-1} \\ \gamma^{-1} f & \gamma^{-1}(t^{n-1} + \sum_{i=1}^{n-1} \lambda_i t^{i-1}) & \cdots & \gamma^{-1}(t^2 + \lambda_{n-1} t + \lambda_{n-2}) & \gamma^{-1}(t + \lambda_{n-1}) \end{pmatrix}.$$

By left multiplication with

$$P \coloneqq \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

one can exchange the first and last row of A' to obtain a triangular matrix. In $\mathrm{K}_1(\Lambda[[G]])$, we have

$$\begin{aligned} [\, \cdot E \circlearrowleft \Lambda[[G]]^n \,] &= 1, \\ [\, \cdot P \circlearrowleft \Lambda[[G]]^n \,] &= [\, -1 \circlearrowleft \Lambda[[G]] \,]^{n-1}. \end{aligned}$$

We conclude

$$\begin{aligned} s_\gamma([M])^{-1} &= [\, \cdot A \circlearrowleft \Lambda[[G]]^n \,] \\ &= [\, \cdot A' \circlearrowleft \Lambda[[G]]^n \,] \\ &= [\, \cdot P \circlearrowleft \Lambda[[G]]^n \,]^{-1} [\, \cdot(-\gamma^{-1}) \circlearrowleft \Lambda[[G]] \,]^{n-1} [\, \cdot \gamma^{-1} f \circlearrowleft \Lambda[[G]] \,] \\ &= [\, \cdot \gamma^{-n} f \circlearrowleft \Lambda[[G]] \,]. \end{aligned}$$

The section $s_\gamma \colon \mathrm{K}_0(\Lambda[[G]], S) \to \mathrm{K}_1(\Lambda[[G]]_S)$ also commutes with the homomorphisms $\circledast \colon \mathrm{K}_0(\Lambda[[G]]) \to \mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]])$, $\circledast \colon \mathrm{K}_1(\Lambda[[G]]_S) \to \mathrm{K}_0(\Lambda^{\mathrm{op}}[[G]]_S)$ from Definition 2.7.1 in the following sense.

Proposition 2.9.3. *For any element* $A \in \mathrm{K}_0(\Lambda[[G]], S)$,

$$s_{\gamma^{-1}}(A)^{\circledast} = s_\gamma(A^{\circledast})$$

in $\mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]]_S)$.

Proof. Since $\mathrm{K}_0(\mathbf{SP}(\Lambda[[H]], G))$ surjects onto $\mathrm{K}_0(\Lambda[[G]], S)$ by Proposition 2.5.9, it suffices to prove the formula for $C_\gamma(M)$ with M a $\Lambda[[G]]$-module that is finitely generated and projective over $\Lambda[[H]]$. The equality is then a direct consequence of the diagram in Lemma 2.7.5. □

Remark 2.9.4. Note that

$$s_\gamma([C_\gamma(M)]) = s_{\gamma^{-1}}([C_\gamma(M)])[-(\cdot \gamma^{-1} \otimes \gamma \cdot) \circlearrowleft \Lambda[[G]] \otimes_{\Lambda[[H]]} M]$$

for any topological generator γ of Γ and any $\Lambda[[G]]$-module M that is finitely generated and projective over $\Lambda[[H]]$.

2.10. Regular Coefficient Rings

Assume that R is a commutative, local, and regular adic $\mathbb{Z}_\ell$-algebra. By the Cohen structure theorem [**Bou89**, Ch. VIII, §5, Thm. 2], we have

$$R \cong R_0[[X_1, \dots, X_n]]$$

where R_0 is either a finite field of characteristic ℓ or the valuation ring of a finite field extension of $\mathbb{Q}_\ell$ and $X_1, \dots, X_n$ are indeterminates. In particular, we may identify R with the profinite group algebra of $\mathbb{Z}_\ell^n$ with coefficients in R_0.

If $G = H \rtimes \Gamma$ is an ℓ-adic Lie group without elements of order ℓ, then the rings $R[[G]]$ and $R[[H]]$ are both noetherian and of finite global dimension [**Bru66**, Thm. 4.1]. Let $\mathbf{N}_H(R[[G]])$ denote the abelian category of finitely generated $R[[G]]$-modules which are also finitely generated as $R[[H]]$-modules. Note that

$$\begin{aligned} &\mathrm{K}_0(\mathbf{SP}^{w_H}(R[[G]])) \to \mathrm{K}_0(\mathbf{N}_H(R[[G]])), \\ &[P^\bullet] \mapsto \sum_{s=-\infty}^{\infty} (-1)^s [\mathrm{H}^s(P^\bullet)] \end{aligned} \tag{2.10.1}$$

is an isomorphism. The inverse is given by the construction in Remark 2.5.11. The same argument also shows that

$$\mathrm{K}_0(\mathbf{SP}(R[[H]], G)) \cong \mathrm{K}_0(\mathbf{N}_H(R[[G]])),$$

providing some evidence to the conjectured isomorphism in Remark 2.5.10.

If the quotient field of R is of characteristic 0, one may also consider the abelian category $\mathbf{M}_H(R[[G]])$ of finitely generated $R[[G]]$-modules whose ℓ-torsionfree part is finitely generated as $R[[H]]$-module and the left denominator set

$$S^* := \bigcup_n \ell^n S \subset R[[G]].$$

Still assuming that G has no element of order ℓ it is known that the natural maps

$$\mathrm{K}_1(R[[G]]_S) \to \mathrm{K}_1(R[[G]]_{S^*}), \qquad \mathrm{K}_0(\mathbf{N}_H(R[[G]])) \to \mathrm{K}_0(\mathbf{M}_H(R[[G]]))$$

are split injective [**BV11**, Prop. 3.4] and fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}_1(R[[G]]) \arrow[r] \arrow[d, "="] & \mathrm{K}_1(R[[G]]_S) \arrow[r, "d"] \arrow[d] & \mathrm{K}_0(\mathbf{N}_H(R[[G]])) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{K}_1(R[[G]]) \arrow[r] & \mathrm{K}_1(R[[G]]_{S^*}) \arrow[r, "d"] & \mathrm{K}_0(\mathbf{M}_H(R[[G]])) \arrow[r] & 0 \end{tikzcd}$$

In particular, an identity of the type $f = dg$ in $\mathrm{K}_0(\mathbf{N}_H(R[[G]]))$ will imply a corresponding identity in $\mathrm{K}_0(\mathbf{M}_H(R[[G]]))$. It is $\mathbf{M}_H(R[[G]])$ which plays a central role in the original formulation of the non-commutative Iwasawa Main Conjecture [**CFK**$^+$**05**]. However, we will not make use of $\mathbf{M}_H(R[[G]])$ in the following.

Assume that T is a $R[[G]]$-module that is finitely generated as R-module. Quite often, the class $[T]$ is zero in $\mathrm{K}_0(R[[G]], S)$ if H is infinite. However, this is not always the case. Since the forgetful functor from $\mathbf{N}_H(R[[G]])$ to the category of finitely generated $R[[H]]$-modules induces a homomorphism $\mathrm{K}_0(R[[G]], S) \to \mathrm{K}_0(R[[H]])$, a necessary condition is that $[T]$ is zero in $\mathrm{K}_0(R[[H]])$. For this condition, we can formulate the following useful criterion, which is essentially due to Serre (see also [**AW08**, §1.3]). In particular, we see that this condition is not satisfied by the group $H = \mathbb{Z}_\ell^d \rtimes \mu_{\ell-1}$ with the group of $\ell-1$-th roots of units acting by multiplication on $\mathbb{Z}_\ell^d$ if $\ell > 2$.

Recall that an ℓ-adic Lie group H is called *virtually solvable* if its Lie algebra $L(H)$ is solvable.

LEMMA 2.10.1. *Let H be a compact ℓ-adic Lie group without any element of order ℓ and R a commutative, local, regular adic $\mathbb{Z}_\ell$-algebra. The class $[T]$ of every $R[[H]]$-module T which is finitely generated as R-module is zero in $\mathrm{K}_0(R[[H]])$ precisely if the centraliser of every element of finite order in H has infinitely many elements. This condition is satisfied if H is a pro-ℓ-group or if H is not virtually solvable.*

PROOF. By the Cohen structure theorem [**Bou89**, Ch. VIII, §5, Thm. 2], we have

$$R \cong R_0[[X_1, \dots, X_n]]$$

where R_0 is either a finite field of characteristic ℓ or the valuation ring of a finite field extension of $\mathbb{Q}_\ell$ and $X_1, \dots, X_n$ are indeterminates. We do induction on n.

Assume that $n = 0$ and that R_0 is a finite field. By [**Ser98**, Cor. to Thm. C] the Euler characteristic of every $R_0[[H]]$-module T which is of finite dimension over R_0 is trivial precisely if the centraliser of every element of H has infinitely many elements. For any element of infinite order this is clearly an empty condition. Now the proof of [**AW06**, Thm. 8.2, $(a) \Rightarrow (b)$] shows that the vanishing of the Euler characteristics is equivalent with the vanishing of the classes $[T]$.

If H is a pro-ℓ-group without any element of order ℓ, then there are no elements of finite order at all. If H is not virtually solvable, then its Lie algebra $L(H)$ is not solvable. Since any element $h \in H$ of finite order has order prime to ℓ, the image of h in the automorphism group of $L(H)$ must be semi-simple. By an old result of Borel and Mostow [**BM55**, Thm. 4.5] (the author thanks S. Wadsley for pointing out this reference to him), h fixes a non-trivial subspace of $L(H)$, which implies that the centraliser of h in H must be infinite.

Now assume that R_0 is the valuation ring of a finite field extension of $\mathbb{Q}_\ell$. Let $\pi \in R_0$ be a uniformiser and k the residue field of R_0. By Quillen's dévissage theorem [**Qui73**, Thm. 4], we may identify $\mathrm{K}_0(k[[H]])$ with the K-group of the abelian category of finitely generated $R_0[[H]]$-modules that are annihilated by a power of π. Under this identification, classes of those $R_0[[H]]$-modules which are additionally finitely generated over R_0 are mapped to the subgroup of $\mathrm{K}_0(k[[H]])$ generated by the classes of those $k[[H]]$-modules that are finitely generated over k. By Quillen's localisation theorem [**Qui73**, Thm. 5] we thus obtain an exact sequence

$$\mathrm{K}_0(k[[H]]) \to \mathrm{K}_0(R_0[[H]]) \to \mathrm{K}_0(R_0[[H]][\frac{1}{\pi}]) \to 0,$$

noting that all rings in this sequence are of finite global dimension. For any $R_0[[H]]$-module T which is finitely generated over R_0, there exists an exact sequence of $R_0[[H]]$-modules, finitely generated over R_0,

$$0 \to T' \to T \to T'' \to 0$$

where T' is annihilated by a power of π and π is a non-zero divisor on T''.

Assume that the centraliser of every element of finite order in H has infinitely many elements. We already know that $[T'] = 0$. Hence, we may assume that π is a non-zero divisor on T. In particular,

$$k[[H]] \otimes_{R_0[[H]]} T = k \otimes_{R_0} T$$

agrees with the derived tensor product with $k[[H]]$ over $R_0[[H]]$ and is finitely generated over k. Hence, $[k \otimes_{R_0} T] = 0$ in $\mathrm{K}_0(k[[H]])$. Since π is in the Jacobson radical of $R_0[[H]]$, the derived tensor product with $k[[H]]$ induces an isomorphism

$$\mathrm{K}_0(R_0[[H]]) \to \mathrm{K}_0(k[[H]]).$$

Hence $[T] = 0$ in $\mathrm{K}_0(R_0[[H]])$.

Conversely, if H does not satisfy the above property, we may find a $k[[H]]$-module T which is finitely generated over k and which has non-trivial class $[T]$ in $\mathrm{K}_0(k[[H]])$. The image of H in the automorphism group of T is a finite group Δ. By [**Ser77**, Thm. 33], $[T]$ has a preimage in $\mathrm{K}_0(R_0[[H]])$ consisting of a linear combination of classes of finitely generated $R_0[\Delta]$-modules which are free as R_0-modules. Hence, there exist $R_0[[H]]$-modules which are finitely generated and free over R_0 and have non-trivial class in $\mathrm{K}_0(R_0[[H]])$.

The same argumentation still works for R_0 replaced by R, k replaced by $R' \coloneqq R_0[[X_1, \dots, X_{n-1}]]$ and π replaced by X_n. The lifting argument in the last step becomes a bit easier. If T is a $R'[[H]]$-module which is finitely generated and free over R', then $T' \coloneqq R \otimes_{R'} T$ is a $R[[H]]$-module which is finitely generated and free over R and satisfies $R' \otimes_R T' \cong T$. This completes the induction step. □

However, the vanishing in $\mathrm{K}_0(R[[H]])$ is not sufficient. Here is an example. Assume that $G =< \tau, \gamma >\cong \mathbb{Z}_\ell \rtimes \mathbb{Z}_\ell$ with $\gamma^{-1}\tau\gamma = \tau^{1+\ell}$. Set $H \coloneqq< \tau >$ and consider the constant $\mathbb{Z}_\ell[[G]]$-module $\mathbb{Z}_\ell$. Clearly, $[\mathbb{Z}_\ell] = 0$ in $\mathrm{K}_0(\mathbb{Z}_\ell[[H]])$ according to Lemma 2.10.1. However, $[\mathbb{Z}_\ell] \neq 0$ in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], S)$. Indeed, the complex

$$\mathbb{Z}_\ell[[G]] \xrightarrow{v\mapsto(v-v\tau^{1+\ell},v-v(\sum_{i=0}^{\ell}\tau^i)\gamma)} \mathbb{Z}_\ell[[G]]^2 \xrightarrow{(v,w)\mapsto v-v\gamma-w+w\tau^{1+\ell}} \mathbb{Z}_\ell[[G]]$$

is a projective resolution of $\mathbb{Z}_\ell$. Hence, the image of $[\mathbb{Z}_\ell]$ in

$$\mathrm{K}_0(\mathbb{Z}_\ell[[\Gamma]], S) = \mathbb{Z}_\ell[[\Gamma]]_S^\times / \mathbb{Z}_\ell[[\Gamma]]^\times$$

under the natural projection map is given by the class of

$$\frac{1-(\ell+1)\gamma}{1-\gamma} \in \mathbb{Z}_\ell[[\Gamma]]_S^\times,$$

which is not in $\mathbb{Z}_\ell[[\Gamma]]^\times$.

A sufficient criterion for the vanishing of the class $[T]$ in $\mathrm{K}_0(R[[G]], S)$ is given in [**FK06**, Prop. 4.3.17]. Here is another one, inspired by [**Záb10**, Prop. 4.2].

Proposition 2.10.2. *Let $G = H \rtimes \Gamma$ be an ℓ-adic Lie group without elements of order ℓ. Assume that there exists a closed normal subgroup $N \subset G$ such that*

(1) *G/N has no elements of order ℓ and the centraliser of every element in G/N of finite order has infinitely many elements,*
(2) *the image of H in G/N is open.*

Let further R be a commutative, local, regular adic $\mathbb{Z}_\ell$-algebra. Then the class of every $R[[G]]$-module which is finitely generated as R-module is zero in $\mathrm{K}_0(R[[G]], S)$.

Proof. By assumption (1) and Lemma 2.10.1 the constant $R[[G/N]]$-module R has trivial class in $\mathrm{K}_0(R[[G/N]])$. Set $G' \coloneqq G/N \times G/H$. Every $R[[G/N]]$-module may be considered as $R[[G']]$-module by letting G/H act trivially. Thus, we see that $[R] = 0$ in $\mathrm{K}_0(\mathbf{N}_{G/N}(R[[G']]))$, as well. By assumption (2) every finitely generated $R[[G']]$-module which is finitely generated as $R[[G/N]]$-module may be considered via

$$G \to G', \qquad g \mapsto (gN, gH)$$

as a finitely generated $R[[G]]$-module which is also finitely generated as $R[[H]]$-module. This induces an exact functor $\mathbf{N}_{G/N}(R[[G']]) \to \mathbf{N}_H(R[[G]])$ and hence, a homomorphism between the corresponding K-groups. We conclude that $[R] = 0$ also in $\mathrm{K}_0(\mathbf{N}_H(R[[G]]))$.

If T is a $R[[G]]$-module which is finitely generated and free as R-module and M is any module in $\mathbf{N}_H(R[[G]])$, we let $\mathrm{Tor}_i^R(T, M)$ denote the i-th left derived functor of the tensor product $T \otimes_R M$ with the diagonal action of G. Since R is noetherian, any finitely generated, projective $R[[H]]$-module is flat as R-module and the same is true for $R[[G]]$. In particular, it does not matter if we compute $\mathrm{Tor}_i^R(T, M)$ in the category of finitely generated $R[[G]]$-modules or $R[[H]]$-modules or R-modules. Hence, $\mathrm{Tor}_i^R(T, M)$ is again in $\mathbf{N}_H(R[[G]])$ and it is finitely generated over R if M is a finitely generated R-module. We thus obtain an endomorphism

$$\mathrm{K}_0(\mathbf{N}_H(R[[G]])) \to \mathrm{K}_0(\mathbf{N}_H(R[[G]])), \qquad [M] \mapsto \sum_{i=0}^{\infty}(-1)^i[\mathrm{Tor}_i^R(T, M)],$$

which maps $[R]$ to $[T]$. In particular, $[T] = 0$. $\square$

Corollary 2.10.3. *Let $G = H \rtimes \Gamma$ be an ℓ-adic Lie group. Assume that*

(1) *H is not virtually solvable and has no elements of order ℓ,*

(2) *the Lie algebra* $L(G)$ *of* G *decomposes as*

$$L(G) = L(H) \oplus V$$

with $L(H)$ *the Lie algebra of* H *and some ideal* V *of* $L(G)$,

(3) $\ell - 1 > 2\dim_{\mathbb{Q}_\ell} L(H)$.

Let further R *be a commutative, local, regular adic* $\mathbb{Z}_\ell$*-algebra. Then the class of every* $R[[G]]$*-module which is finitely generated as* R*-module is zero in* $\mathrm{K}_0(R[[G]], S)$.

PROOF. By assumption (2) there exist a characteristic open subgroup $H' \subset H$ and a closed subgroup $\Gamma' \cong \mathbb{Z}_\ell$ of the centraliser $Z_G(H')$ of H' in G such that $H' \cap \Gamma' = 1$ and $H'\Gamma'$ is open in G. We may also assume that H' is a uniformly powerful pro-ℓ-group in the sense of [**DdSMS99**, Def. 4.1]. Set $d \coloneqq \dim_{\mathbb{Q}_\ell} L(H)$. By [**DdSMS99**, Cor. 4.18] the automorphism group of H' is isomorphic to a closed subgroup of $\mathrm{Gl}_d(\mathbb{Z}_\ell)$, which does not have elements of order ℓ by assumption (3). In particular, this is also true for $G/Z_G(H')$, as this group acts faithfully on H' by conjugation. Since $\Gamma' \subset Z_G(H')$, the image of H' must be open in $G/Z_G(H')$. This image is just the quotient of H' by its centre $Z(H')$. Since H' is not virtually solvable, the same must be true for $H'/Z(H')$ and therefore, also for $G/Z_G(H')$. We may thus apply Proposition 2.10.2 with $N = Z_G(H')$. □

REMARK 2.10.4. In fact, one can replace condition (1) in Proposition 2.10.2 by

(1)' $[R] = 0$ in $\mathrm{G}_0(R[[G/N]])$.

where $\mathrm{G}_0(R[[G/N]])$ is the Grothendieck group of all finitely generated $R[[G/N]]$-modules, dropping the assumption that G/N has no elements of order ℓ. Presumably, $(1)'$ is satisfied for all G/N which are not virtually solvable. If this is true, then one can drop assumption (3) in Corollary 2.10.3.

CHAPTER 3

Perfect Complexes of Adic Sheaves

We will use étale cohomology instead of Galois cohomology to formulate the main conjecture. The main advantage is that we have a little bit more flexibility in choosing our coefficient systems. Instead of being restricted to locally constant sheaves corresponding to Galois modules, we can work with constructible sheaves. An alternative would be the use of cohomology for Galois modules with local conditions, in the style of [**Nek06**].

As Waldhausen models for the derived categories of complexes of constructible sheaves, we will use the Waldhausen categories of complexes of Λ-adic sheaves introduced in [**Wit08**, § 5.4–5.5] for separated schemes of finite type over a finite field. We will need them in the case of subschemes U of a smooth and proper curve X with function field F. The same constructions still work with some minor changes if we consider subschemes U of the spectrum X of the ring of integers of a number field F. We will give the definition in Section 3.1. Moreover, we will define derived direct images, exceptional direct images, inverse images, exceptional inverse images as well as derived tensor products as Waldhausen exact functors. In Section 3.2, we consider local and global duality theorems for smooth Λ-adic sheaves.

We then recall in Section 3.3 the notion of an *admissible extension* F_∞/F of a global field F. By definition, F_∞ contains the cyclotomic $\mathbb{Z}_\ell$-extension F_{cyc} of F, such that the Galois group G of F_∞/F may be written as the semi-direct product of the Galois group H of $F_\infty/F_{\mathrm{cyc}}$ and a subgroup $\Gamma \cong \mathbb{Z}_\ell$. If U is an open subscheme of X such that F_∞/F is unramified over U, then we may associate to each perfect complex of Λ-adic sheaves $\mathcal{F}^\bullet$ on U a compactly induced complex of $\Lambda[[G]]$-adic sheaves $f_!f^*\mathcal{F}^\bullet$ on U.

Section 3.4 contains the proof of a key assertion of the main conjecture. We prove that the total complex of cohomology with proper support of $f_!f^*\mathcal{F}^\bullet$ is not only perfect as complex of $\Lambda[[G]]$-modules, but also as complex of $\Lambda[[H]]$-modules. If $\Lambda[[H]]$ is noetherian, this signifies that the cohomology groups with proper support are S-torsion if S denotes Venjakob's canonical Ore set. In the number field case, we have to restrict to totally real fields and assume the vanishing of Iwasawa's μ-invariant. We also consider a local variant of this S-torsion property.

This local variant permits us to introduce the notion of non-commutative Euler factors by producing canonical characteristic elements for the complexes $\mathrm{R}\Gamma(x, i^*\,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ for the embedding $k\colon U \to W$ and a closed point $i\colon x \to W$ of W. Comparing Euler factors with the non-commutative algebraic L-functions of these complexes, we obtain certain elements in $\mathrm{K}_1(\Lambda[[G]])$, which we call local modification factors. In the same way, we also introduce the notion of dual non-commutative Euler factors by producing canonical characteristic elements for the complexes $\mathrm{R}\Gamma(x, i^!k_!\mathcal{F}^\bullet)$ and the corresponding dual local modification factors. The investigation of the Euler factors and local modification factors is carried out in Section 3.5 and Section 3.6, first in general, then in the special case of the cyclotomic extension.

3.1. Adic Sheaves

Let F be a global field with ring of integers $\mathcal{O}_F$. We also fix a separable closure $\overline{F}$ of F. If F is a number field, we set $X \coloneqq \operatorname{Spec} \mathcal{O}_F$. If F is a function field of characteristic p, we let $X = X_F$ denote the smooth and proper curve associated to F. Further, we write $\mathbb{F}$ and $\overline{\mathbb{F}}$ for the algebraic closure of the prime field $\mathbb{F}_p$ in F and $\overline{F}$, respectively. For any open or closed subscheme U of X, we then write

$$\overline{U} = U \times_{\operatorname{Spec} \mathbb{F}} \operatorname{Spec} \overline{\mathbb{F}}$$

for the base change to the algebraic closure. In particular, $\overline{U}$ is connected if U is an open dense subscheme of X.

Assume that U is an open or closed subscheme of X. Recall that for a finite ring R, a complex $\mathcal{F}^\bullet$ of étale sheaves of left R-modules on U is called *strictly perfect* if it is strictly bounded and each $\mathcal{F}^n$ is constructible and flat. It is *perfect* if it is quasi-isomorphic to a strictly perfect complex. We call it *DG-flat* if for each geometric point of U, the complex of stalks is DG-flat.

Fix a prime ℓ. Let Λ be an adic $\mathbb{Z}_\ell$-algebra.

Definition 3.1.1. The *category* $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ *of perfect complexes of adic sheaves on* U is the following Waldhausen category. The objects of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ are inverse systems $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ such that:

1. for each $I \in \mathfrak{I}_\Lambda$, $\mathcal{F}_I^\bullet$ is DG-flat perfect complex of étale sheaves of Λ/I-modules on U,
2. for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism
$$\varphi_{IJ} : \mathcal{F}_I^\bullet \to \mathcal{F}_J^\bullet$$
of the system induces an isomorphism
$$\Lambda/J \otimes_{\Lambda/I} \mathcal{F}_I^\bullet \xrightarrow{\sim} \mathcal{F}_J^\bullet.$$

Weak equivalences and cofibrations are defined as in Definition 2.4.1.

Definition 3.1.2. Any system $\mathcal{F} = (\mathcal{F}_I)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ consisting of flat, constructible sheaves $\mathcal{F}_I$ of Λ/I-modules on U, regarded as complexes concentrated in degree 0, will be called a Λ*-adic sheaf* on U. If in addition, the $\mathcal{F}_I$ are locally constant, we call $\mathcal{F}$ a *smooth* Λ*-adic sheaf*. We write $\mathbf{S}(U, \Lambda)$ and $\mathbf{S}^{\mathrm{sm}}(U, \Lambda)$ for the full Waldhausen categories of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ consisting of Λ-adic sheaves and smooth Λ-adic sheaves, respectively.

Note that if U is a closed subscheme of X, then every Λ-adic sheaf on U is automatically smooth.

Definition 3.1.3. Assume that $\ell \neq 2$ and that F is a number field. If U is an open dense subscheme of $X = \operatorname{Spec} \mathcal{O}_F$, we will call a complex $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ to be *smooth at* ∞ if for each $I \in \mathfrak{I}_\Lambda$, the stalk of $\mathcal{F}_I^\bullet$ in $\operatorname{Spec} \overline{F}$ is quasi-isomorphic to a strictly perfect complex of Λ/I-modules with trivial action of any complex conjugation $\sigma \in \operatorname{Gal}_F$. The full subcategory of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ of complexes smooth at ∞ will be denoted by

$$\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$$

Since we assume $\ell \neq 2$, it is immediate that if in an exact sequence

$$0 \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$, the complexes $\mathcal{F}^\bullet$ and $\mathcal{H}^\bullet$ are smooth at ∞, then so is $\mathcal{G}^\bullet$. It then follows from [**Wit08**, Prop. 3.1.1] that $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$ is a Waldhausen subcategory of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.

We will write Λ_U for the smooth Λ-adic sheaf on U given by the system of constant sheaves $(\Lambda/I)_{I\in\mathfrak{I}_\Lambda}$ on U. Further, if ℓ is invertible on U, we will write μ_{ℓ^n} for the sheaf of ℓ^n-th roots of unity on U, and

$$(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}(1) = (\varprojlim_n \mu_{\ell^n} \otimes_{\mathbb{Z}_\ell} \mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$$

for the Tate twist of a complex in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.

We will consider Godement resolutions of the complexes in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. To be explicit, we will fix for each place x of F an embedding $\overline{F} \subset \overline{F}_x$ into a fixed separable closure of the local field F_x in x. In particular, we also obtain an embedding of the residue field $k(x)$ of x into the separably closed residue field $\overline{k(x)}$ of $\overline{F}_x$ for each closed point x of U. We write $\hat{x}$ for the corresponding geometric point $\hat{x}\colon \operatorname{Spec} \overline{k(x)} \to U$ over x and let U^0 denote the set of closed points of U.

For each étale sheaf $\mathcal{F}$ on U we set

$$(\mathrm{G}_U\, \mathcal{F})^n \coloneqq \underbrace{\prod_{u\in U^0} \hat{u}_*\hat{u}^* \cdots \prod_{u\in U^0} \hat{u}_*\hat{u}^*}_{n+1} \mathcal{F}$$

and turn $(\mathrm{G}_U\, \mathcal{F})^\bullet$ into a complex by taking as differentials

$$\partial^n\colon (\mathrm{G}_U\, \mathcal{F})^n \to (\mathrm{G}_U\, \mathcal{F})^{n+1}$$

the alternating sums of the maps induced by the natural transformation

$$\mathcal{F} \to \prod_{u\in U^0} \hat{u}_*\hat{u}^*\mathcal{F}.$$

The Godement resolution of a complex of étale sheaves is given by the total complex of the corresponding double complex as in [**Wit08**, Def. 4.2.1]. The Godement resolution of a complex $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ is given by applying the Godement resolution to each of the complexes $\mathcal{F}_I^\bullet$ individually.

If $j\colon U \to V$ is an open immersion, we set

$$j_!(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \coloneqq (j_!\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda},$$
$$\mathrm{R}\, j_*(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \coloneqq (j_*\, \mathrm{G}_U\, \mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}.$$

for any $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. While the extension by zero $j_!$ always gives us a Waldhausen exact functor

$$j_!\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(V,\Lambda),$$

the total direct image

$$\mathrm{R}\, j_*\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(V,\Lambda)$$

is only a well-defined Waldhausen exact functor if ℓ is invertible on $V - U$. If ℓ is not invertible on $V - U$, then $\mathrm{R}\, j_*(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ is still a system of DG-flat complexes compatible in the sense of Definition 3.1.1.(2), but for $I \in \mathfrak{I}_\Lambda$ the cohomology of the complex of stalks of the complexes $\mathrm{R}\, j_*\mathcal{F}_I^\bullet$ in the geometric points of $V - U$ is in general not finite, such that $\mathrm{R}\, j_*\mathcal{F}_I^\bullet$ fails to be a perfect complex. In any case, we may consider $\mathrm{R}\, j_*$ as a Waldhausen exact functor from $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ to the Waldhausen category of complexes over the abelian category of inverse systems of étale sheaves of Λ-modules, indexed by $\mathfrak{I}_\Lambda$.

The inverse image f^* of a morphism of schemes f and the direct image f_* of a finite morphism of schemes are also defined as Waldhausen exact functors by degreewise application. No Godement resolution is needed, since these functors are exact on all étale sheaves.

Assume one of the following conditions:

(C1) F is a number field, without real places if $\ell = 2$,

(C2) F is a function field of characteristic different from ℓ,
(C3) F is a function field of characteristic ℓ and $U = X$,
(C4) U is a finite subscheme of X,

Then we may define the total derived section functor

$$\mathrm{R}\,\Gamma(U,\cdot)\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda)$$

by the formula

$$\mathrm{R}\,\Gamma(U,(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}) = (\Gamma(U,\mathrm{G}_U\,\mathcal{F}_I^\bullet))_{I\in\mathfrak{I}_\Lambda}.$$

This agrees with the usual construction if we consider $(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}$ as an object of the 'derived' category of adic sheaves, e. g. as defined in [**KW01**] for $\Lambda = \mathbb{Z}_\ell$. In addition, however, we see that $\mathrm{R}\,\Gamma(U,\cdot)$ is a Waldhausen exact functor and hence, induces homomorphisms

$$\mathrm{R}\,\Gamma(U,\cdot)\colon \mathrm{K}_n(\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)) \to \mathrm{K}_n(\Lambda)$$

for all n [**Wit08**, Prop. 4.6.6, Def. 5.4.13]. Here, we use the finiteness and the vanishing in large degrees of the étale cohomology groups $\mathrm{H}^s(U,\mathcal{G})$ for constructible sheaves $\mathcal{G}$ of abelian groups in order to assure that $\mathrm{R}\,\Gamma(U,(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda})$ is indeed an object of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. In particular, for each $I\in\mathfrak{I}_\Lambda$, $\mathrm{R}\,\Gamma(U,\mathcal{F}_I^\bullet)$ is a perfect complex of Λ/I-modules. Note that we do not need to assume that ℓ is invertible on U if F is a number field (see the remark after [**Mil06**, Thm. II.3.1]). However, if the characteristic of F is equal to ℓ, the complexes $\mathrm{R}\,\Gamma(U,\mathcal{F}_I^\bullet)$ are no longer perfect if $U \neq X$ is an open dense subscheme. If F is a number field with real places and we had allowed $\ell = 2$, then the complexes $\mathrm{R}\,\Gamma(U,\mathcal{F}_I^\bullet)$ would not need to be cohomologically bounded.

Assume that F has no real places in the case that $\ell = 2$. Let $j\colon U \to X$ be an open immersion into X. We set as a shorthand

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(U,(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}) \coloneqq \mathrm{R}(X,j_!(\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda}).$$

If $\ell \neq 2$ or F has no real places, this agrees with the definition of cohomology with proper support in [**Mil06**, §II.2]. If F is a totally real number field, $\ell \neq 2$, and $(\mathcal{F}_I^\bullet(-1))_{I\in\mathfrak{I}_\Lambda}$ is smooth at ∞, then it also agrees with the definition in [**FK06**, §1.6.3], but in general, the three definitions differ by contributions coming from the archimedean places.

If F is a function field, we define in the same way Waldhausen exact functors

$$\mathrm{R}\,\Gamma(\overline{U},\cdot),\mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{U},\cdot)\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Lambda),$$

replacing U by $\overline{U}$ in the construction.

Remark 3.1.4. Assume that $j'\colon V \to X$ and $k'\colon W \to X$ are two open dense subschemes of X such that $X = V \cup W$. Set $U \coloneqq V \cap W$ and let $j\colon U \to V$ and $k\colon U \to W$ denote the corresponding open immersions. If the characteristic of F is equal to ℓ, we assume that $V = X$. If F is a number field and $\ell = 2$, we assume that F has no real places. For any étale sheaf $\mathcal{G}$ on U, the canonical morphism

$$k'_!k_*\,\mathrm{G}_U\,\mathcal{G} \cong j'_*j_!\,\mathrm{G}_U\,\mathcal{G} \to j'_*\,\mathrm{G}_V\,j_!\mathcal{G}$$

is seen to be a quasi-isomorphism by checking on the stalks. Hence, for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$, there is a weak equivalence

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(V,j_!\mathcal{F}^\bullet).$$

We recall that the righthand complex is in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. Therefore, the same is true for the left-hand complex without any condition on U and W, even if $\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet$ fails to be a perfect complex if F is a number field and ℓ is not invertible of $W-U$. In particular, we may use the two complexes interchangeably in our results.

Assume again that $U \subset X$ is an open or closed subscheme. If Λ' is another adic $\mathbb{Z}_\ell$-algebra and $M^\bullet$ a complex of Λ'-Λ-bimodules which is strictly perfect as complex of Λ'-modules, we may extend $\Psi_{M^\bullet}$ to a Waldhausen exact functor

$$\Psi_{M^\bullet}\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda'),$$
$$(\mathcal{P}_J^\bullet)_{J\in\mathfrak{I}_\Lambda} \mapsto (\varprojlim_{J\in\mathfrak{I}_\Lambda} \Lambda'/I \otimes_\Lambda M^\bullet \otimes_\Lambda \mathcal{P}_J{}^\bullet)_{I\in\mathfrak{I}_{\Lambda'}}$$

such that

$$\Psi_{M^\bullet} \mathrm{R}\,\Gamma(U,\mathcal{P}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(U,\Psi_{M^\bullet}(\mathcal{P}^\bullet))$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda')$ [**Wit08**, Prop. 5.5.7] if one of the conditions (C1)–(C4) is given. In the function field case, we may replace U by $\overline{U}$.

For any closed point x of X and any complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(x,\Lambda)$, we set

$$\mathrm{R}\,\Gamma(\hat{x},\mathcal{F}^\bullet) \coloneqq \Gamma(\operatorname{Spec}\overline{k(x)}, \hat{x}^*\,\mathrm{G}_x\,\mathcal{F}^\bullet)$$

and let $\mathfrak{F}_x \in \mathrm{Gal}(\overline{k(x)}/k(x))$ denote the geometric Frobenius of $k(x)$. We obtain an exact sequence

$$0 \to \mathrm{R}\,\Gamma(x,\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma(\hat{x},\mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \mathrm{R}\,\Gamma(\hat{x},\mathcal{F}^\bullet) \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ [**Wit08**, Prop. 6.1.2]. Note that if $\hat{x}'$ is the geometric point corresponding to another choice of an embedding $\overline{F} \subset \overline{F}_x$ and if $\mathfrak{F}'_x$ denotes the associated geometric Frobenius, then there is a canonical isomorphism

$$\sigma\colon \mathrm{R}\,\Gamma(\hat{x},\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma(\hat{x}',\mathcal{F}^\bullet)$$

such that

$$\sigma \circ (\mathrm{id} - \mathfrak{F}_x) = (\mathrm{id} - \mathfrak{F}'_x) \circ \sigma. \tag{3.1.1}$$

At some point, we will also make use of the categories $\mathbf{PDG}^{\mathrm{cont}}(\operatorname{Spec} F_x,\Lambda)$ for the local fields F_x together with the associated total derived section functors. In this case, one can directly appeal to the constructions in [**Wit08**, Ch. 5]. We write F_x^{nr} for the maximal unramified extension field of F_x in $\overline{F}_x$ and note that we have a canonical identification $\mathrm{Gal}(F_x^{\mathrm{nr}}/F_x) \cong \mathrm{Gal}(\overline{k(x)}/k(x))$.

LEMMA 3.1.5. *Let $j\colon U \to V$ denote the open immersion of two open dense subschemes of X and assume that $i\colon x \to V$ is a closed point in the complement of U such that the characteristic of $k(x)$ is different from ℓ. Write $\eta_x\colon \operatorname{Spec} F_x \to U$ for the map to the generic point of U. Then there exists a canonical chain of weak equivalences*

$$\mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,j_*\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^*\,\mathrm{G}_U\,\mathcal{F}^\bullet) \xleftarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^*\mathcal{F}^\bullet) \tag{3.1.2}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ compatible with the operation of the Frobenius on each complex and hence, a canonical chain of weak equivalences

$$\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,j_*\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x, \eta_x^*\,\mathrm{G}_U\,\mathcal{F}^\bullet) \xleftarrow{\sim} \mathrm{R}\,\Gamma(\operatorname{Spec} F_x, \eta_x^*\mathcal{F}^\bullet) \tag{3.1.3}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

PROOF. From [**Mil80**, Thm. III.1.15] we conclude that for each $I \in \mathfrak{I}_\Lambda$, the complex $\eta_x^*\,\mathrm{G}_U\,\mathcal{F}_I{}^\bullet$ is a complex of flabby sheaves on $\operatorname{Spec} F_x$ and that

$$\mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,j_*\mathcal{F}_I^\bullet) \to \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^*\,\mathrm{G}_U\,\mathcal{F}_I{}^\bullet)$$

is an isomorphism. Write G_{F_x} for the Godement resolution on $\operatorname{Spec} F_x$ with respect to $\operatorname{Spec} \overline{F}_x \to \operatorname{Spec} F_x$. Then

$$\eta_x^*\,\mathrm{G}_U\,\mathcal{F}_I^\bullet \to \mathrm{G}_{F_x}\,\eta_x^*\,\mathrm{G}_U\,\mathcal{F}_I^\bullet \leftarrow \mathrm{G}_{F_x}\,\eta_x^*\mathcal{F}_I^\bullet$$

are quasi-isomorphisms of complexes of flabby sheaves on $\operatorname{Spec} F_x$. Hence, they remain quasi-isomorphisms if we apply the section functor $\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -)$ in each degree. Since the Frobenius acts compatibly on F_x^{nr} and $\overline{k(x)}$, the induced operation on the complexes is also compatible. The canonical exact sequence

$$0 \to \Gamma(\operatorname{Spec} F_x, -) \to \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, -) \to 0$$

on flabby sheaves on $\operatorname{Spec} F_x$ implies that the morphisms in the chain (3.1.3) are also quasi-isomorphisms. □

Remark 3.1.6. Note that if the characteristic of $k(x)$ is equal to ℓ, the proof of the lemma remains still valid, except that the complexes in the chain (3.1.2) do not lie in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

It will be useful to introduce an explicit strictly perfect complex weakly equivalent to $\mathrm{R}\Gamma(\operatorname{Spec} F_x, \eta_x^* \mathcal{F})$ in the case that $\mathcal{F}$ is a Λ-adic sheaf on U. Assume that the characteristic of $k(x)$ is different from ℓ. Let N be the compact $\operatorname{Gal}(\overline{F}_x/F_x)$-module corresponding to $\eta_x^*\mathcal{F}$ and write $F_x^{\mathrm{nr},(\ell)}$ for the maximal pro-ℓ extension of F_x^{nr} inside $\overline{F}_x$, such that $\operatorname{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}}) \cong \mathbb{Z}_\ell$.

We set $N' \coloneqq N^{\operatorname{Gal}(\overline{F}_x/F_x^{\mathrm{nr},(\ell)})}$. Note that N' is a direct summand of the finitely generated, projective Λ-module N, because the ℓ-Sylow subgroups of the Galois group $\operatorname{Gal}(\overline{F}_x/F_x^{\mathrm{nr},(\ell)})$ are trivial by our assumption on the characteristic of $k(x)$. In particular, N' is itself finitely generated and projective over Λ.

Fix a topological generator τ of $\operatorname{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}})$ and a lift $\varphi \in \operatorname{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x)$ of the geometric Frobenius $\mathfrak{F}_x$. Then τ and φ are topological generators of the profinite group $\operatorname{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x)$ and

$$\varphi\tau\varphi^{-1} = \tau^{q^{-1}}$$

with $q = q_x$ the number of elements of $k(x)$ [**NSW00**, Thm. 7.5.3].

Definition 3.1.7. We define a strictly perfect complex $D^\bullet_{\hat{x}}(\mathcal{F})$ of Λ-modules with an action of $\mathfrak{F}_x$ as follows: For $k \neq 0, 1$ we set $D^k_{\hat{x}}(\mathcal{F}) \coloneqq 0$. As Λ-modules we have $D^0_{\hat{x}}(\mathcal{F}) = D^1_{\hat{x}}(\mathcal{F}) = N'$ and the differential is given by $\mathrm{id} - \tau$. The geometric Frobenius $\mathfrak{F}_x$ acts on $D^0_{\hat{x}}(\mathcal{F})$ via φ and on $D^1_{\hat{x}}(\mathcal{F})$ via

$$\varphi\left(\frac{\tau^q - 1}{\tau - 1}\right) \in \Lambda[[\operatorname{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x)]]^\times.$$

Lemma 3.1.8. *Assume that the characteristic of $k(x)$ is different from ℓ. There exists a weak equivalence*

$$D^\bullet_{\hat{x}}(\mathcal{F}) \xrightarrow{\sim} \mathrm{R}\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x^*\mathcal{F})$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ that is compatible with the operation of the geometric Frobenius $\mathfrak{F}_x$ on both sides.

Proof. Clearly, we have

$$\Lambda/I \otimes_\Lambda D^\bullet_{\hat{x}}(\mathcal{F}) \cong D^\bullet_{\hat{x}}(\mathcal{F}_I)$$

for all $I \in \mathfrak{I}_\Lambda$. We may therefore reduce to the case that Λ is a finite $\mathbb{Z}_\ell$-algebra.

By construction, the perfect complex of Λ-modules $\mathrm{R}\Gamma(F_x^{\mathrm{nr}}, \eta_x^*\mathcal{F})$ may be canonically identified with the homogenous cochain complex

$$X^\bullet(\operatorname{Gal}(\overline{F}_x/F_x), N)^{\operatorname{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}$$

(in the notation of [**NSW00**, Ch. I, §2]) of the finite $\operatorname{Gal}(\overline{F}_x/F_x)$-module N. Recall that the elements of $X^n(\operatorname{Gal}(\overline{F}_x/F_x), N)$ are continuous maps

$$f\colon \operatorname{Gal}(\overline{F}_x/F_x)^{n+1} \to N$$

and the operation of $\sigma \in \operatorname{Gal}(\overline{F}_x/F_x)$ on $f \in X^n(\operatorname{Gal}(\overline{F}_x/F_x), N)$ is defined by

$$\sigma f\colon \operatorname{Gal}(\overline{F}_x/F_x)^{n+1} \to N, \qquad (\sigma_0, \dots, \sigma_n) \mapsto \sigma f(\sigma^{-1}\sigma_0, \dots, \sigma^{-1}\sigma_n).$$

The inflation map provides a quasi-isomorphism

$$X^\bullet(\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x),N')^{\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}})} \xrightarrow{\sim} X^\bullet(\mathrm{Gal}(\overline{F}_x/F_x),N)^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})},$$

which is compatible with the operation of $\mathfrak{F}_x$ by a lift to $\mathrm{Gal}(\overline{F}_x/F_x)$ on both sides.

We define a quasi-isomorphism

$$\alpha\colon D^\bullet_{\hat{x}}(\mathcal{F}) \xrightarrow{\sim} X^\bullet(\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x),N')^{\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}})}$$

compatible with the $\mathfrak{F}_x$-operation by

$$\alpha(n)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x) \to N', \qquad \tau^a\varphi^b \mapsto \tau^a n \qquad \text{for } n \in D^0_{\hat{x}}(\mathcal{F}),$$

$$\alpha(n)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x)^2 \to N', \qquad (\tau^a\varphi^b,\tau^c\varphi^d) \mapsto \frac{\tau^c-\tau^a}{1-\tau}n \qquad \text{for } n \in D^1_{\hat{x}}(\mathcal{F}),$$

with $a,c \in \mathbb{Z}_\ell$ and $b,d \in \hat{\mathbb{Z}}$. Note that

$$\frac{\tau^c-\tau^a}{1-\tau} = \tau^c \sum_{n=1}^{\infty} \binom{a-c}{n}(\tau-1)^{n-1}$$

is a well-defined element of $\Lambda[[\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x)]]$ for any $a,c \in \mathbb{Z}_\ell$. □

If $i\colon \Sigma \to V$ is the embedding of a closed subscheme Σ of X into an open subscheme V of X with complement $j\colon U \to V$ and $\mathcal{F}$ is an étale sheaf of abelian groups on V, then we may consider the sheaf

$$i^!\mathcal{F} = \ker(i^*\mathcal{F} \to i^*j_*j^*\mathcal{F})$$

on Σ. Its global sections $i^!\mathcal{F}(\Sigma)$ are the global sections of $\mathcal{F}$ on V with support on Σ. The right derived functor $\mathrm{R}\,i^!$ can also be defined via Godement resolution:

Lemma 3.1.9. *Assume that ℓ is invertible on Σ.*

$$\mathrm{R}\,i^!\colon \mathbf{PDG}^{\mathrm{cont}}(V,\Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(\Sigma,\Lambda), \qquad (\mathcal{F}^\bullet_I)_{I\in\mathfrak{I}_\Lambda} \mapsto (i^!\,\mathrm{G}_V(\mathcal{F}^\bullet_I))_{I\in\mathfrak{I}_\Lambda}$$

is a Waldhausen exact functor and for every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(V,\Lambda)$ there is an exact sequence

$$0 \to i_*\,\mathrm{R}\,i^!\mathcal{F}^\bullet \to \mathrm{G}_V(\mathcal{F}^\bullet) \to \mathrm{R}\,j_*j^*\mathcal{F}^\bullet \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(V,\Lambda)$. In particular, if $i^*\mathcal{F}^\bullet$ is weakly equivalent to 0, then there exists a chain of weak equivalences

$$i^*\,\mathrm{R}\,j_*j^*\mathcal{F}^\bullet \sim \mathrm{R}\,i^!\mathcal{F}^\bullet[1].$$

Proof. Note that for any abelian étale sheaf $\mathcal{F}$ on V, we habe $j^*\,\mathrm{G}_V(\mathcal{F}) = \mathrm{G}_U(\mathcal{F})$. Moreover, by [**AGV72b**, XVII, Prop. 4.2.3], $\mathrm{G}_U(\mathcal{F})$ is a complex of flasque sheaves in the sense of [**AGV72b**, V, Def. 4.1]. In particular, $\mathrm{G}_V(\mathcal{F}) \to j_*j^*\,\mathrm{G}_V(\mathcal{F})$ is surjective in the category of presheaves by [**AGV72a**, V, Prop. 4.7]. If $\mathcal{F}^\bullet$ is a complex of abelian sheaves, $\mathrm{G}_V(\mathcal{F}^\bullet)$ is constructed as the total complex of the double complex obtained by taking the Godement resolution of each individual sheaf. In particular, $\mathrm{G}_V(\mathcal{F}^\bullet)$ is a complex of possibly infinite sums of flasque sheaves. Note that infinite sums of flasque sheaves are not necessarily flasque. Still, as étale cohomology of noetherian schemes commutes with filtered direct limits, $\mathrm{G}_V(\mathcal{F}^\bullet) \to j_*j^*\,\mathrm{G}_V(\mathcal{F}^\bullet)$ is always surjective in the category of presheaves. This proves the exactness of the above sequence. Moreover, it implies that $\mathrm{G}_V(\mathcal{F}^\bullet)$ is a $i^!$-acyclic resolution of $\mathcal{F}^\bullet$ such that $i^!\,\mathrm{G}_V$ preserves quasi-isomorphisms and injections. If $\mathcal{F}^\bullet$ is a perfect complex of sheaves of Λ-modules on V for any finite ring Λ, then $i^!\,\mathrm{G}_V(\mathcal{F}^\bullet)$ is perfect since this is true for $i^*\,\mathrm{G}_V(\mathcal{F}^\bullet)$ and $i^*j_*j^*\,\mathrm{G}_V(\mathcal{F}^\bullet)$. Similarly, we see that $i^!\,\mathrm{G}_V$ commutes with tensor products with finitely generated right Λ-modules. In particular, $\mathrm{R}\,i^!$ does indeed take values in $\mathbf{PDG}^{\mathrm{cont}}(\Sigma,\Lambda)$ for any adic ring Λ. Finally, if $i^*\mathcal{F}^\bullet$ is weakly equivalent to 0, then we obtain the chain of weak equivalences

$$i^*\,\mathrm{R}\,j_*j^*\mathcal{F}^\bullet \xleftarrow{\sim} \mathrm{Cone}\big(\mathrm{R}\,i^!\mathcal{F}^\bullet \to i^*\,\mathrm{G}_V(\mathcal{F}^\bullet)\big) \xrightarrow{\sim} \mathrm{R}\,i^!\mathcal{F}^\bullet[1].$$

□

3.2. Duality for Smooth Adic Sheaves

For any scheme Z, any ring R and any two étale sheaves of R-modules $\mathcal{F}$, $\mathcal{G}$ on Z, let

$$\mathcal{H}om_{R,Z}(\mathcal{F},\mathcal{G})$$

denote the sheaf of R-linear morphisms $\mathcal{F}\to\mathcal{G}$ on Z. As before, we fix an adic $\mathbb{Z}_\ell$-algebra Λ. Let $U\subset X$ be an open or closed subscheme. Unfortunately, we cannot present a construction of a Waldhausen exact functor

$$*\colon \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)^{\mathrm{op}} \to \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda^{\mathrm{op}})$$

that would give rise to the usual total derived $\mathcal{H}om$-functor $\mathcal{F}\mapsto \mathrm{R}\,\mathcal{H}om_{\Lambda,U}(\mathcal{F},\Lambda_U)$ on the 'derived' category of Λ-adic sheaves. Instead, we will construct a Waldhausen exact duality functor on the Waldhausen subcategory $\mathbf{S}^{\mathrm{sm}}(U,\Lambda)$ of smooth Λ-adic sheaves.

For any smooth Λ-adic sheaf $\mathcal{F}$,

$$\begin{aligned}\mathcal{F}^{*_\Lambda} &\coloneqq (\mathcal{H}om_{\Lambda/I,U}(\mathcal{F}_I,(\Lambda/I)_U))_{I\in\mathfrak{I}_\Lambda}\\ &= (\mathcal{H}om_{\mathbb{Z},U}(\mathcal{H}om_{\mathbb{Z},U}((\Lambda/I)_U,(\mathbb{Q}_\ell/\mathbb{Z}_\ell)_U)\otimes_{\Lambda/I}\mathcal{F}_I,(\mathbb{Q}_\ell/\mathbb{Z}_\ell)_U))_{I\in\mathfrak{I}_\Lambda}\end{aligned}$$

is a smooth Λ^{op}-adic sheaf on U. In this way, we obtain a Waldhausen exact equivalence

$$*\colon \mathbf{S}^{\mathrm{sm}}(U,\Lambda)^{\mathrm{op}} \to \mathbf{S}^{\mathrm{sm}}(U,\Lambda^{\mathrm{op}})$$

and, by composing with $I\colon \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U,\Lambda)) \xrightarrow{\cong} \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U,\Lambda)^{\mathrm{op}})$, isomorphisms

$$*\colon \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U,\Lambda)) \to \mathrm{K}_n(\mathbf{S}^{\mathrm{sm}}(U,\Lambda^{\mathrm{op}}))$$

for each $n\geq 0$.

Assume that U is an open dense subscheme of X such that ℓ is invertible on U and that F has no real places if $\ell=2$. If $\mathcal{F}$ is a smooth Λ-adic sheaf on U, we can find a strictly perfect complex of Λ-modules $P^\bullet$ together with a weak equivalence

$$P^\bullet \xrightarrow{\sim} \mathrm{R}\,\Gamma_{\mathrm{c}}(U,\mathcal{F})$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$. As a consequence of Artin-Verdier duality [**Mil06**, Thm. II.3.1], we then also have a weak equivalence

$$(P^\bullet)^* \xrightarrow{\sim} \mathrm{R}\,\Gamma(U,\mathcal{F}^*(1))[-3]. \tag{3.2.1}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$. We refer to Corollary 5.3.3 for a slightly more general statement.

We could proceed in the same way for local duality and duality over finite fields, but instead, we prove the following finer results.

LEMMA 3.2.1. *Assume that U is an open subscheme of X such that ℓ is invertible on U and that $i\colon x\to X$ is a closed point such that the characteristic of $k(x)$ is different from ℓ. For any smooth Λ-adic sheaf $\mathcal{F}$ on U, there exists a weak equivalence*

$$D^\bullet_{\hat{x}}(\mathcal{F})^* \xrightarrow{\sim} \mathrm{R}\,\Gamma(\mathrm{Spec}\,F_x^{\mathrm{nr}},\eta_x\mathcal{F}^*(1))[-1]$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$, compatible with the operation of $\mathfrak{F}_x^*$ on the left and of $\mathfrak{F}_x^{-1}$ on the right.

PROOF. As in the proof of Lemma 3.1.8, we can replace $\mathrm{R}\Gamma(\operatorname{Spec} F_x^{\mathrm{nr}}, \eta_x \mathcal{F}^*(1))$ by the homogenous cochain complex $X^\bullet(\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x), (N')^*(1))^{\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}})}$. By choosing a basis of the free $\mathbb{Z}_\ell$-module $\mathbb{Z}_\ell(1)$, i. e. a compatible system of ℓ^n-th roots of unity, we may identify the underlying Λ-modules of $(N')^*$ and $(N')^*(1)$. The operation of $\sigma \in \mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}})$ on $f \in (N')^*$ is given by

$$\sigma f = f \circ (\sigma^*)^{-1}.$$

The operation of $\mathfrak{F}_x^*$ on $f \in D^1_{\hat{x}}(\mathcal{F})^* = (N')^*$ is then given by

$$\mathfrak{F}_x^*(f) = \left(\frac{\tau^{-q}-1}{\tau^{-1}-1}\right)\varphi^{-1} f$$

and on $g \in D^0_{\hat{x}}(\mathcal{F})^* = (N')^*$ by

$$\mathfrak{F}_x^*(g) = \varphi^{-1} g,$$

with $\varphi, \tau \in \mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x)$ denoting our fixed topological generators and $q \in \Lambda^\times$ denoting the order of the residue field $k(x)$. For $b \in \hat{\mathbb{Z}}$ set

$$s(b) := q^{-b}\left(\frac{\tau^{-1}-1}{\tau^{-q^{-b}}-1}\right) \in \Lambda[[\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}})]]^\times$$

and note that s satisfies the cocycle relation

$$s(b+1) = q^{-1}\varphi s(b)\left(\frac{\tau^{-q}-1}{\tau^{-1}-1}\right)\varphi^{-1} = q^{-1}\varphi s(b)\varphi^{-1}s(1).$$

We define a weak equivalence

$$\beta\colon D^\bullet_{\hat{x}}(\mathcal{F})^* \xrightarrow{\sim} X^\bullet(\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x), (N')^*(1))^{\mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x^{\mathrm{nr}})}[-1]$$

by

$$\beta(f)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x) \to (N')^*(1), \qquad \tau^a\varphi^b \mapsto \tau^a s(b) f$$

for $f \in D^1_{\hat{x}}(\mathcal{F})^*$ and by

$$\beta(g)\colon \mathrm{Gal}(F_x^{\mathrm{nr},(\ell)}/F_x)^2 \to (N')^*(1), \qquad (\tau^a\varphi^b, \tau^c\varphi^d) \mapsto \left(\frac{\tau^a s(b) - \tau^c s(d)}{1-\tau^{-1}}\right) g$$

for $g \in D^0_{\hat{x}}(\mathcal{F})^*$, $a, c \in \mathbb{Z}_\ell$ and $b, d \in \hat{\mathbb{Z}}$.

Using the cocycle relation for s, it is easily checked that

$$\beta \circ \mathfrak{F}_x^* = \mathfrak{F}_x^{-1} \circ \beta,$$

as claimed. □

In particular, if $Q^\bullet$ denotes the cocone of

$$D^\bullet_{\hat{x}}(\mathcal{F}) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} D^\bullet_{\hat{x}}(\mathcal{F}),$$

then $Q^\bullet$ is a strictly perfect complex of Λ-modules and there exist weak equivalences

$$\begin{aligned} Q^\bullet &\xrightarrow{\sim} \mathrm{R}\Gamma(\operatorname{Spec} F_x, \eta_x^*\mathcal{F}), \\ (Q^\bullet)^* &\xrightarrow{\sim} \mathrm{R}\Gamma(\operatorname{Spec} F_x, \eta_x^*\mathcal{F}^*(1))[-2]. \end{aligned} \tag{3.2.2}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$.

Let now $\mathcal{G}$ be a complex in $\mathbf{S}(x,\Lambda) = \mathbf{S}^{\mathrm{sm}}(x,\Lambda)$ and let

$$\mathcal{G}_{\hat{x}} := \varprojlim_{I \in \mathfrak{I}_\Lambda} (\mathcal{G}_I)_{\hat{x}}$$

be the stalk of $\mathcal{G}$ in the geometric point $\hat{x}$ over x. Then $\mathcal{G}_{\hat{x}}$ is a finitely generated, projective Λ-module, equipped with a natural operation of $\mathfrak{F}_x$. Clearly, the natural morphism

$$\mathcal{G}_{\hat{x}} \xrightarrow{\sim} \mathrm{R}\,\Gamma(\hat{x}, \mathcal{G}) \tag{3.2.3}$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ that is compatible with the operation of $\mathfrak{F}_x$ on both sides. In particular, the cocone $C^\bullet$ of

$$\mathcal{G}_{\hat{x}} \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \mathcal{G}_{\hat{x}}$$

is weakly equivalent to $\mathrm{R}\,\Gamma(x, \mathcal{G})$.

LEMMA 3.2.2. *With $\mathcal{G}$ as above, there exists an isomorphism*

$$(\mathcal{G}_{\hat{x}})^* \xrightarrow{\cong} (\mathcal{G}^*)_{\hat{x}}$$

of finitely generated, projective Λ-modules, compatible with the operation of $\mathfrak{F}_x^*$ on the left and of $\mathfrak{F}_x^{-1}$ on the right.

PROOF. Let R be any finite ring. Under the equivalence between the categories of étale sheaves of R-modules on x and of discrete $R[[\mathrm{Gal}(\overline{k(x)}/k(x))]]$-modules, given by $\mathcal{F} \mapsto \mathcal{F}_{\hat{x}}$, the dual sheaf $\mathcal{F}^{*_R}$ corresponds to the R^{op}-module $(\mathcal{F}_{\hat{x}})^{*_R}$ with $\sigma \in \mathrm{Gal}(\overline{k(x)}/k(x))$ acting on $f\colon \mathcal{F}_{\hat{x}} \to R$ by $f \circ (\sigma^{*_R})^{-1}$. □

Consequently, we obtain a weak equivalence

$$(C^\bullet)^* \xrightarrow{\sim} \mathrm{R}\,\Gamma(x, \mathcal{G}^*)[-1] \tag{3.2.4}$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda^{\mathrm{op}})$. If $\mathcal{G} = i^*\mathcal{F}$ with $\mathcal{F}$ a smooth Λ-adic sheaf on U as above, then by the exchange formula [**Fu11**, Thm. 8.4.7], there exists a chain of weak equivalences

$$\begin{aligned}(i^*\mathcal{F})^* &= (\mathrm{R}\,\mathcal{H}om_{\Lambda/I,x}(i^*\mathcal{F}_I, (\Lambda/I)_x))_{I\in\mathfrak{I}_\Lambda}\\ &\sim (\mathrm{R}\,\mathcal{H}om_{\Lambda/I,x}(i^*\mathcal{F}_I, \mathrm{R}\,i^!(\Lambda/I)_U(1)[-2]))_{I\in\mathfrak{I}_\Lambda}\\ &\sim (\mathrm{R}\,i^!\,\mathcal{H}om_{\Lambda/I,U}(\mathcal{F}_I, (\Lambda/I)_U(1))[-2])_{I\in\mathfrak{I}_\Lambda}\\ &= \mathrm{R}\,i^!\mathcal{F}^*(1)[-2]\end{aligned} \tag{3.2.5}$$

in $\mathbf{PDG}^{\mathrm{cont}}(x, \Lambda^{\mathrm{op}})$.

3.3. Admissible Extensions

As before, we fix a global field F and a prime ℓ. Assume that F_∞/F is a possibly infinite Galois extension unramified over an open or closed subscheme $U = U_F$ of X. Let $G = \mathrm{Gal}(F_\infty/F)$ be its Galois group. We also assume that G has a topologically finitely generated, open pro-ℓ-subgroup, such that for any adic $\mathbb{Z}_\ell$-algebra Λ, the profinite group ring $\Lambda[[G]]$ is again an adic ring [**Wit14**, Prop. 3.2]. For any intermediate field K of F_∞/F, we will write U_K for the base change with X_K and $f_K\colon U_K \to U$ for the corresponding Galois covering of U, such that we obtain a system of Galois coverings $(f_K\colon U_K \to U)_{F\subset K\subset F_\infty}$, which we denote by

$$f\colon U_{F_\infty} \to U.$$

As in [**Wit14**, Def. 6.1] we make the following construction.

DEFINITION 3.3.1. Let Λ be any adic $\mathbb{Z}_\ell$-algebra. For $\mathcal{F}^\bullet \in \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ we set

$$f_!f^*\mathcal{F}^\bullet \coloneqq (\varprojlim_{I\in\mathfrak{I}_\Lambda} \varprojlim_{F\subset K\subset F_\infty} \Lambda[[G]]/J \otimes_{\Lambda[[G]]} f_{K\,!}f_K^*\mathcal{F}_I{}^\bullet)_{J\in\mathfrak{I}_{\Lambda[[G]]}}$$

As in [**Wit14**, Prop. 6.2] one verifies that we thus obtain a Waldhausen exact functor

$$f_! f^* \colon \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G]]).$$

We recall how the functor $f_! f^*$ transforms under the change of the extension F_∞/F and under changes of the coefficient ring Λ.

Proposition 3.3.2. *Let $f \colon U_{F_\infty} \to U$ be the system of Galois coverings of the open or closed subscheme U of X associated to the extension F_∞/F with Galois group G which is unramified over U. Let further Λ be an adic $\mathbb{Z}_\ell$-algebra and $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.*

(1) *Let Λ' be another adic $\mathbb{Z}_\ell$-algebra and let $P^\bullet$ be a complex of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules. Then there exists a natural isomorphism*
$$\Psi_{P[[G]]^\delta{}^\bullet} f_! f^* \mathcal{F}^\bullet \cong f_! f^* \Psi_{P^\bullet} f_! f^* \mathcal{F}^\bullet$$
(2) *Let $F'_\infty \subset F_\infty$ be a subfield such that F'_∞/F is a Galois extension with Galois group G' and let $f' \colon U_{F'_\infty} \to U$ denote the corresponding system of Galois coverings. Then there exists a natural isomorphism*
$$\Psi_{\Lambda[[G']]} f_! f^* \mathcal{F}^\bullet \cong (f')_! (f')^* \mathcal{F}^\bullet$$
in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G']])$.

(3) *Let F'/F be a finite extension inside F_∞/F, let $f_{F'} \colon U_{F'} \to U$ denote the associated étale covering of U and let $g \colon U_{F_\infty} \to U'_F$ be the restriction of the system of coverings f to $U_{F'}$. Write $G' \subset G$ for the corresponding open subgroup and view $\Lambda[[G]]$ as a $\Lambda[[G']]$-$\Lambda[[G]]$-bimodule. Then there exists a natural isomorphism*
$$\Psi_{\Lambda[[G]]} f_! f^* \mathcal{F}^\bullet \cong f_{F'*} (g_! g^*) f^*_{F'} \mathcal{F}^\bullet$$
in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G']])$.

(4) *With the notation of (3), let $\mathcal{G}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda)$ and view $\Lambda[[G]]$ as a $\Lambda[[G]]$-$\Lambda[[G']]$-bimodule. Then there exists a natural isomorphism*
$$\Psi_{\Lambda[[G]]} f_{F'*} g_! g^* \mathcal{G}^\bullet \cong f_! f^* (f_{F'*} \mathcal{G}^\bullet)$$
in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G]])$.

Proof. Part (1) – (3) are proved in [**Wit14**, Prop. 6.5, 6.7]. We prove (4). First, note that for any finite Galois extension F''/F with $F' \subset F'' \subset F_\infty$ and any $I \in \mathfrak{I}_\Lambda$ the canonical map

$$g_{F''!} g^*_{F''} (\Lambda/I)_{U_{F'}} \to f^*_{F'} f_{F''!} f^*_{F''} (\Lambda/I)_U$$

induces an isomorphism

$$\Lambda/I[\mathrm{Gal}(F''/F)] \otimes_{\Lambda/I[\mathrm{Gal}(F''/F')]} g_{F''!} g^*_{F''} (\Lambda/I)_{U_{F'}} \cong f^*_{F'} f_{F''!} f^*_{F''} (\Lambda/I)_U.$$

Hence,

$$\Psi_{\Lambda[[G]]} (g_! g^* \Lambda_{U_{F'}}) \cong f^*_{F'} f_! f^* \Lambda_U$$

in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda[[G]])$. We further recall that in the notation of [**Wit14**, Prop. 6.3], there exists an isomorphism

$$f_! f^* f_{F'*} \mathcal{G}^\bullet \cong \Psi_{f_! f^* \Lambda_U} f_{F'*} \mathcal{G}^\bullet.$$

The projection formula then implies

$$\begin{aligned}
\Psi_{f_! f^* \Lambda_U} f_{F'*} \mathcal{G}^\bullet &\cong f_{F'*} (\Psi_{f^*_{F'} f_! f^* \Lambda_U} (\mathcal{G}^\bullet)) \\
&\cong f_{F'*} (\Psi_{\Psi_{\Lambda[[G]]} (g_! g^* \Lambda_{U_{F'}})} (\mathcal{G}^\bullet)) \\
&\cong f_{F'*} (\Psi_{\Lambda[[G]]} (g_! g^* \mathcal{G}^\bullet))
\end{aligned}$$

as desired. □

To understand Part (1) of this proposition, note that if ρ is a representation of G on a finitely generated and projective Λ-module and $\rho^\sharp$ is the corresponding Λ-$\mathbb{Z}_\ell[[G]]$-bimodule as in Example 2.6.3, then

$$\eta_*(\rho) \coloneqq \Psi_{\rho^\sharp} f_! f^* (\mathbb{Z}_\ell)_U \tag{3.3.1}$$

is simply the smooth Λ-adic sheaf on U associated to ρ [**Wit14**, Prop. 6.8]. In general,

$$\Psi_{\tilde{P}^\bullet} \mathcal{F}^\bullet \coloneqq \Psi_{P^\bullet} f_! f^* \mathcal{F}^\bullet \tag{3.3.2}$$

should be understood as the derived tensor product over Λ of the complex of sheaves associated to $P^\bullet$ and the complex $\mathcal{F}^\bullet$.

Assume that $\mathcal{F}$ is a smooth Λ-adic sheaf on U. As before, we write

$$\mathcal{F}^{*_\Lambda} \coloneqq (\mathcal{H}om_{\Lambda/I,U}(\mathcal{F}_I, \Lambda/I))_{I\in\mathfrak{I}_{\Lambda/I}} \in \mathbf{PDG}^{\mathrm{cont}}(U, \Lambda^{\mathrm{op}})$$

for the Λ-dual of $\mathcal{F}$ and $\Lambda^{\mathrm{op}}[[G]]^\sharp$ for the $\Lambda^{\mathrm{op}}[[G]]$-$\Lambda[[G]]^{\mathrm{op}}$-bimodule with $g \in G$ acting by g^{-1} from the right. We then have a natural isomorphism

$$f_! f^* \mathcal{F}^{*_\Lambda} \cong \Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp} (f_! f^* \mathcal{F})^{*_{\Lambda[[G]]}}. \tag{3.3.3}$$

This can then be combined with the duality assertions (3.2.1), (3.2.2), and (3.2.4). For example, we may find a strictly perfect complex of $\Lambda^{\mathrm{op}}[[G]]$-modules $P^\bullet$ and weak equivalences

$$\begin{aligned} P^\bullet &\xrightarrow{\sim} \mathrm{R}\,\Gamma_c(U, f_! f^* \mathcal{F}^{*_\Lambda}(1)), \\ {}^\sharp(P^\bullet)^{*_{\Lambda^{\mathrm{op}}[[G]]}} &\xrightarrow{\sim} \mathrm{R}\,\Gamma(U, f_! f^* \mathcal{F})[-3] \end{aligned} \tag{3.3.4}$$

if ℓ is invertible on U and F has no real places in the case that $\ell = 2$.

Let F_{cyc}/F denote the cyclotomic $\mathbb{Z}_\ell$-extension of F, i. e.

$$F_{\mathrm{cyc}} = \bigcup_{n\in\mathbb{N}} \mathbb{F}_{p^{\ell^n}} F$$

if F is a function field of characteristic p and F_{cyc}/F is the unique $\mathbb{Z}_\ell$-subextension of

$$\bigcup_{n\in\mathbb{N}} F(\zeta_{\ell^n})$$

with ζ_{ℓ^n} denoting an ℓ^n-th root of unity if F is a number field.

Definition 3.3.3. Let F be a global field. An extension F_∞/F inside $\overline{F}$ is called *admissible* if

(1) F_∞/F is Galois and unramified outside a finite set of places,
(2) F_∞ contains the cyclotomic $\mathbb{Z}_\ell$-extension F_{cyc},
(3) $\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ contains a topologically finitely generated, open pro-ℓ subgroup.

An admissible extension F_∞/F of a number field F is called *really admissible* if F_∞ and F are totally real.

If F_∞/F is an admissible extension, we let $G \coloneqq \mathrm{Gal}(F_\infty/F)$ denote its Galois group and set $H \coloneqq \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$, $\Gamma \coloneqq \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. We may then choose a continuous splitting $\Gamma \to G$ to identify G with the corresponding semi-direct product $G = H \rtimes \Gamma$.

Assume that F is a totally real number field and that $\ell \neq 2$. Let M be the maximal abelian ℓ-extension of F_{cyc} unramified outside the places over ℓ. By the validity of the weak Leopoldt conjecture for F_{cyc}, the Galois group $\mathrm{Gal}(M/F_{\mathrm{cyc}})$ is a finitely generated torsion module of projective dimension less or equal 1 over the classical Iwasawa algebra $\mathbb{Z}_\ell[[\mathrm{Gal}(F_{\mathrm{cyc}}/F)]]$ [**NSW00**, Thm. 11.3.2]. Like in

[**Kak13**], we will assume the vanishing of its Iwasawa μ-invariant in the following sense:

CONJECTURE 3.3.4. *For every totally real field F, the Galois group over F_{cyc} of the maximal abelian ℓ-extension of F_{cyc} unramified outside the places over ℓ is a finitely generated $\mathbb{Z}_\ell$-module.*

In particular, for any totally real field F and any finite set Σ of places of F containing the places over ℓ, the Galois group over F_{cyc} of the maximal abelian ℓ-extension of F_{cyc} unramified outside Σ is also a finitely generated $\mathbb{Z}_\ell$-module, noting that no finite place is completely decomposed in F_{cyc}/F [**NSW00**, Cor. 11.3.6]. We also observe that the Galois group $\mathrm{Gal}(F_\Sigma^{(\ell)}/F_{\mathrm{cyc}})$ of the maximal ℓ-extension of F unramified outside Σ is then a free pro-ℓ-group topologically generated by finitely many elements [**NSW00**, Thm. 11.3.7].

REMARK 3.3.5. The notion of really admissible extensions is slightly weaker than the notion of admissible extension used in [**Kak13**, Def. 2.1]: We do not need to require $\mathrm{Gal}(F_\infty/F)$ to be an ℓ-adic Lie group. For example, as a result of the preceding discussion, we see that we could choose $F_\infty = F_\Sigma^{(\ell)}$ for some finite set of places Σ of F containing the places above ℓ, provided that Conjecture 3.3.4 is valid.

If a really admissible extension F_∞/F is unramified over the open dense subscheme $U = W$ of X, $\Lambda = \mathbb{Z}_\ell$ and $\mathcal{F}^\bullet = (\mathbb{Z}_\ell)_U(1)$, then

$$\varprojlim_{I \in \mathfrak{I}_{\mathbb{Z}_\ell[[G]]}} \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* (\mathbb{Z}_\ell)_U(1))[-3]$$

is by Artin-Verdier duality and comparison of étale and Galois cohomology quasi-isomorphic to the complex $C(F_\infty/F)$ featuring in the main conjecture [**Kak13**, Thm. 2.11]. In particular,

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* (\mathbb{Z}_\ell)_U(1))$$

is in fact an object of $\mathbf{PDG}^{\mathrm{cont},w_H}(\mathbb{Z}_\ell[[G]])$ under Conjecture 3.3.4. We will generalise this statement in the next section.

3.4. The S-Torsion Property

Let F be a global field. Assume that F_∞/F is an admissible extension that is unramified over the open dense subscheme U of X and that $k: U \to W$ is the open immersion into another open dense subscheme of X. We also fix an adic $\mathbb{Z}_\ell$-algebra Λ. If F is a number field, we note that ℓ must be invertible on U, because the cyclotomic extension F_{cyc}/F is ramified in all places over ℓ.

Our purpose is to prove:

THEOREM 3.4.1. *Assume that F is a function field of characteristic p. Let $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$. If $p \neq \ell$, then the complexes*

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet(1)), \quad \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. If $\ell = p$, then the complex

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^\bullet)$$

is in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

THEOREM 3.4.2. *Assume that F_∞/F is a really admissible extension and that $\ell \neq 2$ is invertible on W. Let $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$. If Conjecture 3.3.4 is valid, then the complexes*

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet(1)), \quad \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

In the course of the proof, we will also need to consider the following local variant, whose validity is independent of Conjecture 3.3.4 in the number field case.

THEOREM 3.4.3. *Assume that F_∞/F is an admissible extension of a global field F with $k\colon U \to W$ as above. Let $i\colon \Sigma \to W$ denote a closed subscheme of W and assume that $\mathcal{F}^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. If ℓ is invertible on Σ, the complexes*

$$\mathrm{R}\,\Gamma(\Sigma, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet), \mathrm{R}\,\Gamma(\Sigma, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. If the characteristic of F is ℓ and Σ is a closed subscheme of U, then

$$\mathrm{R}\,\Gamma(\Sigma, i^* f_! f^* \mathcal{F}^\bullet)$$

is in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

Using [**Wit14**, Prop. 4.8] we may at once reduce to the case that Λ is a finite simi-simple $\mathbb{Z}_\ell$-algebra and that $F_\infty/F_{\mathrm{cyc}}$ is a finite extension. It then suffices to show that the complexes appearing in the above theorems have finite cohomology groups. We may then replace $\mathcal{F}^\bullet$ by a quasi-isomorphic strictly perfect complex. Using stupid truncation and induction on the length of the strictly perfect complex we may assume that $\mathcal{F}$ is in fact a flat and constructible sheaf (unramified in ∞). Note further that the cohomology groups

$$\begin{aligned}
\mathrm{H}^s_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) &= \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}^s_{\mathrm{c}}(W_K, \mathrm{R}\, k_* f_K^* \mathcal{F}(1)),\\
\mathrm{H}^s(W, k_! f_! f^* \mathcal{F}) &= \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}^s(W_K, k_! f_K^* \mathcal{F}),\\
\mathrm{H}^s(\Sigma, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}) &= \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}^s(\Sigma_K, i^* \mathrm{R}\, k_* f_K^* \mathcal{F})\\
\mathrm{H}^s(\Sigma, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}) &= \varprojlim_{F \subset K \subset F_\infty} \mathrm{H}^s(\Sigma_K, \mathrm{R}\, i^! k_! f_K^* \mathcal{F})
\end{aligned}$$

do not change if we replace F by a finite extension of F inside F_∞. So, we may assume that $F_\infty = F_{\mathrm{cyc}}$ and that no place in Σ splits in F_∞/F. Further, we may reduce to the case that Σ consists of a single place x. In particular, x does not split or ramify in F_∞/F.

We consider Theorem 3.4.3 in the case that $x \in U$ and write $i'\colon x \to U$ for the inclusion map. Under the above assumptions on x, there exists a chain of weak equivalences

$$\mathrm{R}\,\Gamma(x, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}) \xleftarrow{\sim} \mathrm{R}\,\Gamma(x, i'^* f_! f^* \mathcal{F}) \xrightarrow{\sim} \mathrm{R}\,\Gamma(x, g_! g^* i'^* \mathcal{F})$$

where $g\colon x_\infty \to x$ is the unique $\mathbb{Z}_\ell$-extension of x. We can now refer directly to [**Wit14**, Thm. 8.1] or identify

$$\mathrm{H}^s(x, g_! g^* i'^* \mathcal{F}) = \mathrm{H}^s(\mathrm{Gal}_{k(x)}, \mathbb{F}_\ell[[\Gamma]]^\sharp \otimes_{\mathbb{F}_\ell} M)$$

with $\mathrm{Gal}_{k(x)}$ the absolute Galois group of the residue field $k(x)$ of x, M the stalk of $\mathcal{F}$ in a geometric point over x and $\mathbb{F}_\ell[[\Gamma]]^\sharp$ being the $\mathrm{Gal}_{k(x)}$-module $\mathbb{F}_\ell[[\Gamma]]$ with $\sigma \in \mathrm{Gal}_{k(x)}$ acting by right multiplication with the image of σ^{-1} in Γ. It is then clear that the only non-vanishing cohomology group is $H^1(\mathrm{Gal}_{k(x)}, \mathbb{F}_\ell[[\Gamma]]^\sharp \otimes_{\mathbb{F}_\ell} M)$, of order bounded by the order of M.

Assume that the characteristic of $k(x)$ is different from ℓ. Write $U' = U - \{x\}$ and let $j'\colon U' \to U$ denote the inclusion morphism. Then there is an exact sequence

$$0 \to \mathrm{R}\, i^! k_! j'_! j'^* f_! f^* \mathcal{F} \to \mathrm{R}\, i^! k_! f_! f^* \mathcal{F} \to \mathrm{R}\, i^! i_* i^* k_! \mathcal{F} \to 0.$$

Moreover, there exists a chain of weak equivalences

$$i'^* f_! f^* \mathcal{F} \xrightarrow{\cong} i^* k_! f_! f^* \mathcal{F} \xrightarrow{\sim} \mathrm{R}\, i^! i_* i^* k_! f_! f^* \mathcal{F}.$$

Since we already know that the groups $\mathrm{H}^s(x, i'^* f_! f^* \mathcal{F})$ are finite, it is sufficient to prove that $\mathrm{H}^s(x, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F})$ is finite in the case that $x \in W - U$.

Now we prove Theorem 3.4.3 in the case that $x \in W - U$, assuming that the characteristic of F is different from ℓ. First, note that the complex $\mathrm{R}\, i^! \,\mathrm{R}\, k_* f_! f^* \mathcal{F}$ is quasi-isomorphic to 0. Hence, there is a chain of weak equivalences

$$i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F} \sim \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}[1]$$

by Lemma 3.1.9. So, it suffices to consider the left-hand complex. By Lemma 3.1.5 and the smooth base change theorem there exists a chain of weak equivalences

$$\mathrm{R}\,\Gamma(x, i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}) \sim \mathrm{R}\,\Gamma(\operatorname{Spec} F_x, h_! h^* \eta_x^* \mathcal{F}),$$

where F_x is the local field in x with valuation ring $\mathcal{O}_{F_x}$, $\eta_x \colon \operatorname{Spec} F_x \to U$ is the map to the generic point of U, and $h \colon \operatorname{Spec}(F_x)_{\mathrm{cyc}} \to \operatorname{Spec} F_x$ is the unique $\mathbb{Z}_\ell$-extension of F_x inside $\overline{F}_x$. We may now identify

$$\mathrm{H}^s(x, i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}) = \mathrm{H}^s(\mathrm{Gal}_{F_x}, \mathbb{F}_\ell[[\Gamma]]^\sharp \otimes_{\mathbb{F}_\ell} M)$$

with Gal_{F_x} the absolute Galois group of the local field F_x in x, M the finite Gal_{F_x}-module corresponding to $\eta_x^* \mathcal{F}$ and $\mathbb{F}_\ell[[\Gamma]]^\sharp$ being the Gal_{F_x}-module $\mathbb{F}_\ell[[\Gamma]]$ with $\sigma \in \mathrm{Gal}_{F_x}$ acting by right multiplication with the image of σ^{-1} in Γ. The finiteness of the cohomology group on the righthand side is well-known: We can use local duality to identify it with the Pontryagin dual of

$$\mathrm{H}^{2-s}(\mathrm{Gal}_{(F_x)_{\mathrm{cyc}}}, M^\vee(1))$$

where $M^\vee(1)$ is the first Tate twist of the Pontryagin dual of M.

Next, we prove Theorem 3.4.1 and Theorem 3.4.2. By [**Wit14**, Thm. 8.1], we know that the complex

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F})$$

is in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ if F is a function field. This settles in particular the case that the characteristic p of F is equal to ℓ. So, let us assume that ℓ is invertible on W and $\ell \neq 2$ if F is a totally real field.

We begin with the case of étale cohomology with proper support. Letting $i \colon \Sigma \to W$ denote the complement of U in W, we have the exact excision sequence

$$\begin{aligned} 0 \to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, k_! k^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) &\to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \\ &\to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, i_* i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) \to 0 \end{aligned}$$

and chains of weak equivalences

$$\begin{aligned} \mathrm{R}\,\Gamma_{\mathrm{c}}(W, k_! k^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) &\sim \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}(1)), \\ \mathrm{R}\,\Gamma_{\mathrm{c}}(W, i_* i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}(1)) &\sim \mathrm{R}\,\Gamma(\Sigma, i^* \,\mathrm{R}\, k_* f_! f^* \mathcal{F}(1)). \end{aligned}$$

By Theorem 3.4.3, we may thus reduce to the case $W = U$. In particular, this settles the function field case.

Furthermore, we may shrink U ad libitum. Hence, we may assume that F_∞/F is really admissible, $\mathcal{F}$ is locally constant on U and smooth at ∞. Consequently, there exists a finite Galois extension F'/F such that F' is totally real, $g_{F'} \colon U_{F'} \to U$ is étale and $g_{F'}^* \mathcal{F}$ is constant. Then F'_{cyc}/F is an admissible extension and

$$\rho \coloneqq g_{F'}^* \mathcal{F}(U_{F'})$$

may be viewed as a continuous representation of $G = \mathrm{Gal}(F'_{\mathrm{cyc}}/F)$ on a finitely generated, projective Λ-module. Write $g \colon U_{F'_{\mathrm{cyc}}} \to U$ for the corresponding system of coverings of U and observe that there exists a weak equivalence

$$\Phi_\rho(\mathrm{R}\,\Gamma_{\mathrm{c}}(U, g_! g^* (\mathbb{Z}_\ell)_U(1))) \xrightarrow{\sim} \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}(1))$$

with Φ_ρ being defined by (2.6.2) [**Wit14**, Prop. 5.9, 6.3, 6.5, 6.7]. Since Φ_ρ takes complexes in $\mathbf{PDG}^{\mathrm{cont},w_H}(\mathbb{Z}_\ell[[G]])$ to complexes in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[\Gamma]])$, it remains to show that the cohomology groups $\mathrm{H}^s_{\mathrm{c}}(U, g_! g^*(\mathbb{Z}_\ell)_U(1))$ are finitely generated as $\mathbb{Z}_\ell$-modules.

Let M denote the maximal abelian ℓ-extension of F'_{cyc} unramified over U. Then

$$\mathrm{H}^s_{\mathrm{c}}(U, g_! g^*(\mathbb{Z}_\ell)_U(1)) = \begin{cases} 0 & \text{if } s \neq 2,3, \\ \mathrm{Gal}(M/F'_{\mathrm{cyc}}) & \text{if } s = 2, \\ \mathbb{Z}_\ell & \text{if } s = 3 \end{cases}$$

by [**Kak13**, p. 548]. At this point, we make use of Conjecture 3.3.4 on the vanishing of the μ-invariant to finish the proof for the first complex.

We now turn to the complex $\mathrm{R}\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet)$. We still assume that Λ is a finite ring. Write $\Sigma = W - U$, $V = U \cup (X - W)$ and $j: U \to V$, $j': V \to X$, $i: \Sigma \to X$ for the natural immersions. As mentioned in Remark 3.1.4, the exists a chain of weak equivalences

$$\mathrm{R}\Gamma_{\mathrm{c}}(V, \mathrm{R} j_* f_! f^* \mathcal{F}) \sim \mathrm{R}\Gamma(W, k_! f_! f^* \mathcal{F}).$$

In the function field case, we are thus reduced to the case already treated above. The proof of Theorem 3.4.1 is now complete.

So, let us again assume that F_∞/F is really admissible. Using the exact sequence

$$0 \to j'_! \mathrm{R} j_* f_! f^* \mathcal{F} \to \mathrm{R}(j' \circ j)_* f_! f^* \mathcal{F} \to i_* i^* \mathrm{R} j_* f_! f^* \mathcal{F} \to 0$$

and Theorem 3.4.3 we may reduce to the case that $V = X$, $W = U$ and $\mathcal{F}$ locally constant on U and smooth at ∞.

Let $P^\bullet$ be a strictly perfect complex of $\Lambda^{\mathrm{op}}[[G]]$-modules quasi-isomorphic to $\mathrm{R}\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^{*_\Lambda}(1))$. By what we have proved above, $P^\bullet$ is also perfect as complex of $\Lambda^{\mathrm{op}}[[H]]$-modules. By (3.3.4) we obtain a weak equivalence

$$^\sharp(P^\bullet)^{*_{\Lambda^{\mathrm{op}}[[G]]}} \xrightarrow{\sim} \mathrm{R}\Gamma(U, f_! f^* \mathcal{F}).$$

We conclude that $\mathrm{R}\Gamma(U, f_! f^* \mathcal{F})$ is in $\mathbf{PDG}^{\mathrm{cont},w_H}(U, \Lambda[[G]])$ by applying Proposition 2.7.4. This completes the proof of Theorem 3.4.2.

3.5. Non-Commutative Euler Factors

Assume as before that F_∞/F is an admissible extension of a global field F which is unramified over a dense open subscheme U of X and write $f: U_{F_\infty} \to U$ for the system of Galois coverings of U corresponding to F_∞/F. If the characteristic of F is different from ℓ, we let W be another dense open subscheme of X containing U, such that ℓ is invertible on W. If the characteristic of F is equal to ℓ, we choose $W = U$. Write $k: U \to W$ for the corresponding open immersion. We consider a complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$. As the complexes

$$\mathrm{R}\Gamma(x, i^* \mathrm{R} k_* f_! f^* \mathcal{F}^\bullet)$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ for $i: x \to W$ a closed point, we conclude that the endomorphism

$$\mathrm{R}\Gamma(\hat{x}, i^* \mathrm{R} k_* f_! f^* \mathcal{F}^\bullet) \xrightarrow{\mathrm{id} - \mathfrak{F}_x} \mathrm{R}\Gamma(\hat{x}, i^* \mathrm{R} k_* f_! f^* \mathcal{F}^\bullet)$$

is in fact a weak equivalence in $w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$. Hence, it gives rise to an element in $\mathrm{K}_1(\Lambda[[G]]_S)$.

DEFINITION 3.5.1. The non-commutative Euler factor $\mathcal{L}_{F_\infty/F}(x, \mathrm{R} k_* \mathcal{F}^\bullet)$ of $\mathrm{R} k_* \mathcal{F}^\bullet$ at x is the inverse of the class of the above weak equivalence in $\mathrm{K}_1(\Lambda[[G]]_S)$:

$$\mathcal{L}_{F_\infty/F}(x, \mathrm{R} k_* \mathcal{F}^\bullet) = [\mathrm{id} - \mathfrak{F}_x \circlearrowright \mathrm{R}\Gamma(\hat{x}, i^* \mathrm{R} k_* f_! f^* \mathcal{F}^\bullet)]^{-1}$$

Note that $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$ is independent of our specific choice of a geometric point above x. Indeed, by (3.1.1) and relation (R5) in the definition of $\mathcal{D}_\bullet(\mathbf{W})$, we conclude that the classes $[\mathrm{id} - \mathfrak{F}_x]$ and $[\mathrm{id} - \mathfrak{F}'_x]$ agree in $\mathrm{K}_1(\Lambda[[G]]_S)$. Moreover, $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$ does not change if we enlarge W by adding points with residue field characteristic different from ℓ or shrink U by removing a finite set of points different from x.

PROPOSITION 3.5.2. *The non-commutative Euler factor is a characteristic element for* $\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)$*:*

$$d\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet) = -[\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)]$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

PROOF. The complex $\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)$ is weakly equivalent to the cone of the endomorphism

$$\mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_x} \mathrm{R}\,\Gamma(\hat{x}, i^*\,\mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)$$

shifted by one. Hence, the result follows from the explicit description of d given in (2.3.2). □

DEFINITION 3.5.3. For a topological generator $\gamma \in \Gamma$, we define the local modification factor at x to be the element

$$\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet) \coloneqq \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet) s_\gamma([\mathrm{R}\,\Gamma(x, i^*\,\mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)]).$$

in $\mathrm{K}_1(\Lambda[[G]])$.

We obtain the following transformation properties.

PROPOSITION 3.5.4. *With* $k\colon U \to W$ *as above, let* Λ *be any adic* $\mathbb{Z}_\ell$*-algebra and let* $\mathcal{F}^\bullet$ *be a complex in* $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.

(1) *Let* Λ' *be another adic* $\mathbb{Z}_\ell$*-algebra. For any complex* $P^\bullet$ *of* Λ'-$\Lambda[[G]]$*-bimodules which is strictly perfect as complex of* Λ'*-modules we have*

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = \mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

in $\mathrm{K}_1(\Lambda'[[G]]_S)$ *and*

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = \mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

in $\mathrm{K}_1(\Lambda'[[G]])$.

(2) *Let* F'_∞/F *be an admissible subextension of* F_∞/F *with Galois group* G'*. Then*

$$\Psi_{\Lambda[[G']]}(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = \mathcal{L}_{F'_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$ *and*

$$\Psi_{\Lambda[[G']]}(\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)) = \mathcal{M}_{F'_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']])$.

(3) *Let* F'/F *be a finite extension inside* F_∞/F*. Set* $r \coloneqq [F' \cap F_{\mathrm{cyc}} : F]$*. Write* $f_{F'}\colon U_{F'} \to U$ *for the corresponding étale covering and* $x_{F'}$ *for the fibre in* $X_{F'}$ *above* x*. Let* $G' \subset G$ *be the Galois group of the admissible extension* F_∞/F' *and consider* $\Lambda[[G]]$ *as a* $\Lambda[[G']]$-$\Lambda[[G]]$*-bimodule. Then*

$$\Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{L}_{F_\infty/F'}(y, \mathrm{R}\,k_* f^*_{F'}\mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$ *and*

$$\Psi_{\Lambda[[G]]}\big(\mathcal{M}_{F_\infty/F,\gamma}(x, \mathrm{R}\,k_*\mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{M}_{F_\infty/F',\gamma^r}(y, \mathrm{R}\,k_* f^*_{F'}\mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']])$.

(4) *With the notation of* (3), *assume that* $\mathcal{G}^\bullet$ *is a complex in* $\mathbf{PDG}^{\mathrm{cont}}(U_{F'},\Lambda)$ *and consider* $\Lambda[[G]]$ *as a* $\Lambda[[G]]$-$\Lambda[[G']]$*-bimodule. Then*

$$\prod_{y\in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F'}(y,\mathrm{R}\,k_*\mathcal{G}^\bullet)\big) = \mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_* f_{F'*}\mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$ *and*

$$\prod_{y\in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{M}_{F_\infty/F',\gamma^r}(y,\mathrm{R}\,k_*\mathcal{G}^\bullet)\big) = \mathcal{M}_{F_\infty/F',\gamma}(x,\mathrm{R}\,k_* f_{F'*}\mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]])$.

Proof. Note that the functor Ψ commutes up to weak equivalences with $\mathrm{R}\,\Gamma$, i^*, and $\mathrm{R}\,k_*$ [**Wit08**, 5.5.7] and apply Proposition 3.3.2 and Proposition 2.9.1. Part (1) and (2) are direct consequences.

For Part (3), we additionally need the same reasoning as in the proof of [**Wit14**, Thm. 8.4.(3)] to verify that for any $\mathcal{G}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U_{F'},\Lambda)$

$$[\mathrm{id}-\mathfrak{F}_x \circlearrowleft \mathrm{R}\,\Gamma(y\times_x\hat{x},\mathrm{R}\,k_*g_!g^*\mathcal{G}^\bullet)] = [\mathrm{id}-\mathfrak{F}_y \circlearrowleft \mathrm{R}\,\Gamma(\hat{y},\mathrm{R}\,k_*g_!g^*f_{F'}^*\mathcal{G}^\bullet)] \tag{3.5.1}$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$. Here, $g\colon U_{F_\infty}\to U_{F'}$ denotes the system of coverings induced by f. This implies the formula for $\Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{F}^\bullet)\big)$. Moreover, we have a weak equivalence

$$\Psi_{\Lambda[[G]]}\,\mathrm{R}\,\Gamma(x,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet)\xrightarrow{\sim}\mathrm{R}\,\Gamma(x_{F'},\mathrm{R}\,k_*g_!g^*f_{F'}^*\mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G']])$. In particular,

$$s_{\gamma^r}([\Psi_{\Lambda[[G]]}\,\mathrm{R}\,\Gamma(x,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet)]) = \prod_{y\in x_{F'}} s_{\gamma^r}([\mathrm{R}\,\Gamma(y,\mathrm{R}\,k_*g_!g^*f_{F'}^*\mathcal{F}^\bullet)])$$

from which the formula for $\Psi_{\Lambda[[G]]}\big(\mathcal{M}_{F_\infty/F,\gamma}(x,\mathrm{R}\,k_*\mathcal{F}^\bullet)\big)$ follows.

For Part (4) we use (3.5.1) to show

$$\begin{aligned}\prod_{y\in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{L}_{F_\infty/F'}(y,\mathrm{R}\,k_*\mathcal{G}^\bullet)\big) &= \\ &= \Psi_{\Lambda[[G]]}([\mathrm{id}-\mathfrak{F}_x \circlearrowleft \mathrm{R}\,\Gamma(x_{F'}\times_x\hat{x},\mathrm{R}\,k_*g_!g^*\mathcal{G}^\bullet)]^{-1})\\ &= [\mathrm{id}-\mathfrak{F}_x \circlearrowleft \mathrm{R}\,\Gamma(\hat{x},\mathrm{R}\,k_*f_!f^*f_{F'*}\mathcal{G}^\bullet)]^{-1}\\ &= \mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*f_{F'*}\mathcal{G}^\bullet).\end{aligned}$$

On the other hand, we also have a weak equivalence

$$\Psi_{\Lambda[[G]]}\,\mathrm{R}\,\Gamma(x_{F'},\mathrm{R}\,k_*g_!g^*\mathcal{G}^\bullet)\xrightarrow{\sim}\mathrm{R}\,\Gamma(x,\mathrm{R}\,k_*f_!f^*f_{F'*}\mathcal{G}^\bullet),$$

thence the formula for the local modification factors. □

For the rest of this section, we assume that the characteristic of F is different from ℓ. If $\mathcal{G}$ is a smooth Λ-adic sheaf on U and x is a point in U, it makes sense to consider the element

$$\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{G}^{*_\Lambda}(1))^{\circledast}\in\mathrm{K}_1(\Lambda[[G]],S)$$

as an alternative Euler factor, which does not agree with $\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{G})$ in general. We shall show below that

$$\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{G}^{*_\Lambda}(1))^{\circledast} = [\mathrm{id}-\mathfrak{F}_x^{-1}\circlearrowleft \mathrm{R}\,\Gamma(\hat{x},\mathrm{R}\,i^!k_!f_!f^*\mathcal{G})]$$

and take this as a definition for arbitrary complexes $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.

Definition 3.5.5. The dual non-commutative Euler factor of $k_!\mathcal{F}^\bullet$ at $x\in W$ is the element

$$\mathcal{L}^{\circledast}_{F_\infty/F}(x,k_!\mathcal{F}^\bullet) \coloneqq [\mathrm{id}-\mathfrak{F}_x^{-1}\circlearrowleft \mathrm{R}\,\Gamma(\hat{x},\mathrm{R}\,i^!k_!f_!f^*\mathcal{F}^\bullet)]$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$.

PROPOSITION 3.5.6. *The inverse of the dual non-commutative Euler factor is a characteristic element for* $\mathrm{R}\Gamma(x, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}^\bullet)$*:*

$$d\mathcal{L}^{\circledast}_{F_\infty/F}(x, k_! \mathcal{F}^\bullet) = [\mathrm{R}\Gamma(x, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}^\bullet)]$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

PROOF. The complex $\mathrm{R}\Gamma(x, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet)$ is weakly equivalent to the cone of the endomorphism

$$\mathrm{R}\Gamma(\hat{x}, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}^\bullet) \xrightarrow{\mathrm{id} - \mathfrak{F}_x^{-1}} \mathrm{R}\Gamma(\hat{x}, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}^\bullet)$$

shifted by one. Hence, the result follows from the explicit description of d given in [**Wit14**, Thm. A.5]. □

DEFINITION 3.5.7. For a topological generator $\gamma \in \Gamma$, the dual local modification factor $k_! \mathcal{F}^\bullet$ at x is the element

$$\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x, k_! \mathcal{F}^\bullet) := \mathcal{L}^{\circledast}_{F_\infty/F}(x, k_! \mathcal{F}^\bullet) s_{\gamma^{-1}}([\mathrm{R}\Gamma(x, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F}^\bullet)])^{-1}.$$

We obtain the following transformation properties.

PROPOSITION 3.5.8. *With* $k: U \to W$ *as above, let* Λ *be any adic* $\mathbb{Z}_\ell$*-algebra and let* $\mathcal{F}^\bullet$ *be a complex in* $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$.

(1) *Let* Λ' *be another adic* $\mathbb{Z}_\ell$*-algebra. For any complex* $P^\bullet$ *of* Λ'-$\Lambda[[G]]$*-bimodules which is strictly perfect as complex of* Λ'*-modules we have*

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{L}^{\circledast}_{F_\infty/F}(x, k_! \mathcal{F}^\bullet)) = \mathcal{L}^{\circledast}_{F_\infty/F}(x, k_! \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

in $\mathrm{K}_1(\Lambda'[[G]]_S)$ *and*

$$\Psi_{P[[G]]^{\delta\bullet}}(\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x, k_! \mathcal{F}^\bullet)) = \mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x, k_! \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

in $\mathrm{K}_1(\Lambda'[[G]])$.

(2) *Let* F'_∞/F *be an admissible subextension of* F_∞/F *with Galois group* G'*. Then*

$$\Psi_{\Lambda[[G']]}(\mathcal{L}^{\circledast}_{F_\infty/F}(x, k_! \mathcal{F}^\bullet)) = \mathcal{L}^{\circledast}_{F'_\infty/F}(x, k_! \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$ *and*

$$\Psi_{\Lambda[[G']]}(\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x, k_! \mathcal{F}^\bullet)) = \mathcal{M}^{\circledast}_{F'_\infty/F,\gamma}(x, k_! \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']])$.

(3) *Let* F'/F *be a finite extension inside* F_∞/F*. Set* $r := [F' \cap F_{\mathrm{cyc}} : F]$*. Write* $f_{F'}: U_{F'} \to U$ *for the corresponding étale covering and* $x_{F'}$ *for the fibre in* $X_{F'}$ *above* x*. Let* $G' \subset G$ *be the Galois group of the admissible extension* F_∞/F' *and consider* $\Lambda[[G]]$ *as a* $\Lambda[[G']]$-$\Lambda[[G]]$*-bimodule. Then*

$$\Psi_{\Lambda[[G]]}\big(\mathcal{L}^{\circledast}_{F_\infty/F}(x, k_! \mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{L}^{\circledast}_{F_\infty/F'}(y, k_! f^*_{F'} \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']]_S)$ *and*

$$\Psi_{\Lambda[[G]]}\big(\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x, k_! \mathcal{F}^\bullet)\big) = \prod_{y \in x_{F'}} \mathcal{M}^{\circledast}_{F_\infty/F',\gamma^r}(y, k_! f^*_{F'} \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G']])$.

(4) *With the notation of* (3)*, assume that* $\mathcal{G}^\bullet$ *is a complex in* $\mathbf{PDG}^{\mathrm{cont}}(U_{F'}, \Lambda)$ *and consider* $\Lambda[[G]]$ *as a* $\Lambda[[G]]$-$\Lambda[[G']]$*-bimodule. Then*

$$\prod_{y \in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{L}^{\circledast}_{F_\infty/F'}(y, k_! \mathcal{G}^\bullet)\big) = \mathcal{L}^{\circledast}_{F_\infty/F}(x, k_! f_{F'*} \mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$ *and*

$$\prod_{y \in x_{F'}} \Psi_{\Lambda[[G]]}\big(\mathcal{M}^{\circledast}_{F_\infty/F',\gamma^r}(y, k_! \mathcal{G}^\bullet)\big) = \mathcal{M}^{\circledast}_{F_\infty/F',\gamma}(x, k_! f_{F'*} \mathcal{G}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[G]])$.

PROOF. The arguments are the same as in the proof of Proposition 3.5.4. □

PROPOSITION 3.5.9.

(1) *Let* $\mathcal{G}$ *be a smooth* Λ*-adic sheaf on* U*. Then*

$$(\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{G}^{*_\Lambda}(1)))^{\circledast}=\mathcal{L}^{\circledast}_{F_\infty/F}(x,k_!\mathcal{G})=$$
$$=\begin{cases}[-\mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x},i^*f_!f^*\mathcal{G}(-1))]^{-1}\mathcal{L}_{F_\infty/F}(x,\mathcal{G}(-1))^{-1} & \text{if } x\in U\\ [-\mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x},i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{G})]\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{G}) & \text{if } x\in W-U\end{cases}$$

in $\mathrm{K}_1(\Lambda[[G]],S)$ *and*

$$(\mathcal{M}_{F_\infty/F,\gamma}(x,\mathrm{R}\,k_*\mathcal{G}^{*_\Lambda}(1)))^{\circledast}=\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x,k_!\mathcal{G})$$

in $\mathrm{K}_1(\Lambda[[G]])$.

(2) *Let* $\mathcal{G}$ *be a* Λ*-adic sheaf on* $x\in U$*. Then*

$$(\mathcal{L}_{F_\infty/F}(x,i_*\mathcal{G}^{*_\Lambda}))^{\circledast}=\mathcal{L}^{\circledast}_{F_\infty/F}(x,i_*\mathcal{G})$$
$$=[-\mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x},i^*f_!f^*i_*\mathcal{G})]^{-1}\mathcal{L}_{F_\infty/F}(x,i_*\mathcal{G})^{-1}$$

in $\mathrm{K}_1(\Lambda[[G]],S)$ *and*

$$(\mathcal{M}_{F_\infty/F,\gamma}(x,i_*\mathcal{G}^{*_\Lambda}))^{\circledast}=\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x,i_*\mathcal{G})$$

in $\mathrm{K}_1(\Lambda[[G]])$.

PROOF. We only need to prove the formulas for the non-commutative Euler factors, the formulas for the local modification factors then follow from Proposition 2.9.3.

We begin by proving (1) in the case that $x\in W-U$. By (3.3.3), combined with Lemma 3.1.8 and Lemma 3.1.5, we have

$$\begin{aligned}\mathcal{L}_{F_\infty/F}(x,i_*\,\mathrm{R}\,k_*\mathcal{G}^{*_\Lambda}(1))&=\\ &=\Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp}([\mathrm{id}-\mathfrak{F}_x \circlearrowright \mathrm{R}\,\Gamma(\hat{x},(i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{G})^{*_{\Lambda[[G]]}}(1))]^{-1})\\ &=\Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp}([\mathrm{id}-\mathfrak{F}_x \circlearrowright D^\bullet_{\hat{x}}((\mathrm{R}\,k_*f_!f^*\mathcal{G})^{*_{\Lambda[[G]]}}(1))]^{-1}).\end{aligned}$$

From the Definition 2.7.1 of $\circledast$, Lemma 3.2.1, and again Lemma 3.1.5, we conclude

$$\begin{aligned}{}^\sharp\big(\Psi_{\Lambda^{\mathrm{op}}[[G]]^\sharp}([\mathrm{id}-\mathfrak{F}_x \circlearrowright D^\bullet_{\hat{x}}((\mathrm{R}\,k_*f_!f^*\mathcal{G})^{*_{\Lambda[[G]]}}(1))]^{-1})\big)^{*_{\Lambda^{\mathrm{op}}[[G]]}}&=\\ &=[\mathrm{id}-\mathfrak{F}_x^{*_{\Lambda[[G]]^{\mathrm{op}}}} \circlearrowright D^\bullet_{\hat{x}}((\mathrm{R}\,k_*f_!f^*\mathcal{G})^{*_{\Lambda[[G]]}}(1))^{*_{\Lambda[[G]]^{\mathrm{op}}}}]\\ &=[\mathrm{id}-\mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\,\Gamma(\hat{x},i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{G})]^{-1}\\ &=[-\mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\,\Gamma(\hat{x},i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{G})]^{-1}\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{G}).\end{aligned}$$

Finally,

$$[\mathrm{id}-\mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\,\Gamma(\hat{x},i^*\,\mathrm{R}\,k_*f_!f^*\mathcal{G})]^{-1}=\mathcal{L}^{\circledast}_{F_\infty/F}(x,k_!\mathcal{G})$$

by Lemma 3.1.9.

The validity of the first equality in (2) follows similarly from Lemma 3.2.2 and the exchange formula (3.2.5):

$$\begin{aligned}(\mathcal{L}_{F_\infty/F}(x,i_*\mathcal{G}^{*_\Lambda}))^{\circledast}&=([\mathrm{id}-\mathfrak{F}_x \circlearrowright (i^*f_!f^*i_*\mathcal{G}^{*_\Lambda})_{\hat{x}}]^{-1})^{\circledast}\\ &=\Psi_{\Lambda[[G]]^\sharp}([\mathrm{id}-\mathfrak{F}_x^{*_{\Lambda[[G]]^{\mathrm{op}}}} \circlearrowright ((i^*f_!f^*i_*\mathcal{G}^{*_\Lambda})_{\hat{x}})^{*_{\Lambda[[G]]^{\mathrm{op}}}}])\\ &=\Psi_{\Lambda[[G]]^\sharp}([\mathrm{id}-\mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\,\Gamma(\hat{x},(i^*f_!f^*i_*\mathcal{G}^{*_\Lambda}))^{*_{\Lambda[[G]]^{\mathrm{op}}}}])\\ &=[\mathrm{id}-\mathfrak{F}_x^{-1} \circlearrowright \mathrm{R}\,\Gamma(\hat{x},\mathrm{R}\,i^!f_!f^*(i_*\mathcal{G}^{*_\Lambda}))^{*_{\Lambda^{\mathrm{op}}}}]\\ &=\mathcal{L}^{\circledast}_{F_\infty/F}(x,i_*\mathcal{G})\end{aligned}$$

Further, write $j' \colon U' \to U$ for the complement of x in U. Then

$$\mathrm{R}\, j'_* j'^* f_! f^* i_* \mathcal{G} \cong \mathrm{R}\, j'_* f_! f^* j'^* i_* \mathcal{G} = 0$$

and hence,

$$\mathrm{R}\, i^! f_! f^* i_* \mathcal{G} \cong i^* f_! f^* i_* \mathcal{G},$$

from which the second equality in (2) follows.

For the proof of (1) in the case that $x \in U$, we observe that

$$\begin{aligned}(\mathcal{L}_{F_\infty/F}(x, \mathcal{G}^{*\Lambda}(1)))^{\circledast} &= (\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, j'_* j'^* \mathcal{G}^{*\Lambda}(1)))^{\circledast} (\mathcal{L}_{F_\infty/F}(x, i_* \mathrm{R}\, i^! \mathcal{G}^{*\Lambda}(1)))^{\circledast} \\ &= (\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, j'_* j'^* \mathcal{G}^{*\Lambda}(1)))^{\circledast} (\mathcal{L}_{F_\infty/F}(x, i_* (i^* \mathcal{G})^{*\Lambda}))^{\circledast} \\ &= \mathcal{L}^{\circledast}_{F_\infty/F}(x, j'_! j'^* \mathcal{G}) \mathcal{L}^{\circledast}_{F_\infty/F}(x, i^* \mathcal{G}) \\ &= \mathcal{L}^{\circledast}_{F_\infty/F}(x, \mathcal{G})\end{aligned}$$

by what we have proved above. For the second equality, we use that by absolute purity [**Mil06**, Ch. II, Cor. 1.6], there exists chain of weak equivalences

$$i^* f_! f^* \mathcal{G}(-1) \sim \mathrm{R}\, i^! f_! f^* \mathcal{G}[2].$$

□

3.6. Euler Factors for the Cyclotomic Extension

In the case $F_\infty = F_{\mathrm{cyc}}$, we can give a different description of $\mathcal{L}_{F_\infty/F}(x, \mathrm{R}\, k_* \mathcal{F}^\bullet)$. We will undergo the effort to allow arbitrary adic $\mathbb{Z}_\ell$-algebras Λ as coefficient rings, but in the end, we will use the results only in the case that Λ is the valuation ring in a finite extension of $\mathbb{Q}_\ell$. If one restricts to this case, some of the technical constructions that follow may be skipped.

Let $\Lambda[t]$ be the polynomial ring over Λ in the indeterminate t that is assumed to commute with the elements of Λ. In Appendix A we define a Waldhausen category $w_t\mathbf{P}(\Lambda[t])$: The objects are perfect complexes of $\Lambda[t]$-modules and cofibrations are injective morphism of complexes such that the cokernel is again perfect. A weak equivalence is a morphism $f \colon P^\bullet \to Q^\bullet$ of perfect complexes of $\Lambda[t]$-modules such that $\Lambda \otimes^{\mathbb{L}}_{\Lambda[t]} f$ is a quasi-isomorphism of complexes of Λ-modules. Here, Λ is considered as a Λ-$\Lambda[t]$-bimodule via the augmentation map and $\Lambda \otimes^{\mathbb{L}}_{\Lambda[t]} \cdot$ denotes the total derived tensor product as functor between the derived categories.

If Λ is noetherian, then the subset

$$S_t \coloneqq \{ f(t) \in \Lambda[t] \mid f(0) \in \Lambda^\times \} \subset \Lambda[t]$$

is a left and right denominator set, the localisation $\Lambda[t]_{S_t}$ is semi-local and $\Lambda[t] \to \Lambda[t]_{S_t}$ induces an isomorphism

$$\mathrm{K}_1(w_t\mathbf{P}(\Lambda[t])) \cong \mathrm{K}_1(\Lambda[t]_{S_t})$$

(Proposition A.1). By [**War93**, Cor. 36.35], commutative adic rings are always noetherian . In this case, we may further identify

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \cong \Lambda[t]^\times_{S_t}$$

via the determinant map. In general, S_t is not a left or right denominator set. We then take

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \coloneqq \mathrm{K}_1(w_t\mathbf{P}(\Lambda[t]))$$

as a definition.

For any adic $\mathbb{Z}_\ell$-algebra Λ and any $\gamma \in \Gamma \cong \mathbb{Z}_\ell$, the ring homomorphism

$$\mathrm{ev}_\gamma \colon \Lambda[t] \mapsto \Lambda[[\Gamma]], \quad f(t) \mapsto f(\gamma).$$

induces a homomorphism

$$\mathrm{ev}_\gamma \colon \mathrm{K}_1(\Lambda[t]_{S_t}) \to \mathrm{K}_1(\Lambda[[\Gamma]]_S)$$

(Proposition A.2). In the noetherian case, the proof boils down to a verification that $\mathrm{ev}_\gamma(S_t) \subset S$.

Assume as before that $W = U$ if the characteristic p of F is equal to ℓ and that ℓ is an odd prime invertible on W if F is a number field.

DEFINITION 3.6.1. For $\mathcal{F}^\bullet = (\mathcal{F}_I^\bullet)_{I\in\mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ we define

$$Z(x, \mathrm{R}\, k_* \mathcal{F}^\bullet, t) \coloneqq [\mathrm{id} - t\mathfrak{F}_x \circlearrowleft P^\bullet]^{-1} \in \mathrm{K}_1(\Lambda[t]_{S_t})$$

for

$$P^\bullet \coloneqq \Lambda[t] \otimes_\Lambda \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\hat{x}, i^* \mathrm{R}\, k_* \mathcal{F}^\bullet).$$

If $p \neq \ell$, we set

$$Z^{\circledast}(x, k_! \mathcal{F}^\bullet, t) \coloneqq [\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowleft Q^\bullet] \in \mathrm{K}_1(\Lambda[t]_{S_t})$$

where

$$Q^\bullet \coloneqq \Lambda[t] \otimes_\Lambda \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\, i^! k_! \mathcal{F}^\bullet).$$

For any $1 \neq \gamma \in \Gamma$, we write $Z(x, \mathrm{R}\, k_* \mathcal{F}^\bullet, \gamma)$ and $Z^{\circledast}(x, k_! \mathcal{F}^\bullet, \gamma)$ for the images of $Z(x, \mathrm{R}\, k_* \mathcal{F}^\bullet, t)$ and $Z^{\circledast}(x, k_! \mathcal{F}^\bullet, t)$ under

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow{\mathrm{ev}_\gamma} \mathrm{K}_1(\Lambda[[\Gamma]]_S).$$

Since the endomorphism $\mathrm{id} - t\mathfrak{F}_x$ is canonical, it follows easily from the relations in the definition of $\mathcal{D}_\bullet(\mathbf{W})$ that $Z(x, \mathcal{F}^\bullet, t)$ does only depend on the weak equivalence class of $\mathcal{F}^\bullet$ and is multiplicative on exact sequences. So, it defines a homomorphism

$$Z(x, \mathrm{R}\, k_*(-), t) \colon \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)) \to \mathrm{K}_1(\Lambda[t]_{S_t}).$$

The same is also true for $Z^{\circledast}(x, k_! \mathcal{F}^\bullet, t)$.

PROPOSITION 3.6.2. *Let $\gamma_x \in \Gamma$ be the image of $\mathfrak{F}_x$ in Γ. Then*

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\, k_* \mathcal{F}^\bullet) = Z(x, \mathrm{R}\, k_* \mathcal{F}^\bullet, \gamma_x^{-1}).$$

If $p \neq \ell$, then

$$\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, k_! \mathcal{F}^\bullet) = Z^{\circledast}(x, k_! \mathcal{F}^\bullet, \gamma_x).$$

PROOF. Since ℓ is invertible on W in the number field case, the extension F_{cyc}/F is unramified over W. Assume $p \neq \ell$. By the smooth base change theorem applied to the étale morphism $f_K \colon W_K \to W$ for each finite subextension K/F of F_{cyc}/F and the quasi-compact morphism $k \colon U \to W$ there exists a weak equivalence

$$f_! f^* \mathrm{R}\, k_* \mathcal{F}^\bullet \xrightarrow{\sim} \mathrm{R}\, k_* f_! f^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(W,\Lambda)$. By the proper base change theorem, there exists also an isomorphism

$$f_! f^* k_! \mathcal{F}^\bullet \cong k_! f_! f^* \mathcal{F}^\bullet$$

Hence, we may assume $x \in U = W$ and drop the assumption $p \neq \ell$.

For any finite subextension K/F in F_{cyc}/F write x_K for the set of places of K lying over x and $g \colon x_{F_{\mathrm{cyc}}} \to x$ for the corresponding system of Galois covers. (We note that this system might not be admissible in the sense of [**Wit14**, Def. 2.6] for any base field $\mathbb{F} \subset k(x)$: for example if $F = \mathbb{Q}$ and $x = (p)$ with $p \neq \ell$ splitting in the cyclotomic $\mathbb{Z}_\ell$-extension of $\mathbb{Q}$.) By the proper base change theorem there exists an isomorphism

$$i^* f_! f^* \mathcal{F}^\bullet \cong g_! g^* i^* \mathcal{F}^\bullet.$$

From Lemma 3.1.9 we can then also infer the existence of a weak equivalence

$$\mathrm{R}\, i^! f_! f^* \mathcal{F}^\bullet \xrightarrow{\sim} g_! g^* \mathrm{R}\, i^! \mathcal{F}^\bullet$$

if $p \neq \ell$.

We will now concentrate on the proof of the equality

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathcal{F}^\bullet) = Z(x, \mathcal{F}^\bullet, \gamma_x^{-1}).$$

The proof of the equality for the dual Euler factors follows along the same lines, with $\mathfrak{F}_x$ replaced by $\mathfrak{F}_x^{-1}$ and γ_x^{-1} replaced by γ_x.

By our choice of the embedding $\overline{F} \subset \overline{F}_x$, we have a compatible system of morphisms $\operatorname{Spec} \overline{k(x)} \to x_K$ for each $K \subset F_{\mathrm{cyc}}$ and hence, distinguished isomorphisms

$$\alpha \colon \mathbb{Z}[\mathrm{Gal}(K/F)] \otimes_{\mathbb{Z}} \mathcal{M}_{\hat{x}} \to (g_{K!} g_K^* \mathcal{M})_{\hat{x}}$$

for the stalk $\mathcal{M}_{\hat{x}}$ in $\hat{x}$ of any étale sheaf $\mathcal{M}$ on x. The action of the Frobenius $\mathfrak{F}_x$ on the righthand side corresponds to the operation of $\cdot\gamma_x^{-1} \otimes \mathfrak{F}_x$ on the left-hand side. By compatibility, we may extend α to an isomorphism

$$\alpha \colon \Psi_{\Lambda[[\Gamma]]} \operatorname{R}\Gamma(\hat{x}, i^* \mathcal{F}^\bullet) \cong \operatorname{R}\Gamma(\hat{x}, g_! g^* i^* \mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[\Gamma]])$. Hence,

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathcal{F}^\bullet) = [\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Psi_{\Lambda[[\Gamma]]} \operatorname{R}\Gamma(\hat{x}, i^* \mathcal{F}^\bullet)]^{-1}$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. Furthermore, we may choose a strictly perfect complex of Λ-modules $P^\bullet$ with an endomorphism f and a quasi-isomorphism

$$\beta \colon P^\bullet \to \varprojlim_{I \in \mathfrak{I}_\Lambda} \operatorname{R}\Gamma(\hat{x}, \mathcal{G}^\bullet)$$

under which f and $\mathfrak{F}_x$ are compatible up to chain homotopy [**Wit08**, Lem. 3.3.2]. The endomorphism

$$\mathrm{id} - tf \colon \Lambda[t] \otimes_\Lambda P^\bullet \to \Lambda[t] \otimes_\Lambda P^\bullet$$

is clearly a weak equivalence in $w_t\mathbf{P}(\Lambda[t])$. By [**Wit08**, Lem. 3.1.6], homotopic weak auto-equivalences have the same class in the first K-group. Hence, we may conclude

$$[\mathrm{id} - tf \circlearrowright \Lambda[t] \otimes_\Lambda P^\bullet]^{-1} = Z(x, \operatorname{R} k_* \mathcal{F}^\bullet, t)$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$Z(x, \operatorname{R} k_* \mathcal{F}^\bullet, \gamma_x^{-1}) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \operatorname{R} k_* \mathcal{F}^\bullet)$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. □

We will make this construction a little more explicit in the case that $\mathcal{F}$ is a Λ-adic sheaf on U. If $x \in U$, recall from (3.2.3) that there is a weak equivalence

$$\mathcal{F}_{\hat{x}} \xrightarrow{\sim} \operatorname{R}\Gamma(\hat{x}, i^* \operatorname{R} k_* \mathcal{F})$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ compatible with the operation of the Frobenius $\mathfrak{F}_x$ on both sides.

Hence, we have

$$Z(x, \operatorname{R} k_* \mathcal{F}, t) = [\Lambda[t] \otimes_\Lambda \mathcal{F}_{\hat{x}} \xrightarrow{\mathrm{id} - t\mathfrak{F}_x} \Lambda[t] \otimes_\Lambda \mathcal{F}_{\hat{x}}]^{-1} \tag{3.6.1}$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \operatorname{R} k_* \mathcal{F}) = [\Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}_{\hat{x}} \xrightarrow{\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x} \Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}_{\hat{x}}]^{-1}$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. In particular, if Λ is commutative, then the isomorphism

$$\mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow{\det} \Lambda[t]_{S_t}^\times$$

sends $Z(x, \operatorname{R} k_* \mathcal{F}, t)$ to the inverse of the reverse characteristic polynomial of the geometric Frobenius operation on $\mathcal{F}_{\hat{x}}$.

Assume now that $p \neq \ell$. If $\mathcal{F}$ is smooth in $x \in U$, then by absolute purity [**Mil06**, Ch. II, Cor. 1.6], there exists chain of weak equivalences

$$\mathcal{F}(-1)_{\hat{x}} \sim \operatorname{R}\Gamma(\hat{x}, \operatorname{R} i^! k_! \mathcal{F})[2].$$

Hence,

$$Z^{\circledast}(x, k_! \mathcal{F}, t) = [\Lambda[t] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}} \xrightarrow{\mathrm{id} - t\mathfrak{F}_x^{-1}} \Lambda[t] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}}] \tag{3.6.2}$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, k_! \mathcal{F}) = [\Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}} \xrightarrow{\mathrm{id} - \gamma_x \otimes \mathfrak{F}_x^{-1}} \Lambda[[\Gamma]] \otimes_\Lambda \mathcal{F}(-1)_{\hat{x}}]$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$. If $\mathcal{F} = i_* \mathcal{G}$ for some $\mathcal{G}$ in $\mathbf{PDG}^{\mathrm{cont}}(x, \Lambda)$, then there exists a weak equivalence

$$\mathcal{G}_{\hat{x}} \xrightarrow{\sim} \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\, i^! k_! \mathcal{F}).$$

Hence,

$$Z^{\circledast}(x, k_! \mathcal{F}, t) = [\Lambda[t] \otimes_\Lambda \mathcal{G}_{\hat{x}} \xrightarrow{\mathrm{id} - t\mathfrak{F}_x^{-1}} \Lambda[t] \otimes_\Lambda \mathcal{G}_{\hat{x}}] \tag{3.6.3}$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, k_! \mathcal{F}) = [\Lambda[[\Gamma]] \otimes_\Lambda \mathcal{G}_{\hat{x}} \xrightarrow{\mathrm{id} - \gamma_x \otimes \mathfrak{F}_x^{-1}} \Lambda[[\Gamma]] \otimes_\Lambda \mathcal{G}_{\hat{x}}]$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$.

If $x \in W - U$, there exists by Lemma 3.1.5, Lemma 3.1.8 and Lemma 3.1.9 a chain of weak equivalences

$$D^{\bullet}_{\hat{x}}(\mathcal{F}) \sim \mathrm{R}\,\Gamma(\hat{x}, i^* \mathrm{R}\, k_* f_! f^* \mathcal{F}) \sim \mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\, i^! k_! f_! f^* \mathcal{F})[1]$$

compatible with the Frobenius operation.

We conclude that for $x \in W - U$,

$$\begin{aligned} Z(x, \mathrm{R}\, k_* \mathcal{F}, t) &= [\mathrm{id} - t\mathfrak{F}_x \circlearrowright \Lambda[t] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1} [\mathrm{id} - t\mathfrak{F}_x \circlearrowright \Lambda[t] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})], \\ Z^{\circledast}(x, k_! \mathcal{F}, t) &= [\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowright \Lambda[t] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1} [\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowright \Lambda[t] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})] \end{aligned}$$

in $\mathrm{K}_1(\Lambda[t]_{S_t})$ and

$$\begin{aligned} \mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\, k_* \mathcal{F}) =& [\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1} \\ & [\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})], \\ \mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, k_! \mathcal{F}) =& [\mathrm{id} - \gamma_x \otimes \mathfrak{F}_x^{-1} \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda D^0_{\hat{x}}(\mathcal{F})]^{-1} \\ & [\mathrm{id} - \gamma_x \otimes \mathfrak{F}_x^{-1} \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda D^1_{\hat{x}}(\mathcal{F})]. \end{aligned}$$

Let N be the stalk of $\mathcal{F}$ in the geometric point $\operatorname{Spec} \overline{F}$, viewed as $\mathrm{Gal}(\overline{F}_x/F_x)$-module. If the image of $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$ in the automorphism group of N has trivial ℓ-Sylow subgroups, then $N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})} = D^0_{\hat{x}}(\mathcal{F})$ and the differential of $D^{\bullet}_{\hat{x}}(\mathcal{F})$ is trivial. Our formula then simplifies to

$$\begin{aligned} Z(x, \mathrm{R}\, k_* \mathcal{F}, t) &= [\mathrm{id} - t\mathfrak{F}_x \circlearrowright \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}]^{-1} \\ & \quad [\mathrm{id} - tq_x\mathfrak{F}_x \circlearrowright \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}], \\ Z^{\circledast}(x, k_! \mathcal{F}, t) &= [\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowright \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}]^{-1} \\ & \quad [\mathrm{id} - tq_x^{-1}\mathfrak{F}_x^{-1} \circlearrowright \Lambda[t] \otimes_\Lambda N^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}], \end{aligned} \tag{3.6.4}$$

where q_x is the order of the residue field $k(x)$. In particular, this is the case if N is unramified in x.

Conversely, assume that the differential of $D^{\bullet}_{\hat{x}}(\mathcal{F})$ is an isomorphism. Since the operation of $\mathfrak{F}_x$ commutes with the differential, we then have

$$Z(x, \mathrm{R}\, k_* \mathcal{F}, t) = Z^{\circledast}(x, k_! \mathcal{F}, t) = 1.$$

If $\Lambda = \mathcal{O}_C$ is the valuation ring of a finite field extension C of $\mathbb{Q}_\ell$, then we may replace $C \otimes_{\mathcal{O}_C} N$ by its semi-simplification $(C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}}$ as a $\mathrm{Gal}(\overline{F}_x/F_x)$-module and obtain a corresponding decomposition $(C \otimes_{\mathcal{O}_C} D^\bullet_{\hat{x}}(\mathcal{F}))^{\mathrm{ss}}$ of the complex $D^\bullet_{\hat{x}}(\mathcal{F})$. Note that

$$(C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}} = ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})} \oplus V$$

with $V^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})} = 0$. In particular, on each simple part of $(C \otimes_{\mathcal{O}_C} D^\bullet_{\hat{x}}(\mathcal{F}))^{\mathrm{ss}}$, the differential is either trivial or an isomorphism. We conclude

$$\begin{aligned}
\det Z(x, \mathrm{R}\,k_*\mathcal{F}, t) &= \det(\mathrm{id} - t\mathfrak{F}_x \circlearrowleft ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})})^{-1} \\
&\qquad \det(\mathrm{id} - tq_x\mathfrak{F}_x \circlearrowleft ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}), \\
\det Z^{\circledast}(x, k_!\mathcal{F}, t) &= \det(\mathrm{id} - t\mathfrak{F}_x^{-1} \circlearrowleft ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})})^{-1} \\
&\qquad \det(\mathrm{id} - tq_x^{-1}\mathfrak{F}_x^{-1} \circlearrowleft ((C \otimes_{\mathcal{O}_C} N)^{\mathrm{ss}})^{\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})}),
\end{aligned} \tag{3.6.5}$$

in the units $(C[t]_{(t)})^\times$ of the localisation of $C[t]$ at the prime ideal (t).

Remark 3.6.3. In the case that F is a function field of characteristic ℓ and Λ is any adic ring, we will use (3.6.4) as a definition for $Z(x, \mathrm{R}\,k_*\mathcal{F}, t)$ for $x \in W - U$, provided that N is at most tamely ramified in x. If $\Lambda = \mathcal{O}_C$, we can use (3.6.5) instead, without any condition on N. In both cases, we set

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_*\mathcal{F}) \coloneqq Z(x, \mathrm{R}\,k_*\mathcal{F}, \gamma_x^{-1}).$$

We can also give more explicit formulas for the local modification factors. We will not make use of the following calculations in any other part of the text.

Let M be one of the finitely generated and projective Λ-modules $\mathcal{F}_{\hat{x}}$, $\mathcal{F}(-1)_{\hat{x}}$, $D^0_{\hat{x}}(\mathcal{F})$, $D^1_{\hat{x}}(\mathcal{F})$. Then M comes equipped with a continuous action of the Galois group $\mathrm{Gal}(\overline{k(x)}/k(x))$. Let $k(x)_{\mathrm{cyc}}$ denote the unique $\mathbb{Z}_\ell$-extension of $k(x)$ and let r be the index of the image of $\Gamma' \coloneqq \mathrm{Gal}(k(x)_{\mathrm{cyc}}/k(x))$ in $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. Fix a topological generator $\gamma \in \Gamma$.

Clearly,

$$[\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M] = \Psi_{\Lambda[[\Gamma]]}([\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma']] \otimes_\Lambda M])$$

in $\mathrm{K}_1(\Lambda[[\Gamma]]_S)$, while

$$\begin{aligned}
\Psi_{\Lambda[[\Gamma]]}(s_{\gamma^r}([\mathrm{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma']] \otimes_\Lambda M)])) &= \\
s_\gamma(\Psi_{\Lambda[[\Gamma]]}([\mathrm{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma']] \otimes_\Lambda M)])) & \\
= s_\gamma([\mathrm{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowleft \Lambda[[\Gamma]] \otimes_\Lambda M)])&
\end{aligned}$$

by Proposition 2.9.1. Hence, it suffices to consider the case that x does not split in F_{cyc}/F. So, we assume from now on that $r = 1$.

The image of $\mathrm{Gal}(\overline{k(x)}/k(x)_{\mathrm{cyc}})$ in the automorphism group of M is a finite commutative group Δ of order d prime to ℓ. Write

$$e_\Delta \coloneqq \frac{1}{d}\sum_{\delta \in \Delta} \delta$$

for the corresponding idempotent in the endomorphism ring. We thus obtain a canonical decomposition

$$M \cong M' \oplus M''$$

of M with $M' \coloneqq e_\Delta M$ and $M'' \coloneqq (\mathrm{id} - e_\Delta)M$.

Since

$$(\mathrm{id} - \delta)(\mathrm{id} - e_\Delta)m = (\mathrm{id} - \delta)m$$

for every $\mathrm{id} \neq \delta \in \Delta$ and every $m \in M$, the action of Δ on M'' is faithful. Since d is prime to ℓ, the action of Δ on $M''/\mathrm{Jac}(\Lambda)M''$ is still faithful. Indeed, the kernel K of $\mathrm{id} - \delta \circlearrowleft M''$ is a direct summand of M'' with $K \neq M''$. The Nakayama lemma then implies $K/\operatorname{Jac}(\Lambda)K \neq M''/\operatorname{Jac}(\Lambda)M''$ such that $\delta \circlearrowright M''/\operatorname{Jac}(\Lambda)M''$ cannot be the identity.

Note that K is trivial if δ is a generator of Δ. In this case,

$$\mathrm{id} - \delta\colon M''/\operatorname{Jac}(\Lambda)M'' \to M''/\operatorname{Jac}(\Lambda)M''$$

must be an automorphism. We may apply this to a suitable ℓ^n-th power of $\mathfrak{F}_x$ to infer that $\mathrm{id} - \mathfrak{F}_x \circlearrowright M''/\operatorname{Jac}(\Lambda)M''$ is an automorphism. By the Nakayama lemma for $\Lambda[[\Gamma]]$ we conclude that the endomorphism $\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x$ of $\Lambda[[\Gamma]] \otimes_\Lambda M''$ is also an automorphism.

The Λ-module M' can be seen as $\Lambda[[\Gamma]]$-module with γ_x acting as $\mathfrak{F}_x$ and by Example 2.9.2 we have

$$[\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda M''] = s_{\gamma_x}([\operatorname{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda M')])^{-1}$$

We conclude that

$$\begin{aligned} &[\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda M] s_\gamma([\operatorname{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda M)]) = \\ &[\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda M''] \frac{s_\gamma}{s_{\gamma_x}}([\operatorname{coker}(\mathrm{id} - \gamma_x^{-1} \otimes \mathfrak{F}_x \circlearrowright \Lambda[[\Gamma]] \otimes_\Lambda M')]). \end{aligned}$$

In particular, if $x \in U$ and $\mathrm{H}^0(\operatorname{Spec} k(x)_{\mathrm{cyc}}, i^* \mathcal{F}) = 0$, then

$$\mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(x, \mathrm{R}\, k_* \mathcal{F}) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\, k_* \mathcal{F}).$$

If $x \in U$, $\mathrm{H}^0(\operatorname{Spec} k(x)_{\mathrm{cyc}}, i^* \mathcal{F}) = \mathcal{F}_{\hat{x}}$ and $\gamma = \gamma_x$, then

$$\mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(x, \mathrm{R}\, k_* \mathcal{F}) = 1.$$

The same considerations apply to the dual local modification factors.

Remark 3.6.4. Write V^0 for the set of closed points of V. Note that the infinite product

$$\prod_{x \in V^0} \mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(x, \mathrm{R}\, k_* \mathcal{F})$$

does not converge in the compact topology of $\mathrm{K}_1(\Lambda[[\Gamma]])$. Indeed, by the Chebotarev density theorem, we may find for each non-trivial finite subextensions F'/F of F_{cyc}/F an infinite subset $S \subset U$ of closed points such that the elements

$$\mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(x, \mathrm{R}\, k_* \mathcal{F}) \in \mathrm{K}_1(\Lambda[[\Gamma]])$$

for $x \in S$ have a common non-trivial image in $\mathrm{K}_1(\Lambda/\operatorname{Jac}(\Lambda)[[\operatorname{Gal}(F'/F)]])$.

CHAPTER 4

Main Conjectures for Perfect Complexes of Adic Sheaves

In this chapter, we will consider the non-commutative Iwasawa main conjecture for perfect complexes of adic sheaves. We begin with a short reminder on L-functions of Artin representations in Section 4.1.

Section 4.2 contains the main results in the case of really admissible extensions F_∞/F. Our objective is to show that Kakde's proof of the main conjecture may be refined in order to obtain a unique choice of a non-commutative L-function as an element of the localised K-group $\mathrm{K}_1(\Lambda[[G]]_S)$.

We use Kakde's non-commutative L-functions and the non-commutative algebraic L-function of the complex $\mathrm{R}\,\Gamma_c(U, f_! f^* \mathbb{Z}_\ell(1))$ to define global modification factors. Changing the open dense subscheme U is reflected by adding or removing local modification factors. This compatibility allows us to pass to field extensions with arbitrarily large ramification. We can then use the results of Section 2.1 to prove the uniqueness of the family of modification factors for all pairs (U, F_∞) with F_∞/F admissible and unramified over U. The corresponding non-commutative L-functions are the product of the global modification factors and the non-commutative algebraic L-functions. We then extend in Theorem 4.2.4 the definition of global modification factors to Λ-adic sheaves smooth at ∞ by requiring a compatibility under twists with certain bimodules. In the same way, we construct global dual modification factors in Theorem 4.2.5. In Theorem 4.2.7 we show that the global modification factors are also compatible under changes of the base field F. The non-commutative L-functions for the complexes $\mathcal{F}^\bullet$ are then defined as the product of the algebraic L-function and the modification factors. Corollary 4.2.9 subsumes the transformation properties of the non-commutative L-functions.

In Section 4.3 we extend our results to the case of CM-extensions of F. Section 4.4 deals with the function field case.

4.1. Artin Representations

Let $\mathcal{O}_C$ be the valuation ring of a finite extension field C of $\mathbb{Q}_\ell$ inside a fixed algebraic closure $\overline{\mathbb{Q}}_\ell$ and assume as before that $\Gamma \cong \mathbb{Z}_\ell$. The augmentation map $\varphi\colon \mathcal{O}_C[[\Gamma]] \to \mathcal{O}_C$ extends to a map

$$\varphi\colon \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S) \xrightarrow[\cong]{\det} \mathcal{O}_C[[\Gamma]]_S^\times \to \mathbb{P}^1(C).$$

Indeed, let $\frac{a}{s} \in \mathcal{O}_C[[\Gamma]]_S^\times$. Since $\mathcal{O}_C[[\Gamma]]$ is a unique factorisation domain and the augmentation ideal is a principal prime ideal, we may assume that not both a and s are contained in the augmentation ideal. Hence, we obtain a well-defined element

$$\varphi\left(\frac{a}{s}\right) \coloneqq [\varphi(a) : \varphi(s)] \in \mathbb{P}^1(C) = C \cup \{\infty\},$$

with $[x : y]$ denoting the standard projective coordinates of $\mathbb{P}^1(C)$. Note that this map agrees with φ' in [**Kak13**, §2.4]. We further note that $\det a^\circledast = ((\det a)^\sharp)^{-1}$

for any $a \in \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S)$. Since $\sharp\colon \mathcal{O}_C[[\Gamma]]_S \to \mathcal{O}_C[[\Gamma]]_S$ maps $\gamma \in \Gamma$ to γ^{-1} and is given by the identity on $\mathcal{O}_C$, we conclude that

$$\varphi(a^{\circledast}) = \varphi(a)^{-1}. \tag{4.1.1}$$

Finally, note that the diagram

$$\begin{tikzcd} \mathrm{K}_1(\mathcal{O}_C[t]_{S_t}) \arrow[r, "\det"] \arrow[d, "t\mapsto\gamma"] & (C[t]_{(t)})^{\times} \arrow[d, "\frac{f}{g}\mapsto[f(1):g(1)]"] \\ \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S) \arrow[r, "\varphi"] & \mathbb{P}^1(C) \end{tikzcd}$$

commutes for any choice of $\gamma \in \Gamma$ with $\gamma \neq 1$. On the right downward pointing map, $\frac{f}{g}$ denotes a reduced fraction.

Let F be a global field of characteristic $p \geq 0$. Consider an Artin representation $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ (i.e. with open kernel) over $\mathcal{O}_C$. We will write

$$\rho^*\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C), \qquad g \mapsto \rho(g^{-1})^t$$

for the dual representation. Any Artin representation may also be considered as a $\mathcal{O}_C$-adic sheaf on $\operatorname{Spec} F$ whose global sections over $\operatorname{Spec} F'$ is the module of invariants of ρ under $\mathrm{Gal}_{F'}$ for each field $F' \subset \overline{F}$. We will not distinguish between the representation ρ and the corresponding $\mathcal{O}_C$-adic sheaf.

Fix two open dense subschemes $U \subset W \subset X$. We set

$$\Sigma \coloneqq X - W, \qquad \mathrm{T} \coloneqq W - U$$

and assume that ℓ is invertible on T. Since the image of $\mathrm{Gal}(\overline{F}_x/F_x)$ in $\mathrm{Gl}_d(\mathcal{O}_C)$ is finite, the base change of ρ to C is automatically a simi-simple representation of $\mathrm{Gal}(\overline{F}_x/F_x)$. For any $x \in X$, we let ρ_x denote the representation of $\mathrm{Gal}(F_x^{\mathrm{nr}}/F_x)$ obtained from ρ by restricting to $\mathrm{Gal}(\overline{F}_x/F_x)$ and then taking invariants under $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$.

For any open dense subscheme V of X, we write V^0 for the set of closed points of V. Let $\alpha\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an embedding of $\overline{\mathbb{Q}}_\ell$ into the complex numbers. We can then associate to the complex Artin representation $\alpha \circ \rho$ the classical Σ-truncated T-modified Artin L-function with the product formula

$$L_{\Sigma,\mathrm{T}}(\alpha\circ\rho, s) \coloneqq \prod_{x\in U^0} \det(1-\alpha\circ\rho_x(\mathfrak{F}_x)q_x^{-s})^{-1} \prod_{x\in\mathrm{T}} \frac{\det(1-\alpha\circ\rho_x(\mathfrak{F}_x)q_x^{1-s})}{\det(1-\alpha\circ\rho_x(\mathfrak{F}_x)q_x^{-s})}$$

for $\operatorname{Re} s > 1$. Note that we follow the geometric convention of using the geometric Frobenius in the definition of the Artin L-function as in [**CL73**].

Write $\eta\colon \operatorname{Spec} F \to U$ for the generic point of U and assume for simplicity that ρ is unramified over U, i.e. for each $x \in U$, the restriction of ρ to $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$ is trivial. Then ρ corresponds to the smooth $\mathcal{O}_C$-adic sheaf $\eta_*(\rho)$ on $U \subset W$ defined by (3.3.1) and therefore, to an object in $\mathbf{PDG}^{\mathrm{cont}}(U, \mathcal{O}_C)$. Analogously, ρ^* corresponds to $\eta_*(\rho^*) = (\eta_*\rho)^{*\mathcal{O}_C}$.

Assume that $\ell \neq p$. From (3.6.1), (3.6.2), and (3.6.5) we conclude

$$\begin{aligned} &\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \mathrm{R}\,k_*\eta_*(\rho)(n))) \\ &\qquad = \begin{cases} [1 : \det(1-\rho_x(\mathfrak{F}_x)q_x^{-n})] & \text{if } x \in U, \\ [\det(1-\rho_x(\mathfrak{F}_x)q_x^{1-n}) : \det(1-\rho_x(\mathfrak{F}_x)q_x^{-n})] & \text{if } x \in \mathrm{T}, \end{cases} \\ &\varphi(\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, k_!\eta_*(\rho)(n))) \\ &\qquad = \begin{cases} [\det(1-\rho^*_x(\mathfrak{F}_x)q_x^{n-1}) : 1] & \text{if } x \in U, \\ [\det(1-\rho^*_x(\mathfrak{F}_x)q_x^{n-1}) : \det(1-\rho^*_x(\mathfrak{F}_x)q_x^{n})] & \text{if } x \in \mathrm{T}, \end{cases} \end{aligned} \tag{4.1.2}$$

where q_x denotes the number of elements of the residue field $k(x)$. Note that $\det(1-\rho_x(\mathfrak{F}_x)q_x^{-n}) = 0$ if and only if $n = 0$ and ρ_x contains the trivial representation as a subrepresentation.

In particular, we have

$$\begin{aligned} L_{\Sigma,\mathrm{T}}(\alpha\circ\rho,n) &= \prod_{x\in W^0}\alpha(\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x,\mathrm{R}\,k_*\eta_*(\rho)(n)))) \\ &= \prod_{x\in W^0}\alpha(\varphi(\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x,k_!\eta_*(\rho^*)(1-n))))^{-1} \end{aligned}$$

for all integers $n > 1$.

If $\ell = p$ and $n = 0$, then we still have

$$\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x,\eta_*(\rho))) = [1 : \det(1-\rho_x(\mathfrak{F}_x))]$$

if $x \in U$. If $x \in \mathrm{T}$, we use the definition of $\mathcal{L}_{F_{\mathrm{cyc}}/F}(x,\mathrm{R}\,k_*\eta_*(\rho))$ from Remark 3.6.3 and obtain

$$\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x,\mathrm{R}\,k_*\eta_*(\rho))) = [\det(1-\rho_x(\mathfrak{F}_x)q_x) : \det(1-\rho_x(\mathfrak{F}_x))].$$

Assume that F is a function field. As before, we let q denote the number of elements of the algebraic closure $\mathbb{F}$ of the prime field $\mathbb{F}_p$ in F. It follows from the work of Weil [**Wei48**] that there exists a unique element

$$Z(W,\mathrm{R}\,k_*\eta_*(\rho),t) \in \mathrm{K}_1(\mathcal{O}_C[t]_{S_t}) \cong (\mathcal{O}_C[t]_{(t,\ell)})^\times = \mathcal{O}_C[[t]]^\times \cap C(t)^\times$$

such that

$$\alpha(Z(W,\mathrm{R}\,k_*\eta_*(\rho),q^{-n})) = L_{\Sigma,\mathrm{T}}(\alpha\circ\rho,n) \in \mathbb{P}^1(\mathbb{C})$$

for all integers n.

Definition 4.1.1. Let F be a function field of characteristic p and n be an integer. If $\ell = p$, we assume $n = 0$. Write $\gamma_{\mathbb{F}}$ for the image of the geometric Frobenius automorphism $\mathfrak{F}_{\mathbb{F}}$ of $\mathbb{F}$ in $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. The ℓ-adic L-function of $\mathrm{R}\,k_*\eta_*(\rho)$ is given by

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\eta_*(\rho)(n)) \coloneqq Z(W,\mathrm{R}\,k_*\eta_*(\rho),q^{-n}\gamma_{\mathbb{F}}^{-1}) \in \mathrm{K}_1(\Lambda[[\Gamma]]_S).$$

From now on, we let F denote a totally real number field and assume $\ell \neq 2$. By [**CL73**, Cor 1.4] there exists for each $n \in \mathbb{Z}$, $n < 0$ a well defined number $L_{\Sigma,\mathrm{T}}(\rho,n) \in C$ such that

$$\alpha(L_{\Sigma,\mathrm{T}}(\rho,n)) = L_{\Sigma,\mathrm{T}}(\alpha\circ\rho,n) \in \mathbb{C}$$

Consequently,

$$L_{\Sigma',\mathrm{T}'}(\rho,n) = L_{\Sigma,\mathrm{T}}(\rho,n)\prod_{x\in\Sigma'\cup\mathrm{T}'-\Sigma\cup\mathrm{T}}\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(x,\mathrm{R}\,k_*\eta_*(\rho)(n)))^{-1}$$

if $\Sigma \subset \Sigma'$ and $\mathrm{T} \subset \mathrm{T}'$ with disjoint subsets Σ' and T' of X such that ρ is unramified over $X - \Sigma' - \mathrm{T}'$ and ℓ is invertible on T'.

Let $\kappa_F\colon \mathrm{Gal}_F \to \mathbb{Z}_\ell^\times$ denote the cyclotomic character such that

$$\sigma(\zeta) = \zeta^{\kappa_F(\sigma)}$$

for every $\sigma \in \mathrm{Gal}_F$ and $\zeta \in \mu_{\ell^\infty}$. Further, we write $\omega_F\colon \mathrm{Gal}_F \to \mu_{\ell-1}$ for the Teichmüller character, i. e. the composition of κ_F with the projection $\mathbb{Z}_\ell^\times \to \mu_{\ell-1}$. Finally, we set $\epsilon_F \coloneqq \kappa_F\omega^{-1}$ and note that ϵ_F factors through $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$.

Assume that ρ factors through the Galois group of a totally real field extension of F. Then $\eta_*(\rho)$ is smooth at ∞. Assume further that $U = W$, such that T is empty. Under Conjecture 3.3.4 it follows from [**Gre83**] and from the validity of the classical main conjecture that for every integer n there exist unique elements

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho\omega_F^n)(1-n)) \in \mathrm{K}_1(\mathcal{O}_C[[\Gamma]]_S)$$

such that

$$\begin{aligned}\Phi_{\epsilon_F^n}(\mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho\omega_F^n)(1-n))) &= \mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho)(1)),\\ \varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho\omega_F^n)(1-n))) &= L_{\Sigma,\varnothing}(\rho\omega_F^n,1-n) \quad \text{if } n>1.\end{aligned} \tag{4.1.3}$$

with $\Phi_{\epsilon_F^n}$ as defined in Example 2.6.3. Beware that Greenberg uses the arithmetic convention for L-functions.

Definition 4.1.2. Let $\gamma \in \Gamma$ be a topological generator. We define the *global modification factor* for $\eta_*(\rho\omega_F^n)(1)$ and $f\colon U_{F_{\mathrm{cyc}}} \to U$ to be the element

$$\mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U,\eta_*(\rho)(1)) \coloneqq \mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho)(1))s_\gamma([\mathrm{R}\,\Gamma_{\mathrm{c}}(U,f_!f^*\eta_*(\rho)(1))])$$

in $\mathrm{K}_1(\mathcal{O}_C[[\Gamma]])$.

If ρ has ramification over U, we will see later in Section 5.1 that we can still associate a $\mathcal{O}_C$-adic sheaf $\eta_*(\rho)$ on U to ρ. In general, this sheaf will not be smooth. Still, all other results in this section can be extended in an obvious manner.

4.2. Non-Commutative L-Functions for Really Admissible Extensions

Let F be a totally real number field and $\ell \neq 2$. Throughout this section, we make use of Conjecture 3.3.4. We recall the main theorem of [**Kak13**].

Theorem 4.2.1. *Let $U \subset X$ be a dense open subscheme with complement Σ and assume that ℓ is invertible on U. Assume that F_∞/F is a really admissible extension which is unramified over U and that $G = \mathrm{Gal}(F_\infty/F)$ is an ℓ-adic Lie group. Then there exists unique elements $\tilde{\mathcal{L}}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_\ell[[G]]_S)/\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[G]])$ such that*

(1)
$$d\tilde{\mathcal{L}}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1)) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(U,f_!f^*(\mathbb{Z}_\ell)_U(1))],$$

(2) *For any Artin representation ρ factoring through G*
$$\Phi_\rho(\tilde{\mathcal{L}}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho)(1))$$

Proof. This is [**Kak13**, Thm. 2.11] translated into our notations. Recall that our $\Phi_{\rho\kappa_F^{-n}}$ corresponds to $\Phi_{\bar{\rho}\kappa_F^n}$ in the notation of the cited article. Moreover, Kakde uses the arithmetic convention in the definition of L-values. Further, note that the ℓ-adic L-function $\mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho)(1))$ is uniquely determined by the values $\varphi(\Phi_{\epsilon_F^n}(\mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho)(1))))$ for $n<0$ and $n \equiv 0 \mod \ell-1$. Finally, Kakde's complex $\mathcal{C}(F_\infty/F)$ corresponds to $\mathrm{R}\,\Gamma_{\mathrm{c}}(U,f_!f^*(\mathbb{Z}_\ell)_U(1))$ shifted by 3 and therefore, the images of the two complexes under d differ by a sign, but at the same time, his definition of d differs by a sign from ours. □

We will improve this theorem as follows. Let $\Xi = \Xi_F$ be the set of pairs (U,F_∞) such that $U \subset X$ is a dense open subscheme with ℓ invertible on U and F_∞/F is a really admissible extension unramified over U.

Theorem 4.2.2. *Let $\gamma \in \Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a topological generator. There exists a unique family of elements*

$$\big(\mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))\big)_{(U,F_\infty)\in\Xi}$$

such that

(1) $\mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]])$,

(2) *if $U \subset U'$ with complement Σ and $(U,F_\infty),(U',F_\infty) \in \Xi$, then*

$$\mathcal{M}_{F_\infty/F,\gamma}(U',(\mathbb{Z}_\ell)_{U'}(1)) = \mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))\prod_{x\in\Sigma}\mathcal{M}_{F_\infty/F,\gamma}(x,(\mathbb{Z}_\ell)_{U'}(1)),$$

(3) *if* $(U, F_\infty), (U, F'_\infty) \in \Xi$ *such that* $F'_\infty \subset F_\infty$ *is a subfield, then*
$$\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1))) = \mathcal{M}_{F'_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1)),$$

(4) *if* $(U, F_\infty) \in \Xi$ *and* $\rho\colon \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_n(\mathcal{O}_C)$ *is an Artin representation, then*
$$\Phi_\rho(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1))) = \mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U, \eta_*(\rho)(1))$$
with $\mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U, \eta_*(\rho)(1))$ *as in Definition 4.1.2.*

Proof. *Uniqueness:* Assume that $m_k(U, F_\infty)$, $k = 1, 2$ are two families with the listed properties. Then
$$d(F_\infty) \coloneqq m_2(U, F_\infty)^{-1} m_1(U, F_\infty)$$
does not depend on U as a consequence of (2).

Let $(U, F_\infty) \in \Xi$ be any pair such that $F_\infty/F_{\mathrm{cyc}}$ is finite and write $f\colon U_{F_\infty} \to U$ for the system of coverings of U associated to F_∞/F. Then (4) implies that the elements
$$m_i(U, F_\infty) s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_\ell)_U(1))])$$
both agree with $\tilde{\mathcal{L}}_{F_\infty/F}(U, (\mathbb{Z}_\ell)_U(1))$ modulo $\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]])$. Hence,
$$d(F_\infty) \in \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]).$$
By Corollary 2.1.4, we may find a pair $(U', F'_\infty) \in \Xi$ such that F'_∞/F_∞ is finite, $U' \subset U$, and
$$\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]}\colon \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]]) \to \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]) \tag{4.2.1}$$
is the zero map. We conclude from (3) that $d(F_\infty) = 1$ for all (U, F_∞) with $F_\infty/F_{\mathrm{cyc}}$ finite. Now for any really admissible extension F_∞/F,
$$\mathrm{K}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]) = \varprojlim_{F'_\infty} \mathrm{K}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]])$$
where F'_∞ runs through the really admissible subextensions of F_∞/F with $F'_\infty/F_{\mathrm{cyc}}$ finite [**FK06**, Prop. 1.5.3]. We conclude that $d(F_\infty) = 1$ in general.

Existence: It suffice to construct the elements for $(U, F_\infty) \in \Xi$ with $F_\infty/F_{\mathrm{cyc}}$ finite. Choose (U', F'_∞) as above such that the map in (4.2.1) becomes trivial. Pick any $m \in \mathrm{K}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]])$ such that
$$m s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_\ell)_U(1))]) \equiv \tilde{\mathcal{L}}_{F'_\infty/F}(U', (\mathbb{Z}_\ell)_{U'}(1)) \mod \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]])$$
Define
$$\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1)) \coloneqq \Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]}(m) \prod_{x \in U - U'} \mathcal{M}_{F_\infty/F,\gamma}(x, (\mathbb{Z}_\ell)_U(1)).$$
By Proposition 2.9.1 and Proposition 3.5.4 we conclude that
$$\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1)) s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^*(\mathbb{Z}_\ell)_U(1))]) \equiv \tilde{\mathcal{L}}_{F_\infty/F}(U, (\mathbb{Z}_\ell)_U(1)) \mod \widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]])$$
and that $\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1))$ satisfies
$$\Phi_\rho(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1))) = \mathcal{M}_{F_{\mathrm{cyc}}/F,\gamma}(U, \eta_*(\rho)(1))$$
for any Artin representation $\rho\colon \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_d(\mathcal{O}_C)$. In particular, the system
$$(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1)))_{(F_\infty, U) \in \Xi}$$
satisfies (1) and (4). By construction and again by Proposition 3.5.4, it is independent of the choices of (U', F'_∞) and m and satisfies (2) and (3). □

COROLLARY 4.2.3. *There exists a unique family of elements*

$$\left(\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))\right)_{(U,F_\infty)\in\Xi}$$

such that

(1) $\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1)) \in \mathrm{K}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]_S)$,

(2) *if* $(U,F_\infty)\in\Xi$ *and* $f\colon U_{F_\infty}\to U$ *denotes the associated system of coverings, then*

$$d\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1)) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(U,f_!f^*(\mathbb{Z}_\ell)_U(1))]$$

(3) *if* $U'\subset U$ *with complement* Σ *and* $(U',F_\infty),(U,F_\infty)\in\Xi$, *then*

$$\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1)) = \mathcal{L}_{F_\infty/F}(U',(\mathbb{Z}_\ell)_{U'}(1))\prod_{x\in\Sigma}\mathcal{L}_{F_\infty/F}(x,(\mathbb{Z}_\ell)_U(1)),$$

(4) *if* $(U,F_\infty),(U,F'_\infty)\in\Xi$ *such that* $F'_\infty\subset F_\infty$ *is a subfield, then*

$$\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))) = \mathcal{L}_{F'_\infty/F}(U,(\mathbb{Z}_\ell)_U(1)),$$

(5) *if* $(U,F_\infty)\in\Xi$ *and* $\rho\colon\mathrm{Gal}(F_\infty/F)\to\mathrm{Gl}_d(\mathcal{O}_C)$ *is an Artin representation, then*

$$\Phi_\rho(\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(U,\eta_*(\rho)(1)).$$

PROOF. Fix a topological generator $\gamma\in\Gamma$ and set

$$\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1)) \coloneqq \mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(U,f_!f^*(\mathbb{Z}_\ell)_U(1))]).$$

If $(\tilde{\mathcal{L}}(U,F_\infty))_{(U,F_\infty)\in\Xi}$ is a second family with the listed properties, then

$$\tilde{\mathcal{L}}(U,F_\infty)s_\gamma([\mathrm{R}\,\Gamma_{\mathrm{c}}(U,f_!f^*(\mathbb{Z}_\ell)_U(1))]) = \mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))$$

by the uniqueness of $\mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))$. □

Let $\Theta=\Theta_F$ be the set of triples (U,F_∞,Λ) such that $U\subset X$ is a dense open subscheme with ℓ invertible on U, F_∞/F is a really admissible extension unramified over U and Λ is an adic $\mathbb{Z}_\ell$-algebra.

THEOREM 4.2.4. *Let* $\gamma\in\Gamma=\mathrm{Gal}(F_{\mathrm{cyc}}/F)$ *be a topological generator. There exists a unique family of homomorphisms*

$$\left(\mathcal{M}_{F_\infty/F,\gamma}(U,(-)(1))\colon\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda))\to\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])\right)_{(U,F_\infty,\Lambda)\in\Theta}$$

such that

(1) *for any* $(U,F_\infty,\mathbb{Z}_\ell)\in\Theta$, $\mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))$ *is the element constructed in Theorem 4.2.2,*

(2) *if* $j\colon U'\to U$ *is an open immersion and* $(U',F_\infty,\Lambda),(U,F_\infty,\Lambda)\in\Theta$, *then*

$$\mathcal{M}_{F_\infty/F,\gamma}(U,\mathcal{F}^\bullet(1)) = \mathcal{M}_{F_\infty/F,\gamma}(U',j^*\mathcal{F}^\bullet(1))\prod_{x\in U-U'}\mathcal{M}_{F_\infty/F,\gamma}(x,\mathcal{F}^\bullet(1)),$$

for any $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$.

(3) *if* $(U,F_\infty,\Lambda),(U,F'_\infty,\Lambda)\in\Theta$ *such that* $F'_\infty\subset F_\infty$ *is a subfield, then*

$$\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{M}_{F_\infty/F,\gamma}(U,\mathcal{F}^\bullet(1))) = \mathcal{M}_{F'_\infty/F,\gamma}(U,\mathcal{F}^\bullet(1)),$$

for any $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$.

(4) *if* $(U,F_\infty,\Lambda),(U,F_\infty,\Lambda')\in\Theta$ *and* $P^\bullet$ *is a complex of* Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-*bimodules, strictly perfect as complex of* Λ'-*modules, then*

$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{M}_{F_\infty/F,\gamma}(U,\mathcal{F}^\bullet(1))) = \mathcal{M}_{F_\infty/F,\gamma}(U,\Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)(1))$$

for any $\mathcal{F}^\bullet$ *in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$.

PROOF. Applying (4) to the Λ/I-$\Lambda[[G]]$-bimodule $\Lambda/I[[G]]$ for any open two-sided ideal I of Λ and using

$$\mathrm{K}_1(\Lambda[[G]]) = \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{K}_1(\Lambda/I[[G]]),$$

we conclude that it is sufficient to consider triples $(U, F_\infty, \Lambda) \in \Theta$ with Λ a finite ring. So, let Λ be finite. Since $\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))$ depends only on the class of $\mathcal{F}^\bullet$ in $\mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda))$, we may assume that $\mathcal{F}^\bullet$ is a bounded complex of flat constructible étale sheaves of Λ-modules. Using (2) we may shrink U until $\mathcal{F}^\bullet$ is a complex of locally constant étale sheaves. Hence, there exists a $(U, F'_\infty, \Lambda) \in \Theta$ such that F_∞/F is a subextension of F'_∞/F and such that the restriction of $\mathcal{F}^\bullet$ to U_K for some finite subextension K/F of F'_∞/F is a complex of constant sheaves. By (3), we may replace F_∞ by F'_∞. We may then find a complex of Λ-$\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]$-bimodules $P^\bullet$, strictly perfect as complex of Λ modules and a weak equivalence

$$\Psi_{P^\bullet} f_! f^*(\mathbb{Z}_\ell)_U(1) \xrightarrow{\sim} \mathcal{F}^\bullet(1)$$

[**Wit14**, Prop. 6.8]. By (4), the only possible definition of $\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))$ is

$$\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) = \Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1))).$$

It is then clear that this construction satisfies the given properties. $\square$

THEOREM 4.2.5. *Let $\gamma \in \Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a topological generator. There exists a unique family of homomorphisms*

$$\left(\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U,-)\colon \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)) \to \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])\right)_{(U,F_\infty,\Lambda)\in\Theta}$$

such that

(1) *for any $(U, F_\infty, \mathbb{Z}_\ell) \in \Theta$,*

$$\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U) = (\mathcal{M}_{F_\infty/F,\gamma}(U, (\mathbb{Z}_\ell)_U(1)))^{\circledast}$$

(2) *if $j\colon U' \to U$ is an open immersion and $(U', F_\infty, \Lambda), (U, F_\infty, \Lambda) \in \Theta$, then*

$$\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet) = \mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U', j^*\mathcal{F}^\bullet) \prod_{x\in U-U'} \mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(x, \mathcal{F}^\bullet),$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$.

(3) *if $(U, F_\infty, \Lambda), (U, F'_\infty, \Lambda) \in \Theta$ such that $F'_\infty \subset F_\infty$ is a subfield, then*

$$\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1))) = \mathcal{M}^{\circledast}_{F'_\infty/F,\gamma}(U, \mathcal{F}^\bullet),$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$.

(4) *if $(U, F_\infty, \Lambda), (U, F_\infty, \Lambda') \in \Theta$ and $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, then*

$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet)) = \mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U, \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$.

Moreover, for any $(U, F_\infty, \Lambda) \in \Theta$ and any smooth Λ-adic sheaf $\mathcal{F}$ on U, we have

$$\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(U, \mathcal{F}) = (\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^{*_\Lambda}(1)))^{\circledast}.$$

PROOF. We proceed as in Theorem 4.2.4 and use Lemma 2.7.3. $\square$

PROPOSITION 4.2.6. *Assume that γ, γ' are two topological generators of Γ. Then*

$$\frac{\mathcal{M}_{F_\infty/F,\gamma}}{\mathcal{M}_{F_\infty/F,\gamma'}}(U, \mathcal{F}^\bullet(1)) = \frac{s_\gamma}{s_{\gamma'}}([\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^\bullet(1))])$$
$$\frac{\mathcal{M}^\circledast_{F_\infty/F,\gamma}}{\mathcal{M}^\circledast_{F_\infty/F,\gamma'}}(U, \mathcal{F}^\bullet) = \frac{s_{(\gamma')^{-1}}}{s_{\gamma^{-1}}}([\mathrm{R}\,\Gamma(U, f_! f^* \mathcal{F}^\bullet)])$$

for any $(U, F_\infty, \Lambda) \in \Theta$ and any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$.

PROOF. By definition, these identities hold for the local modification factors and by Corollary 4.2.3 and Proposition 2.9.3 they hold for $\mathcal{F}^\bullet = (\mathbb{Z}_\ell)_U$. Hence,

$$\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)) = \frac{s_\gamma}{s_{\gamma'}}([\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^\bullet(1))])\mathcal{M}_{F_\infty/F,\gamma'}(U, \mathcal{F}^\bullet(1))$$
$$\mathcal{M}^\circledast_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet) = \frac{s_{(\gamma')^{-1}}}{s_{\gamma^{-1}}}([\mathrm{R}\,\Gamma(U, f_! f^* \mathcal{F}^\bullet)])\mathcal{M}^\circledast_{F_\infty/F,\gamma'}(U, \mathcal{F}^\bullet)$$

by the uniqueness assertion in Theorem 4.2.4 and Theorem 4.2.5. □

THEOREM 4.2.7. *Let F'/F be a finite extension of totally real fields. Set $r \coloneqq [F' \cap F_{\mathrm{cyc}} : F]$ and let $\gamma \in \mathrm{Gal}(F_{\mathrm{cyc}}/F)$ be a topological generator. Assume that $(U, F_\infty, \Lambda) \in \Theta_F$ with $F' \subset F_\infty$ and write $f_{F'}: U_{F'} \to U$ for the associated covering. Then*

(1) *for every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$,*

$$\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'}, f_{F'}^* \mathcal{F}^\bullet(1)) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet(1)),$$
$$\mathcal{M}^\circledast_{F_\infty/F',\gamma^r}(U_{F'}, f_{F'}^* \mathcal{F}^\bullet) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}^\circledast_{F_\infty/F,\gamma}(U, \mathcal{F}^\bullet);$$

(2) *for every $\mathcal{G}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'}, \Lambda)$,*

$$\mathcal{M}_{F_\infty/F,\gamma}(U, f_{F'*} \mathcal{G}^\bullet(1)) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'}, \mathcal{G}^\bullet(1)),$$
$$\mathcal{M}^\circledast_{F_\infty/F,\gamma}(U, f_{F'*} \mathcal{G}^\bullet) = \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}^\circledast_{F_\infty/F',\gamma^r}(U_{F'}, \mathcal{G}^\bullet).$$

PROOF. We prove the identities for the global modification factors; the proof for the global dual modification factors is the same.

We first note that for any complex $P^\bullet$ of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, there exists an obvious isomorphism of complexes of $\Lambda'[[\mathrm{Gal}(F_\infty/F')]]$-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules

$$\Lambda'[[\mathrm{Gal}(F_\infty/F)]] \otimes_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} P[[\mathrm{Gal}(F_\infty/F)]]^{\delta^\bullet} \cong$$
$$P[[\mathrm{Gal}(F_\infty/F')]]^{\delta^\bullet} \otimes_{\Lambda[[\mathrm{Gal}(F_\infty/F')]]} \Lambda[[\mathrm{Gal}(F_\infty/F)]].$$

Hence,

$$\Psi_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} \circ \Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta^\bullet}} = \Psi_{P[[\mathrm{Gal}(F_\infty/F')]]^{\delta^\bullet}} \circ \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]} \tag{4.2.2}$$

as homomorphisms from $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])$ to $\mathrm{K}_1(\Lambda'[[\mathrm{Gal}(F_\infty/F')]])$. Likewise, for a complex $Q^\bullet$ of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F')]]$-bimodules, strictly perfect as complex of Λ'-modules, we have an equality

$$\Psi_{\Lambda'[[\mathrm{Gal}(F_\infty/F)]]} \circ \Psi_{Q[[\mathrm{Gal}(F_\infty/F')]]^{\delta^\bullet}} = \Psi_{Q[[\mathrm{Gal}(F_\infty/F)]]^{\delta^\bullet}} \circ \Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}. \tag{4.2.3}$$

in $\mathrm{Hom}(\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F')]]), \mathrm{K}_1(\Lambda'[[\mathrm{Gal}(F_\infty/F)]]))$.

In particular, we may reduce to the case of finite $\mathbb{Z}_\ell$-algebras Λ by choosing $P^\bullet = \Lambda = Q^\bullet$ with the trivial action of $\mathrm{Gal}(F_\infty/F)$ and $\mathrm{Gal}(F_\infty/F')$, respectively. By Proposition 3.5.4.(4) we may then shrink U until $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ may be assumed to be strictly perfect complexes of locally constant étale sheaves. Using the identities (4.2.2) and (4.2.3) again, we may reduce to the case $\Lambda = \mathbb{Z}_\ell$ and $\mathcal{F}^\bullet = (\mathbb{Z}_\ell)_U$,

$\mathcal{G}^{\bullet} = (\mathbb{Z}_\ell)_{U_{F'}}$. We may then further reduce to the case that $F_\infty/F'_{\mathrm{cyc}}$ is a finite extension.

Setting

$$q \coloneqq \frac{\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'}, f_{F'}^*(\mathbb{Z}_\ell)_{U_{F'}}(1))}{\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))} \in \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F')]]),$$
$$q' \coloneqq \frac{\mathcal{M}_{F_\infty/F,\gamma}(U, f_{F'*}(\mathbb{Z}_\ell)_{U_{F'}}(1))}{\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{M}_{F_\infty/F',\gamma^r}(U_{F'},(\mathbb{Z}_\ell)_U(1))} \in \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]),$$

it suffices to show that $q = 1$ and $q' = 1$.

Let $g\colon U_{F_\infty} \to U_{F'}$ denote the restriction of $f\colon U_{F_\infty} \to U$. Write

$$M = \mathbb{Z}_\ell[\mathrm{Gal}(F_\infty/F')\backslash\mathrm{Gal}(F_\infty/F)]$$

for the $\mathbb{Z}_\ell$-$\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]$-bimodule freely generated as $\mathbb{Z}_\ell$-module by the right cosets $\mathrm{Gal}(F_\infty/F)\sigma$ for $\sigma \in \mathrm{Gal}(F_\infty/F)$ and on which $\tau \in \mathrm{Gal}(F_\infty/F)$ operates by right multiplication. From Proposition 3.3.2 we conclude

$$\begin{aligned}\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]}\,\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_!f^*(\mathbb{Z}_\ell)_U(1)) &\sim \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_{F'*}g_!g^*f_{F'}^*(\mathbb{Z}_\ell)_U(1))\\ &\sim \mathrm{R}\,\Gamma_{\mathrm{c}}(U_{F'}, g_!g^*(\mathbb{Z}_\ell)_{U_{F'}}(1)),\end{aligned}$$
$$\begin{aligned}\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]}\,&\mathrm{R}\,\Gamma_{\mathrm{c}}(U_{F'}, g_!g^*(\mathbb{Z}_\ell)_{U_{F'}}(1))\\ &\sim \Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]}\,\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_{F'*}g_!g^*(\mathbb{Z}_\ell)_{U_{F'}}(1))\\ &\sim \mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_!f^*f_{F'*}(\mathbb{Z}_\ell)_{U_{F'}}(1))\\ &\sim \Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta}\,\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_!f^*(\mathbb{Z}_\ell)_U(1)).\end{aligned}$$

Additionally, we note that

$$\mathcal{M}_{F_\infty/F,\gamma}(U, f_{F'*}(\mathbb{Z}_\ell)_U(1)) = \Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta}\mathcal{M}_{F_\infty/F,\gamma}(U,(\mathbb{Z}_\ell)_U(1))$$

by Theorem 4.2.4.

From this and from Proposition 2.9.1, we conclude

$$q = \frac{\mathcal{L}_{F_\infty/F'}(U_{F'},(\mathbb{Z}_\ell)_{U_{F'}}(1))}{\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))},$$
$$q' = \frac{\Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta}\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))}{\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F'}(U_{F'},(\mathbb{Z}_\ell)_U(1))}.$$

Let $C/\mathbb{Q}_\ell$ be a finite field extension and

$$\rho'\colon \mathrm{Gal}(F_\infty/F') \to \mathrm{Gl}_d(\mathcal{O}_C)$$
$$\rho\colon \mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_d(\mathcal{O}_C)$$

be Artin representations. Write

$$\varphi_F\colon \mathcal{O}_C[[\mathrm{Gal}(F_{\mathrm{cyc}}/F)]] \to \mathcal{O}_C$$
$$\varphi_{F'}\colon \mathcal{O}_C[[\mathrm{Gal}(F'_{\mathrm{cyc}}/F')]] \to \mathcal{O}_C$$

for the augmentation maps. We denote by $\mathrm{Ind}_{F'}^F\rho'$ and $\mathrm{Res}_F^{F'}\rho$ the induced and restricted representations, respectively.

Then for every $n \in \mathbb{Z}$

$$\varphi_{F'} \circ \Phi_{\rho'\epsilon_{F'}^n} \circ \Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]} = \varphi_F \circ \Phi_{\mathrm{Ind}_{F'}^F\rho'\epsilon_{F'}^n} = \varphi_F \circ \Phi_{\epsilon_F^n\mathrm{Ind}_{F'}^F\rho'}$$

as maps from $\mathrm{K}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]_S)$ to $\mathbb{P}^1(C)$ and

$$\varphi_F \circ \Phi_{\rho\epsilon_F^n} \circ \Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]} = \varphi_{F'} \circ \Phi_{\mathrm{Res}_F^{F'}\rho\epsilon_F^n} = \varphi_{F'} \circ \Phi_{\epsilon_{F'}^n\mathrm{Res}_F^{F'}\rho}$$

as maps from $\mathrm{K}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F')]]_S)$ to $\mathbb{P}^1(C)$. From (4.1.3) and the transformation properties of the complex Artin L-functions with respect to inflation and restriction we conclude that for $n<-1$ and $\Sigma = X - U$

$$\begin{aligned}
\varphi_{F'}\circ\Phi_{\rho'\epsilon_{F'}^n}(\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))) &=\\
&= L_{\Sigma,\varnothing}(\omega_F^{-n}\operatorname{Ind}_{F'}^F\rho',1+n)\\
&= L_{\Sigma_{F'},\varnothing}(\rho'\omega_{F'}^{-n},1+n)\\
&= \varphi_{F'}\circ\Phi_{\rho'\epsilon_{F'}^n}(\mathcal{L}_{F_\infty/F'}(U_{F'},(\mathbb{Z}_\ell)_{U_{F'}}(1))),\\
\varphi_F\circ\Phi_{\rho\epsilon_F^n}(\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}\mathcal{L}_{F_\infty/F'}(U_{F'},(\mathbb{Z}_\ell)_{U_{F'}}(1)) &=\\
&= L_{\Sigma_{F'},\varnothing}(\omega_{F'}^{-n}\operatorname{Res}_F^{F'}\rho,1+n)\\
&= L_{\Sigma,\varnothing}(\omega_F^{-n}\operatorname{Ind}_F^{F'}\operatorname{Res}_F^{F'}\rho,1+n)\\
&= \varphi_F\circ\Phi_{\epsilon_F^n\operatorname{Ind}_F^{F'}\operatorname{Res}_F^{F'}\rho}(\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))))\\
&= \varphi_F\circ\Phi_{\rho\epsilon_F^n}(\Psi_{M[[\mathrm{Gal}(F_\infty/F)]]^\delta}\mathcal{L}_{F_\infty/F}(U,(\mathbb{Z}_\ell)_U(1))).
\end{aligned}$$

From [**Bur15**, Lem. 3.4] we conclude that $\Phi_{\rho'}(q)=1$ in $\mathrm{K}_1(\mathcal{O}_C[[\Gamma]])$ and thus $\varphi_{F'}(\Phi_{\rho'}(q))=1$ in C for every Artin representation ρ' of $\mathrm{Gal}(F_\infty/F')$. In particular, with K running through the finite Galois extension fields of F in F_∞, the images of q in the groups $\mathrm{K}_1(\mathbb{Q}_\ell[\mathrm{Gal}(K/F)])$ are trivial. This implies

$$q\in\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F')]]).$$

Using Corollary 2.1.4 we find a suitable admissible extension L_∞/F unramified over $U'\subset U$ such that

$$\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F')]]}\colon\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(L_\infty/F')]])\to\widehat{\mathrm{SK}}_1(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F')]])$$

is the zero map. As

$$q=\Psi_{\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F')]]}\left(\frac{\mathcal{M}_{L_\infty/F',\gamma^r}(U'_{F'},f_{F'}^*(\mathbb{Z}_\ell)_{U'_{F'}}(1))}{\Psi_{\Lambda[[\mathrm{Gal}(L_\infty/F)]]}\mathcal{M}_{L_\infty/F,\gamma}(U',(\mathbb{Z}_\ell)_{U'}(1))}\right),$$

we conclude $q=1$. The proof that $q'=1$ follows the same pattern. □

DEFINITION 4.2.8. Let F be a totally real field, $k\colon U\to W$ be an open immersion of open dense subschemes of $X=\operatorname{Spec}\mathcal{O}_F$ such that ℓ is invertible on W, and Λ be an adic $\mathbb{Z}_\ell$-algebra. Fix a topological generator $\gamma\in\mathrm{Gal}(F_{\mathrm{cyc}}/F)$. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, and any really admissible extension F_∞/F unramified over U, we set

$$\begin{aligned}
\mathcal{M}_{F_\infty/F,\gamma}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)) &\coloneqq \mathcal{M}_{F_\infty/F',\gamma}(U,\mathcal{F}^\bullet(1))\prod_{x\in W-U}\mathcal{M}_{F_\infty/F',\gamma}(x,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1)),\\
\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet) &\coloneqq \mathcal{M}^{\circledast}_{F_\infty/F',\gamma}(U,\mathcal{F}^\bullet)\prod_{x\in W-U}\mathcal{M}^{\circledast}_{F_\infty/F',\gamma}(x,k_!f_!f^*\mathcal{F}^\bullet)
\end{aligned}$$

in $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]])$ and

$$\begin{aligned}
\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)) &\coloneqq \mathcal{M}_{F_\infty/F,\gamma}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))s_\gamma(-[\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1))]),\\
\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet) &\coloneqq \mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(W,k_!\mathcal{F}^\bullet)s_{\gamma^{-1}}([\mathrm{R}\,\Gamma(W,k_!f_!f^*\mathcal{F}^\bullet)])
\end{aligned}$$

in $\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]_S)$.

Note that we do not assume that F_∞/F is unramified over W. If it is unramified over W, then

$$\begin{aligned}
\mathrm{R}\,\Gamma_{\mathrm{c}}(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1)) &= \mathrm{R}\,\Gamma_{\mathrm{c}}(W,f_!f^*\,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)),\\
\mathrm{R}\,\Gamma(W,k_!f_!f^*\mathcal{F}^\bullet) &= \mathrm{R}\,\Gamma(W,f_!f^*k_!\mathcal{F}^\bullet)
\end{aligned}$$

and the two possible definitions of $\mathcal{M}_{F_\infty/F,\gamma}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ and $\mathcal{M}^{\circledast}_{F_\infty/F,\gamma}(W,k_!\mathcal{F}^\bullet)$ agree. Moreover, by Proposition 4.2.6, $\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ and $\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet)$ do not depend on the choice of γ.

In the following corollary, we compile a list of the transformation properties of $\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))$ and $\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet)$.

COROLLARY 4.2.9. *Let F be a totally real field, $k\colon U\to W$ be an open immersion of open dense subschemes of $X=\operatorname{Spec}\mathcal{O}_F$ such that ℓ is invertible on W, and Λ be an adic $\mathbb{Z}_\ell$-algebra. Fix a $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, and a really admissible extension F_∞/F unramified over U.*

(1) *Write $f\colon U_{F_\infty}\to U$ for the system of coverings associated to F_∞/F. Then*
$$d\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))=-[\mathrm{R}\,\Gamma_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1))],$$
$$d\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet)=[\mathrm{R}\,\Gamma(W,k_!f_!f^*\mathcal{F}^\bullet)]$$

(2) *If $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ are weakly equivalent in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, then*
$$\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))=\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{G}^\bullet(1)),$$
$$\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet)=\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{G}^\bullet).$$

(3) *If $0\to\mathcal{F}'^\bullet\to\mathcal{F}^\bullet\to\mathcal{F}''^\bullet\to 0$ is an exact sequence in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, then*
$$\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))=\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}'^\bullet(1))\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}''^\bullet(1)),$$
$$\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet)=\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}'^\bullet)\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}''^\bullet).$$

(4) *If W' is an open dense subscheme of X on which ℓ is invertible and $k'\colon W\to W'$ is an open immersion, then*
$$\begin{aligned}\mathcal{L}_{F_\infty/F}(W',\mathrm{R}(k'k)_*\mathcal{F}^\bullet(1))=&\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))\\&\prod_{x\in W'-W}\mathcal{L}_{F_\infty/F}(x,\mathrm{R}(k'k)_*\mathcal{F}^\bullet(1)),\\\mathcal{L}^{\circledast}_{F_\infty/F}(W',(k'k)_!\mathcal{F}^\bullet)=&\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet)\\&\prod_{x\in W'-W}\mathcal{L}^{\circledast}_{F_\infty/F}(x,(k'k)_!\mathcal{F}^\bullet).\end{aligned}$$

(5) *If $i\colon x\to U$ is a closed point, then*
$$\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*i_*i^*\mathcal{F}^\bullet(1))=\mathcal{L}_{F_\infty/F}(x,\mathcal{F}^\bullet(1)),$$
$$\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!i_*\mathrm{R}\,i^!\mathcal{F}^\bullet)=\mathcal{L}^{\circledast}_{F_\infty/F}(x,\mathcal{F}^\bullet).$$

(6) *If F'_∞/F is a really admissible subextension of F_∞/F, then*
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)))=\mathcal{L}_{F'_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet))=\mathcal{L}^{\circledast}_{F'_\infty/F}(W,k_!\mathcal{F}^\bullet).$$

(7) *If $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, for another adic $\mathbb{Z}_\ell$-algebra Λ', then*
$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)))=\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)(1)),$$
$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet))=\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet)).$$

(8) *If F'/F is a finite extension inside F_∞ and $f_{F'}\colon U_{F'}\to U$ the associated covering, then*
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)))=\mathcal{L}_{F_\infty/F'}(W_{F'},\mathrm{R}\,k_*f_{F'}^*\mathcal{F}^\bullet(1)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet))=\mathcal{L}^{\circledast}_{F_\infty/F'}(W_{F'},k_!f_{F'}^*\mathcal{F}^\bullet).$$

(9) *With the notation of* (8), *if* $\mathcal{G}^\bullet$ *is in* $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'},\Lambda)$, *then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}_{F_\infty/F'}(W_{F'},\mathrm{R}\,k_*\mathcal{G}^\bullet(1))) = \mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_* f_{F'*}\mathcal{G}^\bullet(1)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\circledast}_{F_\infty/F'}(W_{F'},k_!\mathcal{G}^\bullet)) = \mathcal{L}^{\circledast}_{F_\infty/F}(W,k_! f_{F'*}\mathcal{G}^\bullet).$$

(10) *If* $\mathcal{F}$ *is a smooth* Λ*-adic sheaf on* U *which is smooth at* ∞, *then*

$$\mathcal{L}^{\circledast}_{F_\infty/F}(W,k_!\mathcal{F}) = (\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^{*\Lambda}(1)))^{\circledast}.$$

(11) *If* $C/\mathbb{Q}_\ell$ *is a finite field extension and* $\rho\colon\mathrm{Gal}(F_\infty/F)\to\mathrm{Gl}_d(\mathcal{O}_C)$ *is an Artin representation, then*

$$\Phi_{\rho\epsilon_F^{-n}}(\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*(\mathbb{Z}_\ell)_U(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\eta_*(\rho\omega_F^n)(1-n)),$$
$$\Phi_{\rho\epsilon_F^{n}}(\mathcal{L}^{\circledast}_{F_\infty/F}(W,\mathrm{R}\,k_*(\mathbb{Z}_\ell)_U)) = \mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\eta_*(\rho\omega_F^{-n})(n)),$$

for any integer n.

(12) *If* $C/\mathbb{Q}_\ell$ *is a finite field extension and* $\rho\colon\mathrm{Gal}_F\to\mathrm{Gl}_d(\mathcal{O}_C)$ *is an Artin representation which factors through a totally real field and which is unramified over* U, *then*

$$\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\eta_*(\rho\omega_F^n)(1-n))) = L_{\Sigma,\mathrm{T}}(\rho\omega_F^n,1-n),$$
$$\varphi(\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(W,k_!\eta_*(\rho\omega_F^{-n})(n))) = L_{\Sigma,\mathrm{T}}(\rho^*\omega_F^n,1-n)^{-1}$$

with $\Sigma = X-W$, $\mathrm{T}=W-U$ *and any integer* $n>1$.

Proof. Properties (1)–(4) are clear by definition. For Property (5) we notice that for $y\in W$

$$\mathcal{L}_{F_\infty/F}(y,\mathrm{R}\,k_*i_*i^*\mathcal{F}^\bullet(1)) = \begin{cases}\mathcal{L}_{F_\infty/F}(x,\mathcal{F}^\bullet(1)) & \text{if } y=x,\\ 1 & \text{else.}\end{cases}$$

Hence,

$$\mathcal{L}_{F_\infty/F}(W,\mathrm{R}\,k_*i_*i^*\mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(U,i_*i^*\mathcal{F}^\bullet(1)) = \mathcal{L}_{F_\infty/F}(x,\mathcal{F}^\bullet(1))$$

by (4) and by Theorem 4.2.4.(2). The proof for the dual L-function is analogous.

Properties (6) and (7) follow from Theorem 4.2.4 or Theorem 4.2.5 combined with Proposition 2.9.1 and either Proposition 3.5.4 or Proposition 3.5.8. For Properties (8) and (9) one applies Theorem 4.2.7. Property (10) follows from the last part of Theorem 4.2.5 combined with Proposition 2.9.3 and Proposition 3.5.9. Property (11) is just a special case of (7) in a different notation.

It remains to prove (12). The first identity is simply the combination of (4.1.3) and (4.1.2). The second identity follows from (4.1.1), Property (10) and the first identity. □

4.3. CM-Admissible Extensions

Definition 4.3.1. Let F be a totally real number field and F_∞/F an admissible extension. We call F_∞/F *CM-admissible* if F_∞ is totally imaginary and there exists an involution $\iota\in\mathrm{Gal}(F_\infty/F)$ such that the fixed field F_∞^+ of ι is totally real.

Let F be a totally real number field and $\ell\neq 2$ throughout this section. Note that for a CM-admissible extension F_∞/F with Galois group

$$G \coloneqq \mathrm{Gal}(F_\infty/F),$$

the automorphism ι is uniquely determined and commutes with every other field automorphism of F_∞. As usual, we write

$$e_- \coloneqq \frac{1-\iota}{2},\quad e_+ \coloneqq \frac{1+\iota}{2}\in\Lambda[[G]].$$

for the corresponding central idempotents.

The extension F^+_∞/F is Galois and a hence, a really admissible extension. We set $G^+ \coloneqq \mathrm{Gal}(F^+_\infty/F)$. Moreover, we fix as before an immersion $k\colon U \to W$ of open dense subschemes of $X = \operatorname{Spec} F$ such that F_∞/F is unramified over U and $\ell \neq 2$ is invertible on W. Let

$$f^+\colon U_{F^+_\infty} \to U$$

denote the restriction of the family of coverings $f\colon U_{F_\infty} \to U$ to $U_{F^+_\infty}$.

If F_∞ contains the ℓ-th roots of unity and hence, the ℓ^n-th roots of unity for all $n \geq 1$, the cyclotomic character

$$\kappa_F\colon \mathrm{Gal}_F \to \mathbb{Z}_\ell^\times, \quad g\zeta = \zeta^{\kappa_F(g)}, \quad g \in \mathrm{Gal}_F, \zeta \in \mu_{\ell^\infty}$$

factors through $G = \mathrm{Gal}(F_\infty/F)$. We then obtain for every odd $n \in \mathbb{Z}$ a ring isomorphism

$$\Lambda[[G]] \to \Lambda[[G_+]] \times \Lambda[[G_+]], \qquad G \ni g \mapsto (g^+, \kappa_F(g)^n g^+),$$

where g^+ denotes the image of $g \in G$ in G^+. The projections onto the two components corresponds to the decomposition of $\Lambda[[G]]$ with respect to e_+ and e_-.

We will construct the corresponding decomposition of $\mathbf{A}(\Lambda[[G]])$, where

$$\mathbf{A} \in \{\mathbf{PDG}^{\mathrm{cont}}, w_H\mathbf{PDG}^{\mathrm{cont}}, \mathbf{PDG}^{\mathrm{cont}, w_H}\}.$$

Write $\Lambda(\kappa_F^n)^\sharp$ for the Λ-$\Lambda[[G]]$-bimodule Λ with $g \in G$ acting by $\kappa_F^n(g^{-1})$ from the right and $\Lambda(\kappa_F^n)^\sharp[[G]]^\delta$ for the $\Lambda[[G]]$-$\Lambda[[G]]$-bimodule $\Lambda[[G]] \otimes_\Lambda \Lambda(\kappa_F^n)^\sharp$ with the diagonal right action of G. According to Example 2.6.1, we obtain Waldhausen exact functors

$$\Psi_{\Lambda(\kappa_F^n)^\sharp[[G]]^\delta}\colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]]).$$

Moreover, considering $\Lambda[[G^+]]$ as a $\Lambda[[G^+]]$-$\Lambda[[G]]$-bimodule or as a $\Lambda[[G]]$-$\Lambda[[G^+]]$-bimodule, we obtain Waldhausen exact functors

$$\Psi_{\Lambda[[G^+]]}\colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G^+]]), \qquad \Psi_{\Lambda[[G^+]]}\colon \mathbf{A}(\Lambda[[G^+]]) \to \mathbf{A}(\Lambda[[G]]).$$

Note there exists isomorphisms of $\Lambda[[G]]$-$\Lambda[[G]]$-bimodules

$$\begin{aligned} e_+\Lambda[[G]] &\cong \Lambda[[G^+]] \otimes_{\Lambda[[G^+]]} \Lambda[[G^+]] \\ e_-\Lambda[[G]] &\cong \Lambda(\kappa_F^n)^\sharp[[G]]^\delta \otimes_{\Lambda[[G]]} e_+\Lambda[[G]] \otimes_{\Lambda[[G]]} \Lambda(\kappa_F^{-n})^\sharp[[G]]^\delta \end{aligned}$$

for every odd $n \in \mathbb{Z}$ such that the composition

$$\Psi_{\Lambda[[G^+]]} \circ \Psi_{\Lambda[[G^+]]}\colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]])$$

is just the projection onto the e_+-component, whereas the projection onto the e_--component may be written as

$$\Psi_{\Lambda(\kappa_F^n)^\sharp[[G]]^\delta \otimes_{\Lambda[[G]]} \Lambda[[G^+]]} \circ \Psi_{\Lambda[[G^+]] \otimes_{\Lambda[[G]]} \Lambda(\kappa_F^{-n})^\sharp[[G]]^\delta}\colon \mathbf{A}(\Lambda[[G]]) \to \mathbf{A}(\Lambda[[G]]).$$

We further note that

$$\Psi_{\Lambda(\kappa_F^n)^\sharp[[G]]^\delta}(f_! f^* \mathcal{F}^\bullet) \cong f_! f^* \mathcal{F}^\bullet(n).$$

If Λ' is another adic $\mathbb{Z}_\ell$-algebra and $P^\bullet$ is a complex of Λ'-$\Lambda[[G]]$-bimodules, strictly perfect as complex of Λ'-modules, we set

$$P_+^\bullet \coloneqq P^\bullet e_+, \qquad P_-^\bullet \coloneqq P^\bullet e_- \tag{4.3.1}$$

such that ι acts trivially on $P_+^\bullet$ and by -1 on $P_-^\bullet$. Both are again complex of Λ'-$\Lambda[[G]]$-bimodules and strictly perfect as complex of Λ'-modules. In particular, we have an isomorphism of complexes of $\Lambda'[[G]]$-$\Lambda[[G]]$-bimodules

$$P[[G]]^{\delta^\bullet} \cong P_+[[G]]^{\delta^\bullet} \oplus P_-[[G]]^{\delta^\bullet}.$$

Beware that $P_+[[G]]^{\delta^\bullet}$ differs from $P[[G]]^{\delta^\bullet}e_+$. The element ι acts as $\iota\otimes\mathrm{id}$ on the first complex and trivially on the second. In fact, we have

$$\begin{aligned} P_+[[G]]^{\delta^\bullet}e_+ &= e_+P_+[[G]]^{\delta^\bullet}, & P_+[[G]]^{\delta^\bullet}e_- &= e_-P_+[[G]]^{\delta^\bullet},\\ P_-[[G]]^{\delta^\bullet}e_+ &= e_-P_-[[G]]^{\delta^\bullet}, & P_-[[G]]^{\delta^\bullet}e_- &= e_+P_-[[G]]^{\delta^\bullet}. \end{aligned}$$

Moreover, the Waldhausen exact functors

$$\begin{aligned} \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) &\to \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda'), & \mathcal{F}^\bullet &\mapsto \Psi_{\widetilde{P_+}^\bullet}(\mathcal{F}^\bullet),\\ \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda) &\to \mathbf{PDG}^{\mathrm{cont}}(U,\Lambda'), & \mathcal{F}^\bullet &\mapsto \Psi_{\widetilde{P_-}^\bullet}(\mathcal{F}^\bullet)(1) \end{aligned}$$

map complexes in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$ to complexes in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda')$.

Throughout the rest of this section, we make use of Conjecture 3.3.4.

Corollary 4.3.2. *Assume that F_∞/F is any CM-admissible extension unramified over U. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, the complexes*

$$\begin{aligned} &e_+\,\mathrm{R}\Gamma_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1)), && e_-\,\mathrm{R}\Gamma_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet),\\ &e_+\,\mathrm{R}\Gamma(W,k_!f_!f^*\mathcal{F}^\bullet), && e_-\,\mathrm{R}\Gamma(W,k_!f_!f^*\mathcal{F}^\bullet(1)) \end{aligned}$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

Proof. Without loss of generality, we may enlarge F_∞ by adjoining the ℓ-th roots of unity. The claim of the corollary is then an immediate consequence of Theorem 3.4.2 applied to

$$\begin{aligned} \Psi_{\Lambda[[G^+]]}(\mathrm{R}\Gamma_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet(1))) &\sim \mathrm{R}\Gamma_c(W,\mathrm{R}\,k_*(f^+)_!(f^+)^*\mathcal{F}^\bullet(1)),\\ \Psi_{\Lambda[[G^+]]}(\mathrm{R}\Gamma(W,k_!f_!f^*\mathcal{F}^\bullet)) &\sim \mathrm{R}\Gamma(W,k_!(f^+)_!(f^+)^*\mathcal{F}^\bullet),\\ \Psi_{\Lambda[[G^+]]\otimes_{\Lambda[[G]]}\Lambda(\kappa_F)^\sharp[[G]]^\delta}(\mathrm{R}\Gamma_c(W,\mathrm{R}\,k_*f_!f^*\mathcal{F}^\bullet)) &\sim \mathrm{R}\Gamma_c(W,\mathrm{R}\,k_*(f^+)_!(f^+)^*\mathcal{F}^\bullet(1)),\\ \Psi_{\Lambda[[G^+]]\otimes_{\Lambda[[G]]}\Lambda(\kappa_F^{-1})^\sharp[[G]]^\delta}(\mathrm{R}\Gamma(W,k_!f_!f^*\mathcal{F}^\bullet(1))) &\sim \mathrm{R}\Gamma(W,k_!(f^+)_!(f^+)^*\mathcal{F}^\bullet) \end{aligned}$$

□

Assume that F_∞/F is CM-admissible and that F_∞ contains the ℓ-th roots of unity. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$, we set

$$\begin{aligned} \mathcal{L}^+_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1)) &\coloneqq \Psi_{\Lambda[[G^+]]}(\mathcal{L}_{F^+_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))),\\ \mathcal{L}^{\circledast,+}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet) &\coloneqq \Psi_{\Lambda[[G^+]]}(\mathcal{L}^{\circledast}_{F^+_\infty/F}(W,k_!\mathcal{F}^\bullet)),\\ \mathcal{L}^-_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet) &\coloneqq \Psi_{\Lambda(\kappa_F^{-1})^\sharp[[G]]^\delta\otimes_{\Lambda[[G]]}\Lambda[[G^+]]}(\mathcal{L}_{F^+_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1))),\\ \mathcal{L}^{\circledast,-}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet(1)) &\coloneqq \Psi_{\Lambda(\kappa_F)^\sharp[[G]]^\delta\otimes_{\Lambda[[G]]}\Lambda[[G^+]]}(\mathcal{L}^{\circledast}_{F^+_\infty/F}(W,k_!\mathcal{F}^\bullet)) \end{aligned}$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$. We extend this definition to CM-admissible subextensions F'_∞/F with F'_∞ not containing the ℓ-th roots of unity by taking the image of the elements under

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}\colon \mathrm{K}_1(\Lambda[[G]]_S)\to\mathrm{K}_1(\Lambda[[\mathrm{Gal}(F'_\infty/F)]]_S).$$

Furthermore, for $\varepsilon\in\{+,-\}$, $x\in W$ and $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ we set

$$\begin{aligned} \mathcal{L}^{\varepsilon}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{F}^\bullet) &= \Psi_{e_\varepsilon\Lambda[[G]]}(\mathcal{L}_{F_\infty/F}(x,\mathrm{R}\,k_*\mathcal{F}^\bullet)),\\ \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(x,k_!\mathcal{F}^\bullet) &= \Psi_{e_\varepsilon\Lambda[[G]]}(\mathcal{L}^{\circledast}_{F_\infty/F}(x,k_!\mathcal{F}^\bullet)). \end{aligned}$$

We will write $-\varepsilon\in\{+,-\}$ for the opposite sign.

Assume that $C/\mathbb{Q}_\ell$ is a finite field extension and $\rho\colon\mathrm{Gal}_F\to\mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation unramified over U. If $\rho(\sigma)=-\mathrm{id}$ for every complex conjugation $\sigma\in\mathrm{Gal}_F$, then ρ factors through a CM-extension of F. In particular, $\eta_*(\rho\omega_F^{-1})$ is smooth on U and at ∞ and we may define elements

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\eta_*(\rho\omega_F^n)(-n)),\quad \mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(W,k_!\eta_*(\rho\omega_F^n)(1-n)) \tag{4.3.2}$$

by identifying $\eta_*(\rho\omega_F^n)(-n)$ with $\eta_*(\rho\omega_F^{-1}\omega_F^{n+1})(1-(n+1))$. In particular,

$$\varphi(\mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\,k_*\eta_*(\rho\omega_F^n)(-n))) = L_{\Sigma,\mathrm{T}}(\rho\omega_F^n, -n),$$
$$\varphi(\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(W, k_!\eta_*(\rho\omega_F^{-n})(1-n))) = L_{\Sigma,\mathrm{T}}(\rho^*\omega_F^n, -n)^{-1}$$

with $\Sigma = X - W$, $\mathrm{T} = W - U$ and any integer $n > 0$. If ρ is any Artin representation that factors through a CM-extension, then we can decompose it as in (4.3.1) into two subrepresentations ρ_+ and ρ_- such that

$$\rho_+(\sigma) = \mathrm{id}, \qquad \rho_-(\sigma) = -\mathrm{id}$$

for all complex conjugations $\sigma \in \mathrm{Gal}_F$.

Corollary 4.3.3. *Let F be a totally real field, $k: U \to W$ be an open immersion of open dense subschemes of $X = \operatorname{Spec} \mathcal{O}_F$ such that ℓ is invertible on W, and Λ be an adic $\mathbb{Z}_\ell$-algebra. Fix a $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$, and a CM-admissible extension F_∞/F unramified over U. If $\varepsilon = +$, we choose n to be an even integer. We choose n to be odd if $\varepsilon = -$.*

(1) *Write $f: U_{F_\infty} \to U$ for the system of coverings associated to F_∞/F. Then*

$$d\mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1+n)) = -[e_\varepsilon \mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_* f_! f^* \mathcal{F}^\bullet(1+n))],$$
$$d\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{F}^\bullet(n)) = [e_\varepsilon \mathrm{R}\,\Gamma(W, k_! f_! f^* \mathcal{F}^\bullet(n))]$$

(2) *If $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ are weakly equivalent in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$, then*

$$\mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1+n)) = \mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{G}^\bullet(1+n)),$$
$$\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{F}^\bullet(n)) = \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{G}^\bullet(n)).$$

(3) *If $0 \to \mathcal{F}'^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}''^\bullet \to 0$ is an exact sequence in $\mathbf{PDG}^{\mathrm{cont},\infty}(U, \Lambda)$, then*

$$\mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1+n)) = \mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}'^\bullet(1+n))\mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}''^\bullet(1+n)),$$
$$\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{F}^\bullet(n)) = \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{F}'^\bullet(n))\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{F}''^\bullet(n)).$$

(4) *If W' is an open dense subscheme of X on which ℓ is invertible and $k': W \to W'$ is an open immersion, then*

$$\mathcal{L}^{\varepsilon}_{F_\infty/F}(W', \mathrm{R}(k'k)_*\mathcal{F}^\bullet(1+n)) = \mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1+n)) \prod_{x \in W'-W} \mathcal{L}^{\varepsilon}_{F_\infty/F}(x, \mathrm{R}(k'k)_*\mathcal{F}^\bullet(1+n)),$$
$$\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W', (k'k)_!\mathcal{F}^\bullet(n)) = \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{F}^\bullet(n)) \prod_{x \in W'-W} \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(x, (k'k)_!\mathcal{F}^\bullet(n)).$$

(5) *If $i: x \to U$ is a closed point, then*

$$\mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_* i_* i^* \mathcal{F}^\bullet(1+n)) = \mathcal{L}_{F_\infty/F}(x, \mathcal{F}^\bullet(1+n)),$$
$$\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_! i_* \mathrm{R}\, i^! \mathcal{F}^\bullet(n)) = \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(x, \mathcal{F}^\bullet(n)).$$

(6) *If F'_∞/F is a CM-admissible subextension of F_∞/F, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1+n))) = \mathcal{L}^{\varepsilon}_{F'_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, k_!\mathcal{F}^\bullet(n))) = \mathcal{L}^{\circledast,\varepsilon}_{F'_\infty/F}(W, k_!\mathcal{F}^\bullet(n)).$$

If F'_∞/F is a really admissible subextension of F_∞/F, then

$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{+}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1+n))) = \mathcal{L}_{F'_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\circledast,+}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet(n))) = \mathcal{L}^{\circledast}_{F'_\infty/F}(W,k_!\mathcal{F}^\bullet(n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{-}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1+n))) = 1$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}^{\circledast,-}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet(n))) = 1.$$

(7) *If $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, for another adic $\mathbb{Z}_\ell$-algebra Λ', then*

$$\Psi_{P_+[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}^{\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1+n))) = \mathcal{L}^{\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*\Psi_{\widetilde{P_+}^\bullet}(\mathcal{F}^\bullet)(1+n)),$$
$$\Psi_{P_-[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}^{\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1+n))) = \mathcal{L}^{-\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*\Psi_{\widetilde{P_-}^\bullet}(\mathcal{F}^\bullet)(1+n)),$$
$$\Psi_{P_+[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet(n))) = \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W,k_!\Psi_{\widetilde{P_+}^\bullet}(\mathcal{F}^\bullet(n))),$$
$$\Psi_{P_-[[\mathrm{Gal}(F_\infty/F)]]^{\delta\bullet}}(\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet(n))) = \mathcal{L}^{\circledast,-\varepsilon}_{F_\infty/F}(W,k_!\Psi_{\widetilde{P_-}^\bullet}(\mathcal{F}^\bullet(n))).$$

(8) *If F'/F is a finite extension inside F_∞ such that F' is totally real and $f_{F'}\colon U_{F'} \to U$ is the associated covering, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^\bullet(1+n))) = \mathcal{L}^{\varepsilon}_{F_\infty/F'}(W_{F'},\mathrm{R}\,k_*f^*_{F'}\mathcal{F}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W,k_!\mathcal{F}^\bullet)(n)) = \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F'}(W_{F'},k_!f^*_{F'}\mathcal{F}^\bullet(n)).$$

(9) *With the notation of* (8)*, if $\mathcal{G}^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont},\infty}(U_{F'},\Lambda)$, then*

$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\varepsilon}_{F_\infty/F'}(W_{F'},\mathrm{R}\,k_*\mathcal{G}^\bullet(1+n))) = \mathcal{L}^{\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*f_{F'*}\mathcal{G}^\bullet(1+n)),$$
$$\Psi_{\Lambda[[\mathrm{Gal}(F_\infty/F)]]}(\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F'}(W_{F'},k_!\mathcal{G}^\bullet(n))) = \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W,k_!f_{F'*}\mathcal{G}^\bullet(n)).$$

(10) *If $\mathcal{F}$ is a smooth Λ-adic sheaf on U which is smooth at ∞, then*

$$\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W,k_!\mathcal{F}(n)) = (\mathcal{L}^{\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*\mathcal{F}^{*_\Lambda}(1-n)))^{\circledast}.$$

(11) *If $C/\mathbb{Q}_\ell$ is a finite field extension and $\rho\colon\mathrm{Gal}(F_\infty/F) \to \mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation, then*

$$\Phi_\rho(\mathcal{L}^{\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*(\mathbb{Z}_\ell)_U(1+n))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\eta_*(\rho_\varepsilon)(1+n)),$$
$$\Phi_\rho(\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W,\mathrm{R}\,k_*(\mathbb{Z}_\ell)_U(n))) = \mathcal{L}_{F_{\mathrm{cyc}}/F}(W,\mathrm{R}\,k_*\eta_*(\rho_\varepsilon)(n)),$$

PROOF. This is an easy consequence of the preceding remarks and Corollary 4.2.9. □

4.4. Admissible Extensions of Function Fields

Let F be a function field of characteristic p. We write $\mathbb{F}$ for the algebraic closure of $\mathbb{F}_p$ inside F and $\overline{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$ inside $\overline{F}$. Further, let

$$q \coloneqq p^{[\mathbb{F}:\mathbb{F}_p]}$$

denote the number of elements of $\mathbb{F}$. For a fixed prime number ℓ, we let $\mathbb{F}_{\mathrm{cyc}}/\mathbb{F}$ denote the unique $\mathbb{Z}_\ell$-extension of $\mathbb{F}$ and write F_{cyc} for the composite field $\mathbb{F}_{\mathrm{cyc}}F$. As before, we write X for the smooth and proper curve over $\mathbb{F}$ whose closed points are the places of F. For any subscheme Z of X, we write $\overline{Z}$ for the base change to $\overline{\mathbb{F}}$. Further, we fix an immersion of two open dense subschemes

$$k\colon U \to W$$

of X. If $\ell = p$, we will assume $W = U$.

Let F_∞/F be an admissible extension unramified over U and Λ an adic $\mathbb{Z}_\ell$-algebra. Different from the number field case, there exists an explicit construction of the non-commutative L-function. For this, recall that for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$, we obtain a exact sequence

$$(4.4.1)\qquad 0 \to \mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_*\mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}} \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_*\mathcal{F}^\bullet) \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ [**Wit08**, Prop. 6.1.2]. Since the complex $\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)$ is in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ by Theorem 3.4.1, the endomorphism

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet) \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}} \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)$$

is a weak equivalence in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ and hence, we may consider its class

$$[\mathrm{id} - \mathfrak{F}_{\mathbb{F}} \circlearrowright \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)] \in \mathrm{K}_1(\Lambda[[G]]_S).$$

If $\ell \neq p$, we obtain in the same way a class

$$[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}^{-1} \circlearrowright \mathrm{R}\,\Gamma(\overline{W}, k_! f_! f^*\mathcal{F}^\bullet)] \in \mathrm{K}_1(\Lambda[[G]]_S).$$

Definition 4.4.1. Assume that F_∞/F is an admissible extension unramified over U and $\ell \neq p$. Let Λ be any adic $\mathbb{Z}_\ell$-algebra. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$, we set

$$\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet) := [\mathrm{id} - \mathfrak{F}_{\mathbb{F}} \circlearrowright \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_* f_! f^*\mathcal{F}^\bullet)]^{-1}$$
$$\mathcal{L}^{\circledast}_{F_\infty/F}(W, k_!\mathcal{F}^\bullet) := [\mathrm{id} - \mathfrak{F}_{\mathbb{F}}^{-1} \circlearrowright \mathrm{R}\,\Gamma(\overline{W}, k_! f_! f^*\mathcal{F}^\bullet)]$$

If $\ell \neq p$, the dual non-commutative L-function may be related to the non-commutative L-function as follows. Set

$$V = U \cup (X - W)$$

and let $j\colon U \to V$ denote the inclusion map. Recall from Remark 3.1.4 that

$$j'_!\mathrm{R}\,j_*\mathcal{F}^\bullet \sim \mathrm{R}\,k'_* k_!\mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ if $j'\colon V \to X$, $k'\colon W \to X$ denote the inclusion maps.

Definition 4.4.2. Assume $\ell \neq p$ and let Λ be any adic $\mathbb{Z}_\ell$-algebra. If $\mathcal{F}^\bullet$ is in $\mathbf{PDG}^{\mathrm{cont}}(V,\Lambda)$, we define the *global ε-factor* of $\mathcal{F}^\bullet$ on V to be

$$\varepsilon(V,\mathcal{F}^\bullet) := [-\mathfrak{F}_{\mathbb{F}} \circlearrowright \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{V},\mathcal{F}^\bullet)] \in \mathrm{K}_1(\Lambda).$$

Remark 4.4.3. It is expected that the global ε-factor may be expressed as a finite product of local ε-factors. For $\Lambda = \mathbb{Z}_\ell$, this is a theorem of Laumon [**Lau87**, Thm. 3.2.1.1]. In [**FK06**, §3.5.6], Fukaya and Kato sketch how to extend this result to arbitrary adic $\mathbb{Z}_\ell$-algebras.

We then obtain

$$(4.4.2)\qquad \mathcal{L}^{\circledast}_{F_\infty/F}(W, k_!\mathcal{F}^\bullet)\mathcal{L}_{F_\infty/F}(V, \mathrm{R}\,j_*\mathcal{F}^\bullet) = \varepsilon(V, \mathrm{R}\,j_* f_! f^*\mathcal{F}^\bullet).$$

If $\mathcal{F}$ is a smooth Λ-adic sheaf, we will show later in Theorem 5.3.6 that

$$\mathcal{L}^{\circledast}_{F_\infty/F}(W, k_!\mathcal{F}) = (\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^{*_\Lambda}(1)))^{\circledast}.$$

In the case that $\ell = p$, the above elements do not have the right interpolation property. However, we can associate to $\mathcal{F}^\bullet$ an element

$$Q(f_! f^*\mathcal{F}^\bullet, t) \in \varprojlim_{I \in \mathfrak{I}_{\Lambda[[G]]}} \mathrm{K}_1(\Lambda[[G]]/I[t])$$

that measures the failure of the Grothendieck trace formula [**Wit16**, Thm. 4.1]. In particular, we may consider its image $Q(f_! f^*\mathcal{F}^\bullet, 1)$ under the homomomorphism

$$\varprojlim_{I \in \mathfrak{I}_{\Lambda[[G]]}} \mathrm{K}_1(\Lambda[[G]]/I[t]) \to \varprojlim_{I \in \mathfrak{I}_{\Lambda[[G]]}} \mathrm{K}_1(\Lambda[[G]]/I) = \mathrm{K}_1(\Lambda[[G]])$$

induced by the ring homomorphisms

$$\Lambda[[G]]/I[t] \to \Lambda[[G]]/I, \qquad t \mapsto 1.$$

DEFINITION 4.4.4. Assume that F_∞/F is an admissible extension unramified over U and that $\ell = p$. Let Λ be any adic $\mathbb{Z}_\ell$-algebra. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$, we set

$$\mathcal{L}_{F_\infty/F}(U,\mathcal{F}^\bullet) \coloneqq Q(f_! f^* \mathcal{F}^\bullet, 1)[\mathrm{id} - \mathfrak{F}_{\mathbb{F}} \circlearrowright \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{U}, f_! f^* \mathcal{F}^\bullet)]^{-1}$$

It might be worthwhile to notice that the family of non-commutative L-functions is already completely determined by the ℓ-adic L-functions for Artin representations. Let $\Theta = \Theta_F$ be the set of triples (U, F_∞, Λ) such that $U \subset X$ is a dense open subscheme with ℓ invertible on U, F_∞/F is an admissible extension unramified over U and Λ is an adic $\mathbb{Z}_\ell$-algebra.

THEOREM 4.4.5. *Fix a function field F and a prime number ℓ. The family of homomorphisms*

$$\big(\mathcal{L}_{F_\infty/F}(U,(-))\colon \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)) \to \mathrm{K}_1(\Lambda[[\mathrm{Gal}(F_\infty/F)]]_S)\big)_{(U,F_\infty,\Lambda)\in\Theta}$$

is uniquely characterised by following properties.

(1) *For any $(U, F_\infty, \Lambda) \in \Theta$, and any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$,*
$$d\mathcal{L}_{F_\infty/F}(U,\mathcal{F}^\bullet) = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, f_! f^* \mathcal{F}^\bullet)].$$
(2) *If $j\colon U' \to U$ is an open immersion and $(U', F_\infty, \Lambda), (U, F_\infty, \Lambda) \in \Theta$, then*
$$\mathcal{L}_{F_\infty/F}(U,\mathcal{F}^\bullet) = \mathcal{L}_{F_\infty/F}(U', j^*\mathcal{F}^\bullet) \prod_{x\in U-U'} \mathcal{L}_{F_\infty/F}(x,\mathcal{F}^\bullet),$$
for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.
(3) *If $(U, F_\infty, \Lambda), (U, F'_\infty, \Lambda) \in \Theta$ such that $F'_\infty \subset F_\infty$ is a subfield, then*
$$\Psi_{\Lambda[[\mathrm{Gal}(F'_\infty/F)]]}(\mathcal{L}_{F_\infty/F}(U,\mathcal{F}^\bullet)) = \mathcal{L}_{F'_\infty/F}(U,\mathcal{F}^\bullet),$$
for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.
(4) *If $(U, F_\infty, \Lambda), (U, F_\infty, \Lambda') \in \Theta$ and $P^\bullet$ is a complex of Λ'-$\Lambda[[\mathrm{Gal}(F_\infty/F)]]$-bimodules, strictly perfect as complex of Λ'-modules, then*
$$\Psi_{P[[\mathrm{Gal}(F_\infty/F)]]^{\delta}{}^\bullet}(\mathcal{L}_{F_\infty/F}(U,\mathcal{F}^\bullet)) = \mathcal{L}_{F_\infty/F}(U, \Psi_{\tilde{P}^\bullet}(\mathcal{F}^\bullet))$$
for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont},\infty}(U,\Lambda)$.
(5) *If $\mathcal{O}_C$ is the valuation ring of a finite extension field C of $\mathbb{Q}_\ell$ and $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ is an Artin representation unramified over U, then the element $\mathcal{L}_{F_{\mathrm{cyc}}/F}(U, \eta_*(\rho))$ agrees with Definition 4.1.1.*

PROOF. By [**Wit14**, §8] and [**Wit16**, §5] the elements $\mathcal{L}_{F_\infty/F}(U,\mathcal{F}^\bullet)$ satisfy the listed properties both in the case that $\ell \neq p$ and $\ell = p$. In fact, one can even replace U by any scheme of finite type over a finite field. The proof that these properties uniquely characterise the above family of homomorphisms follows along the lines of the proofs of Theorem 4.2.4 and Theorem 4.2.2. □

COROLLARY 4.4.6. *Properties (1)–(9) of Corollary 4.2.9 hold for the non-commutative L-function $\mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet)$ of any complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$. If $\ell \neq p$, then properties (1)–(10) hold for $\mathcal{L}^{\circledast}_{F_\infty/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet)$.*

PROOF. Properties (1)–(9) are proved in the same way as in Corollary 4.2.9. We refer to Theorem 5.3.6 for (10) in the case that $\ell \neq p$. □

In the case of the cyclotomic extension F_{cyc}/F, there exists an alternative description of $\mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\,k_*\mathcal{F}^\bullet)$. Write $s\colon W \to \operatorname{Spec}\mathbb{F}$ for the structure map of W as a scheme over $\operatorname{Spec}\mathbb{F}$. If $\ell = p$, we assume $W = U$. If $\ell \neq p$, we set

$$Q(\mathcal{F}^\bullet, t) = 1.$$

Set

$$Z(\operatorname{Spec}\mathbb{F}, \mathrm{R}\, s_! \,\mathrm{R}\, k_* \mathcal{F}^\bullet, t) \coloneqq [\mathrm{id} - \mathfrak{F}_{\mathbb{F}} t \circlearrowright \Lambda[t] \otimes_\Lambda \mathrm{R}\,\Gamma_c(\overline{W}, \mathrm{R}\, k_* \mathcal{F}^\bullet)]^{-1} \in \mathrm{K}_1(\Lambda[t]_{S_t}),$$

with $\mathrm{K}_1(\Lambda[t]_{S_t})$ as defined in Appendix A. Then

$$Q(\mathcal{F}^\bullet, t) Z(\operatorname{Spec}\mathbb{F}, \mathrm{R}\, s_! \,\mathrm{R}\, k_* \mathcal{F}^\bullet, t) = \prod_{x \in W^0} [\mathrm{id} - \mathfrak{F}_x t \circlearrowright \Lambda[[T]] \otimes_\Lambda \mathrm{R}\, k_* \mathcal{F}^\bullet_{\hat{x}}]^{-1},$$

in $\mathrm{K}_1(\Lambda[[t]])$, where the product runs over the closed point of W [**Wit09**, Thm. 7.2] ($\ell \neq p$), [**Wit16**, Thm. 4.1] ($\ell = p$). Moreover, if $\gamma_{\mathbb{F}} \in \Gamma$ denotes the image of $\mathfrak{F}_{\mathbb{F}}$, then

$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet) = Q(\mathcal{F}^\bullet, \gamma_{\mathbb{F}}^{-1}) Z(\operatorname{Spec}\mathbb{F}, \mathrm{R}\, s_! \,\mathrm{R}\, k_* \mathcal{F}^\bullet, \gamma_{\mathbb{F}}^{-1})$$

[**Wit14**, Thm. 8.6] ($\ell \neq p$) and [**Wit16**, Thm. 5.5] ($\ell = p$). If $\Lambda = \mathcal{O}_C$ for some finite extension C of $\mathbb{Q}_\ell$ and ρ is an Artin representation of Gal_F, then we have

$$Q(\eta_* \rho, t) Z(\operatorname{Spec}\mathbb{F}, \mathrm{R}\, s_! \,\mathrm{R}\, k_* \eta_* \rho, t) = Z(W, \mathrm{R}\, k_* \eta_* \rho, t).$$

CHAPTER 5

Main Conjectures for Galois Representations

If $\mathcal{T}$ is a continuous representation over $\mathbb{Z}_\ell$ of the Galois group Gal_F of a global field F which is ramified in at most a finite set of points of X, then one can associate to $\mathcal{T}$ a constructible ℓ-adic sheaf on $U \subset X$ by taking its direct image $\eta_*\mathcal{T}$ under the inclusion of the generic point

$$\eta\colon \mathrm{Spec}\, F \to U.$$

The stalk of $\eta_*\mathcal{T}$ in a geometric point $\hat{x}$ over $x \in U$ is given by the invariants $\mathcal{T}^{\mathcal{I}_x}$ under the inertia group $\mathcal{I}_x$ of $\hat{x}$. However, there is one subtlety in the construction of $\eta_*\mathcal{T}$ due to the non-exactness of the functor η_*. The naive definition, taking the projective system

$$(\eta_*\mathcal{T}/\ell^n\mathcal{T})_{n\in\mathbb{N}},$$

does not always lead to an ℓ-adic sheaf in the honest sense. Yet, it is isomorphic to the ℓ-adic sheaf

$$\eta_*\mathcal{T} \coloneqq (\varprojlim_{m\geq n} \mathbb{Z}/(\ell^n) \otimes_{\mathbb{Z}/(\ell^m)} \eta_*\mathcal{T}/\ell^m\mathcal{T})_{n\in\mathbb{N}}$$

in the Artin-Rees category [**Gro77**, VI, Lem. 2.2.2].

In Section 5.1, we extend the latter definition to Gal_F-representations $\mathcal{T}$ over arbitrary adic $\mathbb{Z}_\ell$-algebras Λ. We cannot do this without an extra hypothesis. Unless Λ is noetherian and regular of dimension less or equal 2, the Λ-module $\mathcal{T}^{\mathcal{I}_x}$ might not be finitely generated and projective in some $x \in U$. Those points have to be excluded from U. Again due to the non-exactness of η_*, the results do not extend to complexes of Gal_F-representations.

If F_∞/F is an admissible extension unramified over U, we can directly apply the results of the previous chapter to the Λ-adic sheaf $\eta_*\mathcal{T}$. Yet, we can do a little more and allow F_∞/F to have some ramification over U by considering the $\Lambda[[G]]$-adic sheaf $\eta_*\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}$, with $g \in \mathrm{Gal}_F$ acting on $\Lambda[[G]]^\sharp$ by right multiplication with the inverse of its image in G. The main objective of this chapter is formulate and prove a version of the non-commutative main conjecture in this setting. This will be achieved in Section 5.2.

Although the Λ-adic sheaves $\eta_*\mathcal{T}$ are not smooth in general, they still admit a good duality theory, which we will develop in Section 5.3. As a consequence, we can prove in Theorem 5.3.6 a functional equation for the non-commutative L-functions in the function field case.

In Section 5.4 we calculate the cohomology of $\mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})$, where $k\colon U \to W$ denotes the immersion into another open dense subscheme W of X. If $U \neq W \neq X$ and F is a function field of characteristic different from ℓ, then $\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*\mathcal{T})$ is the only non-vanishing cohomology group and a finitely generated projective module over $\Lambda[[H]]$. Similar results also hold for function fields of characteristic ℓ and CM-admissible extensions of totally real fields. This generalises results of Greither and Popescu in [**GP12**] and [**GP15**].

The remaining sections of this chapter deal with special instances of the non-commutative Iwasawa main conjecture for Galois representations. In Section 5.5 we deal with the case that Λ is a regular, noetherian, and commutative ring and that G is an ℓ-adic Lie group without elements of order ℓ, such that we can replace the complex $\mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* \eta_* \Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})$ by its cohomology groups. Finally, in Section 5.6, we prove the function field analogue of the Gl_2 main conjecture of [**CFK⁺05**] for abelian varieties in the case ℓ different from the characteristic of F.

5.1. The Adic Sheaf Associated to a Galois Representation

Fix a prime number ℓ and an admissible extension F_∞/F of a global field F. As before, we set

$$G \coloneqq \mathrm{Gal}(F_\infty/F), \qquad H \coloneqq \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}}), \qquad \Gamma \coloneqq \mathrm{Gal}(F_{\mathrm{cyc}}/F).$$

Further, we write for each $x \in X$

$$\mathcal{I}_x \coloneqq \mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$$

for $\mathrm{Gal}(\overline{F}_x/F_x^{\mathrm{nr}})$ considered as a subgroup of Gal_F via our fixed embedding of $\overline{F}$ into $\overline{F}_x$.

Definition 5.1.1. Let Λ be an adic ring. We call a compact $\Lambda[[\mathrm{Gal}_F]]$-module $\mathcal{T}$ a *finitely ramified representation* of Gal_F over Λ if

(1) it is finitely generated and projective as Λ-module,
(2) $\mathcal{T}$ is unramified outside a finite set of places, i. e. the set

$$\{x \in X \mid \mathcal{T}^{\mathcal{I}_x} \neq \mathcal{T}\}$$

is finite.

Recall that for a finite ring R, taking the stalk in the geometric point $\operatorname{Spec}\overline{F}$ is an equivalence of categories between the category of étale sheaves of R-modules on $\operatorname{Spec} F$ and the category of discrete $R[[\mathrm{Gal}_F]]$-modules [**Mil80**, Thm. II.1.9]. In our notation, we will not distinguish between the discrete $R[[\mathrm{Gal}_F]]$-module and the corresponding sheaf on $\operatorname{Spec} F$.

A finitely ramified representation $\mathcal{T}$ of Gal_F over an adic ring Λ then corresponds to a projective system of sheaves on $\operatorname{Spec} F$. We want to consider the system of direct image sheaves under the inclusion

$$\eta \coloneqq \eta_F \colon \operatorname{Spec} F \to U.$$

of the generic point into an open, dense subscheme U of X. Since the naive definition, applying η_* to each element of the system, does not necessarily lead to an adic sheaf in our sense, we will consider a stabilised version instead, redefining the direct image sheaf as follows.

Definition 5.1.2. Let Λ be an adic ring, $U \subset X$ an open non-empty subscheme and $\mathcal{T}$ a finitely ramified representation of Gal_F over Λ. We define an inverse system of étale sheaves of Λ-modules $\eta_*\mathcal{T} \coloneqq (\eta_*\mathcal{T}_I)_{I\in\mathfrak{I}_\Lambda}$ on U by setting

$$\eta_*\mathcal{T}_I \coloneqq \varprojlim_{J\in\mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda \eta_*\mathcal{T}/J\mathcal{T}.$$

Proposition 5.1.3. *Let Λ be an adic ring and $\mathcal{T}$ be a finitely ramified representation of Gal_F over Λ such that $\mathcal{T}^{\mathcal{I}_x}$ is a finitely generated Λ-module for each closed point x of U. Then $\eta_*\mathcal{T}_I$ is a constructible étale sheaf of Λ/I-modules on U for any $I \in \mathfrak{I}_\Lambda$. If x is a closed point of U, then the stalk of $\eta_*\mathcal{T}_I$ in the geometric point $\hat{x}$ is given by*

$$(\eta_*\mathcal{T}_I)_{\hat{x}} = \Lambda/I \otimes_\Lambda \mathcal{T}^{\mathcal{I}_x}.$$

In particular, $\eta_*\mathcal{T}$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$ if $\mathcal{T}^{\mathcal{I}_x}$ is a finitely generated projective Λ-module for each closed point x of U.

PROOF. Let V be the open complement of U by the set of points x with $\mathcal{T}^{\mathcal{I}_x} \neq \mathcal{T}$. Consider a connected étale open set W of V and let $L \subset \overline{F}$ be the function field of W. Then for any $J \subset I$

$$(\Lambda/I \otimes_\Lambda \eta_* \mathcal{T}/J\mathcal{T})(W) = (\mathcal{T}/I\mathcal{T})^{\mathrm{Gal}_L} = (\eta_* \mathcal{T}/I\mathcal{T})(W).$$

In particular, the restriction of $\Lambda/I \otimes_\Lambda \eta_* \mathcal{T}/J\mathcal{T}$ to V is a locally constant étale sheaf of Λ/I-modules which independent of J. Now the category of étale sheaves of Λ/I-modules on U which are locally constant on V is equivalent to the category of tuples $(M, (M_x, \phi_x)_{x \in U-V})$ where M is a discrete $\Lambda/I[[\mathrm{Gal}_F]]$-module unramified over V, the M_x are discrete $\mathrm{Gal}_{k(x)}$-modules, and $\phi_x \colon M_x \to M^{\mathcal{I}_x}$ are homomorphisms of discrete $\mathrm{Gal}_{k(x)}$-modules [**Mil80**, Ex. II.3.16].

By the above considerations, it is clear that the projective limit of the system $(\Lambda/I \otimes_\Lambda \eta_* \mathcal{T}/J\mathcal{T})_{J \in \mathfrak{I}_\Lambda}$ exists in the category of étale sheaves of Λ/I-modules which are locally constant on V and coincides with the projective limit taken in the category of all étale sheaves of Λ/I-modules. Moreover, it corresponds to the tuple

$$(\mathcal{T}/I\mathcal{T}, (\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (\mathcal{T}/J\mathcal{T})^{\mathcal{I}_x}, \phi_x \colon \varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (\mathcal{T}/J\mathcal{T})^{\mathcal{I}_x} \to (\mathcal{T}/I\mathcal{T})^{\mathcal{I}_x})_{x \in U-V}).$$

Beware that the projective limit

$$\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (\mathcal{T}/J\mathcal{T})^{\mathcal{I}_x}$$

is a priori taken in the category of discrete $\Lambda/I[[\mathrm{Gal}_{k(x)}]]$-modules (i. e. such that the stabiliser of every element is open in $\mathrm{Gal}_{k(x)}$).

In the category of abstract $\Lambda/I[[\mathrm{Gal}_{k(x)}]]$-modules, we have

$$\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (\mathcal{T}/J\mathcal{T})^{\mathcal{I}_x} = \Lambda/I \otimes_\Lambda \varprojlim_{J \in \mathfrak{I}_\Lambda} (\mathcal{T}/J\mathcal{T})^{\mathcal{I}_x} = \Lambda/I \otimes_\Lambda \mathcal{T}^{\mathcal{I}_x}.$$

Here, the first equality is justified because projective limits of finite Λ/I-modules are exact and because Λ/I is finitely presented as Λ^{op}-module: In any adic ring Λ, the Jacobson radical $\mathrm{Jac}(\Lambda)$ is finitely generated both as left and as right ideal [**War93**, Thm. 36.39]. Therefore, the same is true for all open ideals $I \in \mathfrak{I}_\Lambda$ and thus, Λ/I is a finitely presented Λ^{op}-module.

By assumption, $\mathcal{T}^{\mathcal{I}_x}$ is a finitely generated Λ-module. Hence, $\Lambda/I \otimes_\Lambda \mathcal{T}^{\mathcal{I}_x}$ is finite and the equality

$$\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda/I \otimes_\Lambda (\mathcal{T}/J\mathcal{T})^{\mathcal{I}_x} = \Lambda/I \otimes_\Lambda \mathcal{T}^{\mathcal{I}_x}$$

also holds in the category of discrete $\Lambda/I[[\mathrm{Gal}_{k(x)}]]$-modules. This shows that $\eta_* \mathcal{T}_I$ is constructible and that the stalks have the given form.

From the description of the stalks it is also immediate that

$$\Lambda/I \otimes_{\Lambda/J} \eta_* \mathcal{T}_J \cong \eta_* \mathcal{T}_I$$

such that $\eta_* \mathcal{T}$ is indeed an object of $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda)$ if $\mathcal{T}^{\mathcal{I}_x}$ is finitely generated and projective for all closed points x in U. □

REMARK 5.1.4. Note that if Λ is noetherian, $\mathcal{T}^{\mathcal{I}_x}$ is automatically finitely generated. For general adic rings Λ, this is not true. Assume that ℓ is a prime dividing $p-1$ and let Λ be the power series ring over $\mathbb{F}_\ell$ in three non-commuting indeterminates a, b, c, modulo the relations $ab = 0$, $ac = ca$, $(b+1)(c+1) = (c+1)(b+1)^p$. Set $F = \mathbb{F}_p(t)$ and let $F_\infty = F_{\mathrm{cyc}}(\sqrt[\ell^\infty]{t})$ be the Kummer extension of F_{cyc} obtained by adjoining all ℓ^n-th roots of t. Let x be the point of $\operatorname{Spec} \mathbb{F}_p[t]$ corresponding to the prime ideal (t). Then F_∞/F is unramified over the complement U of x in $\operatorname{Spec} \mathbb{F}_p[t]$ and $F_\infty/F_{\mathrm{cyc}}$ is totally and tamely ramified in x. The Galois group $G = \mathrm{Gal}(F_\infty/F)$ is the pro-ℓ-group topologically generated by two elements τ and σ, subject to the relation $\sigma\tau = \tau^p\sigma$. We obtain a finitely ramified representation $\mathcal{T}$ of Gal_F over Λ by letting τ act on Λ by right multiplication with $b+1$ and σ act by right multiplication with $c+1$. Hence, $\mathcal{T}^{\mathcal{I}_x}$ is the kernel of the right multiplication with b, which is the left ideal of Λ topologically generated by ba^i for all $i > 0$. Clearly, this ideal is not finitely generated.

We will now fix an admissible extension F_∞/F with Galois group $G = H \rtimes \Gamma$. For a closed point x of X we will write $\mathcal{K}_x$ and $\mathcal{J}_x$ for the kernel and the image of the homomorphism $\mathcal{I}_x \to G$, respectively. We also fix an open dense subscheme U of X, an adic $\mathbb{Z}_\ell$-algebra Λ and a finitely ramified representation $\mathcal{T}$ of Gal_F over Λ. We let $\Lambda[[G]]^\sharp$ denote the $\Lambda[[G]][[\mathrm{Gal}_F]]$-module $\Lambda[[G]]$ with $g \in \mathrm{Gal}_F$ acting by the image of g^{-1} in G from the right. Note that $\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}$ is a finitely ramified representation of Gal_F over $\Lambda[[G]]$.

Proposition 5.1.5. *Assume that for every closed point x of U one of the following conditions is satisfied:*

(1) $\mathcal{T}^{\mathcal{K}_x} = 0$,
(2) *$\mathcal{J}_x$ contains an element of infinite order,*
(3) *$\mathcal{T}^{\mathcal{K}_x}$ is a finitely generated, projective Λ-module and $\mathcal{J}_x$ contains no element of order ℓ.*

Then $\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G]])$ and for every $I \in \mathfrak{I}_{\Lambda[[G]]}$,

$$(\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})_I)_{\hat{x}} = 0$$

if x satisfies condition (1) *or* (2),

$$(\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})_I)_{\hat{x}} = \Lambda[[G]]/I \otimes_{\Lambda[[G]]} (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x})^{\mathcal{J}_x}$$

if x satisfies condition (3).

Proof. For each compact Λ-module M, write $\mathcal{U}_M$ for the lattice of open submodules of M. We note that $\Lambda[[G]]$ is a projective limit of finitely generated, free Λ^{op}-modules and hence, a projective object in the category of compact Λ^{op}-modules. The completed tensor product

$$\Lambda[[G]]\hat{\otimes}_\Lambda M = \varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} \varprojlim_{U \in \mathcal{U}_M} \Lambda[[G]]/J \otimes_\Lambda M/U$$

is thus an exact functor from the category of compact Λ-modules to the category of compact $\Lambda[[G]]$-modules. Moreover, we have

$$\Lambda[[G]]\hat{\otimes}_\Lambda M = \Lambda[[G]] \otimes_\Lambda M$$

if M is finitely presented [**Wit13b**, Prop. 1.14]. In particular,

$$\Lambda[[G]]\hat{\otimes}_\Lambda \mathcal{T}^{\mathcal{K}_x} \cong (\Lambda[[G]]\hat{\otimes}_\Lambda \mathcal{T})^{\mathcal{K}_x} \cong (\Lambda[[G]] \otimes_\Lambda \mathcal{T})^{\mathcal{K}_x}.$$

If $\mathcal{T}^{\mathcal{K}_x} = 0$, this obviously implies

$$(\Lambda[[G]] \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x} = 0.$$

Assume that $\mathcal{J}_x$ contains an element of infinite order and let M be any finite Λ-module with a continuous $\mathcal{J}_x$-action. We can then find an element τ of infinite order in an ℓ-Sylow subgroup of $\mathcal{J}_x$ which operates trivially on M. Consider the subgroup $\Upsilon \cong \mathbb{Z}_\ell$ of $\mathcal{J}_x$ which is topologically generated by τ. By choosing a continuous map of profinite spaces

$$G/\Upsilon \to G$$

that is a section of the projection map, we can view $\Lambda[[G]]^\sharp$ as a projective limit of finitely generated, free $\Lambda^{\mathrm{op}}[[\Upsilon]]$-modules and conclude that $1-\tau$ acts as non-zero divisor. In particular, we obtain an exact sequence of projective compact Λ^{op}-modules

$$0 \to \Lambda[[G]]^\sharp \xrightarrow{1-\tau} \Lambda[[G]]^\sharp \to \Lambda[[G/\Upsilon]] \to 0$$

The sequence remains exact after taking the tensor product over Λ with M. Hence,

$$(\Lambda[[G]]^\sharp \otimes_\Lambda M)^\Upsilon = \ker\left(\Lambda[[G]]^\sharp \otimes_\Lambda M \xrightarrow{\mathrm{id}-\tau\otimes 1} \Lambda[[G]]^\sharp \otimes_\Lambda M\right) = 0.$$

Recall that the powers of the Jacobson radical $\mathrm{Jac}(\Lambda)$ are finitely generated as left Λ-modules [**War93**, Thm. 36.39]. In particular, any finite Λ-module M is finitely presented: For some k, the kernel K of a surjection

$$P \to M$$

with P a finitely generated, free Λ-module contains the finitely generated module $\mathrm{Jac}(\Lambda)^k P$ as an open submodule. Hence, the tensor product of M with a compact Λ-module agrees with the completed tensor product.

Writing the compact $\Lambda[[\mathcal{J}_x]]$-module $\mathcal{T}^{\mathcal{K}_x}$ as projective limit of finite $\Lambda[[\mathcal{J}_x]]$-modules, we conclude

$$(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x} = (\Lambda[[G]]^\sharp \hat{\otimes}_\Lambda \mathcal{T}^{\mathcal{K}_x})^{\mathcal{J}_x} = 0.$$

Assume that $\mathcal{J}_x$ contains no element of infinite order nor an element of order ℓ and that $\mathcal{T}^{\mathcal{K}_x}$ is a finitely generated, projective Λ-module. Then $\mathcal{J}_x$ is a finite group of order d prime to ℓ. Set

$$e_{\mathcal{J}_x} = \frac{1}{d}\sum_{\sigma\in\mathcal{J}_x}\sigma.$$

Then $e_{\mathcal{J}_x}$ is a central idempotent in $\Lambda[\mathcal{J}_x]$ and

$$(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x} = e_{\mathcal{J}_x}(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x})$$

is a finitely generated and projective $\Lambda[[G]]$-module.

We may now apply Proposition 5.1.3 to conclude that $\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})$ is an object in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda[[G]])$. □

REMARK 5.1.6. If F_∞/F is unramified over U and $f\colon U_{F_\infty} \to U$ is the corresponding system of coverings as in Section 3.3, then

$$\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}) = f_! f^* \eta_* \mathcal{T},$$

see also [**Wit14**, Rem. 6.10].

DEFINITION 5.1.7. Let Λ be an adic $\mathbb{Z}_\ell$-algebra, F_∞/F an admissible extension, x a closed point of X and $\mathcal{T}$ a finitely ramified representation of Gal_F over Λ.

(1) We say that $\mathcal{T}$ has projective stalks in x if $\mathcal{T}^{\mathcal{I}_x}$ is a finitely generated, projective Λ-module.
(2) We say that $\mathcal{T}$ has projective local cohomology in x if $\mathrm{H}^1(\mathcal{I}_x, \mathcal{T})$ is a finitely generated, projective Λ-module and ℓ is different from the characteristic of $k(x)$.
(3) We say that $\mathcal{T}$ has projective stalks in x over F_∞ if $\mathcal{T}^{\mathcal{K}_x}$ is a finitely generated, projective Λ-module.
(4) We say that $\mathcal{T}$ has projective local cohomology in x over F_∞ if $\mathrm{H}^1(\mathcal{K}_x, \mathcal{T})$ is a finitely generated, projective Λ-module and ℓ is different from the characteristic of $k(x)$.
(5) We say that $\mathcal{T}$ has ramification prime to ℓ in x if the image of $\mathcal{I}_x$ in the automorphism group of $\mathcal{T}$ has trivial ℓ-Sylow subgroups.

(6) We say that F_∞/F has ramification prime to ℓ in x if $\mathcal{J}_x$ has trivial ℓ-Sylow subgroups.
(7) We say that F_∞/F has non-torsion ramification if $\mathcal{J}_x$ contains an element of infinite order.

In particular, if $\mathcal{T}$ has projective stalks over U, then $\eta_*\mathcal{T}$ is an object of $\mathbf{PDG}^{\mathrm{cont}}(U,\Lambda)$.

Example 5.1.8. Let $x \in X$ be a closed point and write $p' > 0$ for the characteristic of $k(x)$.

(1) If $\mathcal{T}$ has ramification prime to ℓ in x and $\ell \neq p'$, then it also has projective local cohomology in x. If $\ell = p'$, then it has projective stalks in x.
(2) If $\mathcal{T}$ has projective local cohomology in x (in x over F_∞), then it also has projective stalks in x (in x over F_∞).
(3) Assume $\ell \neq p'$ and that Λ has small finitistic projective dimension 0, i. e. every finitely generated Λ-module of finitely generated projective dimension is projective. Then $\mathcal{T}$ has projective local cohomology in x if and only if it has projective stalks in x. For example, this is true if Λ is finite and commutative [**Bas60**, Thm. (Kaplansky)]. More generally, for any finite Λ, it is true precisely if the left annihilator of every proper right ideal of Λ is non-zero [**Bas60**, Thm. 6.3]. It is not true for
$$\Lambda = \begin{pmatrix} \mathbb{Z}/(\ell^2) & (\ell)/(\ell^2) \\ \mathbb{Z}/(\ell^2) & \mathbb{Z}/(\ell^2) \end{pmatrix}$$
[**KKS92**, Cor. 1.12].
(4) If Λ is noetherian of global dimension less or equal to 2, then $\mathcal{T}$ has projective stalks in all closed points x of X, as $\mathcal{T}^{\mathcal{I}_x}$ is the kernel of the continuous homomorphism of projective compact Λ-modules
$$\mathcal{T} \to \prod_{\sigma \in \mathcal{I}_x} \mathcal{T}, \qquad t \mapsto (t - \sigma t)_{\sigma \in \mathcal{I}_x}.$$
As the global dimension is assumed to be less or equal to 2, $\mathcal{T}^{\mathcal{I}_x}$ is projective as compact Λ-module. As Λ is noetherian, $\mathcal{T}^{\mathcal{I}_x}$ is finitely generated and therefore, also projective as abstract Λ-module. The same argument shows that $\mathcal{T}$ has projective stalks over F_∞ in all closed points x of X.
(5) Assume that $\mathcal{T}$ has projective stalks over F_∞ and F_∞/F has ramification prime to ℓ in x. Then $\mathcal{T}$ has projective stalks in x. Moreover, $\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}$ also has projective stalks in x. The same remains true if one replaces "projective stalks" by "projective local cohomology".
(6) It may happen that $\mathcal{T}$ has projective stalks, but does not have projective stalks over F_∞ in x. For example, $\mathcal{T}^{\mathcal{I}_x}$ can be trivial, while $\mathcal{T}^{\mathcal{K}_x}$ is a non-trivial Λ-module that is not projective.
(7) If $\ell \neq p'$, then F_∞/F has non-torsion ramification in x if and only if $\mathcal{J}_x$ is infinite. If F is a number field and $\ell = p'$, then F_∞/F always has non-torsion ramification in x. Indeed, F_∞/F contains the cyclotomic $\mathbb{Z}_\ell$-extension, which is ramified in x. In the equal characteristic function field case, it may happen that $\mathcal{J}_x$ is an infinite torsion group.

Remark 5.1.9. If Λ is a noetherian adic $\mathbb{Z}_\ell$-algebra of finite global dimension, then one can modify Definition 5.1.2 by choosing for each of the finitely many points x for which $\mathcal{T}^{\mathcal{I}_x}$ is not projective a resolution $P^\bullet$ of $\mathcal{T}^{\mathcal{I}_x}$ by finitely generated, projective Λ-modules and replacing the stalk of $\eta_*\mathcal{T}_I$ in x by $\Lambda/I \otimes_\Lambda P^\bullet$ for each open two-sided ideal I of Λ.

LEMMA 5.1.10. *Assume that F_∞/F has ramification prime to ℓ over U. Assume further that Λ and Λ' are two adic $\mathbb{Z}_\ell$-algebras and that M is a Λ'-$\Lambda[[G]]$-bimodule which is finitely generated and projective as Λ'-module. Write $\mathcal{M}$ for the finitely ramified representation of Gal_F over Λ' given by M, with $g \in \mathrm{Gal}_F$ acting by the inverse of its image in G.*

(1) *If $\mathcal{T}$ has projective local cohomology over F_∞ in all closed points of U, then $\mathcal{M} \otimes_\Lambda \mathcal{T}$ has projective local cohomology over U and*
$$\Psi_M(\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \to \eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})$$
is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda')$.

(2) *If $\mathcal{T}$ has ramification prime to ℓ over U, then $\mathcal{M} \otimes_\Lambda \mathcal{T}$ has ramification prime to ℓ over U and*
$$\Psi_M(\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \to \eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})$$
is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda')$.

(3) *If $\mathcal{T}$ has projective stalks over F_∞ in all closed points of U and M is projective as compact Λ^{op}-module, then $\mathcal{M} \otimes_\Lambda \mathcal{T}$ has projective stalks over U and the canonical morphism*
$$\Psi_M(\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \to \eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})$$
is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda')$.

PROOF. By Proposition 5.1.3 we need to prove that
$$M \otimes_{\Lambda[[G]]} (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x} = (\mathcal{M} \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x}$$
for all closed points $x \in U$. Since F_∞/F has ramification prime to ℓ in x, the ℓ-Sylow subgroup of $\mathcal{J}_x$ is trivial such that taking invariants under $\mathcal{J}_x$ is an exact functor on the category of compact $\mathbb{Z}_\ell[[\mathcal{J}_x]]$-modules. Moreover, $\mathcal{T}^{\mathcal{K}_x}$ is finitely generated and projective as Λ-module by assumption. Hence,
$$M \otimes_{\Lambda[[G]]} (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x})^{\mathcal{J}_x} = (\mathcal{M} \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x})^{\mathcal{J}_x}.$$

If M is projective as compact Λ^{op}-module, then taking the completed tensor product with $\mathcal{M}$ over Λ is an exact functor on compact Λ-modules. Moreover, the completed tensor product commutes with arbitrary direct products and agrees with the usual tensor product on finitely presented modules [**Wit13b**, Prop. 1.7, Prop. 1.14]. By taking the completed tensor product with $\mathcal{M}$ of the left exact sequence
$$0 \to \mathcal{T}^{\mathcal{K}_x} \to \mathcal{T} \xrightarrow{x \mapsto (x - \sigma x)_{\sigma \in \mathcal{K}_x}} \prod_{\sigma \in \mathcal{K}_x} \mathcal{T}$$
we obtain
$$\mathcal{M} \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x} \cong \mathcal{M} \hat{\otimes}_\Lambda \mathcal{T}^{\mathcal{K}_x} \cong (\mathcal{M} \otimes_\Lambda \mathcal{T})^{\mathcal{K}_x},$$
as desired.

If ℓ is different from the characteristic of F, the Tor spectral sequence for the derived tensor product of $\mathcal{M}$ with the cochain complex of the $\mathcal{K}_x$-module $\mathcal{T}$ gives us an exact sequence
$$0 \to \mathrm{Tor}_2^\Lambda(\mathcal{M}, \mathrm{H}^1(\mathcal{K}_x, \mathcal{T})) \to \mathcal{M} \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x} \to \mathrm{Tor}_1^\Lambda(\mathcal{M}, \mathrm{H}^1(\mathcal{K}_x, \mathcal{T})) \to 0$$
Hence, if $\mathrm{H}^1(\mathcal{K}_x, \mathcal{T})$ is finitely generated and projective as a Λ-module, then
$$\mathcal{M} \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x} \cong (\mathcal{M} \otimes_\Lambda \mathcal{T})^{\mathcal{K}_x}.$$

If $\mathcal{T}$ has ramification prime to ℓ, then one can replace $\mathcal{K}_x$ by its image in the automorphism group of $\mathcal{T}$. Since this group is of order prime to ℓ, the natural map
$$\mathcal{M} \otimes_\Lambda \mathcal{T}^{\mathcal{K}_x} \to (\mathcal{M} \otimes_\Lambda \mathcal{T})^{\mathcal{K}_x}$$
is again an isomorphism. This completes the proof of the lemma. $\square$

LEMMA 5.1.11. *Assume that F_∞/F has ramification prime to ℓ and that $\mathcal{T}$ has projective stalks over F_∞ in the closed point x of U. Write $i: x \to U$ for the closed immersion. Then*

$$\mathrm{R}\,\Gamma(x, i^*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$$

is in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

PROOF. Choose an open pro-ℓ-subgroup H' of H which is normal in G. By [**Wit14**, Prop. 4.8], it suffices to show that

$$\Psi_{\Lambda/\operatorname{Jac}(\Lambda)[[G/H']]}(\mathrm{R}\,\Gamma(x, i^*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})))$$

has finite cohomology groups.

The complex $\mathrm{R}\,\Gamma(x, i^*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ may be identified with the strictly perfect complex of $\Lambda[[G]]$-modules

$$C^\bullet\colon \quad (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x} \xrightarrow{\mathrm{id}-\mathfrak{F}_x} (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x}$$

sitting in degree 0 and 1. Let Z be the centre of $\Lambda/\operatorname{Jac}(\Lambda)$, which is a finite product of finite fields of characteristic ℓ. Consider

$$\begin{aligned} P &\coloneqq \Lambda/\operatorname{Jac}(\Lambda)[[G/H']] \otimes_{\Lambda[[G]]} (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}^{\mathcal{I}_x}) \\ &\cong (\Lambda/\operatorname{Jac}(\Lambda)[[G/H']]^\sharp \otimes_{\Lambda/\operatorname{Jac}(\Lambda)} \mathcal{T}^{\mathcal{K}_x}/\operatorname{Jac}(\Lambda)\mathcal{T}^{\mathcal{K}_x})^{\mathcal{J}_x} \end{aligned}$$

as finitely generated, projective $Z[[\Gamma]]$-module. Choose n large enough, such that $\mathfrak{F}_x^n$ operates trivially on the finite groups $\mathcal{J}_x$ and $\mathcal{T}^{\mathcal{K}_x}/\operatorname{Jac}(\Lambda)\mathcal{T}^{\mathcal{K}_x}$. Then

$$\mathrm{id} - \mathfrak{F}_x^n = (\mathrm{id} - \mathfrak{F}_x)(\sum_{s=0}^{n-1} \mathfrak{F}_x^s)$$

is an injective endomorphism of P. The same is then also true for $\mathrm{id} - \mathfrak{F}_x$. We conclude from the elementary divisor theorem that the cokernel of $\mathrm{id} - \mathfrak{F}_x$ is finite, as desired. ☐

In particular, we may extend our previous definition of non-commutative Euler factors introduced in Section 3.5.

DEFINITION 5.1.12. Assume that F_∞/F has ramification prime to ℓ and $\mathcal{T}$ has projective stalks over F_∞ in x. The non-commutative Euler factor of $\eta_*\mathcal{T}$ in $x \in U$ is the element

$$\mathcal{L}_{F_\infty/F}(x, \eta_*\mathcal{T}) \coloneqq [\mathrm{id} - \mathfrak{F}_x \circlearrowright (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x}]^{-1}$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$. If ℓ is invertible on U, the non-commutative dual Euler factor in $x \in U$ is the element

$$\mathcal{L}^{\circledast}_{F_\infty/F}(x, \eta_*\mathcal{T}) \coloneqq [\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowright (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x}]\mathcal{L}^{\circledast}_{F_\infty/F}(x, \beta_!\eta'_*\mathcal{T})^{-1}$$

with $\beta\colon U' \to U$ an open, dense subscheme not containing x such that F_∞/F is unramified over U' and $\eta'\colon \operatorname{Spec} F \to U'$ the generic point of U'.

For the dual non-commutative Euler factor, note that the complex

$$(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x} \to \mathcal{D}^0_{\hat{x}}(\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \xrightarrow{\mathrm{id}-\tau} \mathcal{D}^1_{\hat{x}}(\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$$

sitting in degrees 0, 1 and 2 is a strictly perfect complex of $\Lambda[[G]]$-modules weakly equivalent to

$$\mathrm{R}\,\Gamma(\hat{x}, \mathrm{R}\,i^!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \sim \mathrm{H}^1(\mathcal{I}_x, \Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})[-2]$$

for $i\colon x \to U$.

Propositions 3.5.2, 3.5.6, and Parts (2), (3) of Propositions 3.5.4, 3.5.8 extend verbatimly. Part (1) must be replaced by:

PROPOSITION 5.1.13. *Assume that F_∞/F has ramification prime to ℓ in x. Let Λ' be another adic $\mathbb{Z}_\ell$-algebra and let M be a Λ'-$\Lambda[[G]]$-bimodule, finitely generated and projective as Λ'-module. Write $\mathcal{M}$ for the finitely ramified Gal_F-representation given by M, with $g \in \mathrm{Gal}_F$ acting by the image of its inverse in G. If $\mathcal{T}$ has ramification prime to ℓ in x or if $\mathcal{T}$ has projective local cohomology over F_∞ in x or if M is projective as compact Λ-module, then*

$$\Psi_{M[[G]]^\delta}(\mathcal{L}_{F_\infty/F}(x, \eta_*\mathcal{T})) = \mathcal{L}_{F_\infty/F}(x, \eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T}))$$

and, if $\ell \neq p$,

$$\Psi_{M[[G]]^\delta}(\mathcal{L}^{\circledast}_{F_\infty/F}(x, \eta_*\mathcal{T})) = \mathcal{L}^{\circledast}_{F_\infty/F}(x, \eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})).$$

PROOF. This is an easy consequence of Lemma 5.1.10. $\square$

Proposition 3.5.9 does only extend under extra hypotheses.

PROPOSITION 5.1.14. *Assume $\ell \neq p$ and that F_∞/F has ramification prime to ℓ in x. Let further $\mathcal{O}_C$ denote the valuation ring of a finite extension C of $\mathbb{Q}_\ell$.*

(1) *If $\mathcal{T}$ has projective local cohomology over F_∞ in x, then*

$$\mathcal{L}_{F_\infty/F}(x, \eta_*\mathcal{T}^{*_\Lambda}(1))^{\circledast} = \mathcal{L}^{\circledast}_{F_\infty/F}(x, \eta_*\mathcal{T}).$$

If $\Lambda = \mathcal{O}_C$ and $F_\infty = F_{\mathrm{cyc}}$, then the same is true for any finitely ramified representation $\mathcal{T}$ over $\mathcal{O}_C$.

(2) *If $\mathcal{T}$ has ramification prime to ℓ in x, then*

$$\mathcal{L}^{\circledast}_{F_\infty/F}(x, \eta_*\mathcal{T}) = [-\mathfrak{F}_x \circlearrowleft \mathrm{R}(\hat{x}, i^*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(-1)))]^{-1}\mathcal{L}_{F_\infty/F}(x, \eta_*\mathcal{T}(-1))^{-1}$$

If $\Lambda = \mathcal{O}_C$ and $F_\infty = F_{\mathrm{cyc}}$, then the same is true if $\mathcal{T}$ is a finitely ramified representation over $\mathcal{O}_C$ such that the base change of $\mathcal{T}$ to C is a semi-simple $\mathrm{Gal}(\bar{F}_x/F_x)$-representation.

PROOF. If $\mathcal{T}$ has projective local cohomology over F_∞ in x, then $\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}$ has projective local cohomology in x and the explicit version of local duality from Lemma 3.2.1 shows that

$$^\sharp((\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda}(1))^{\mathcal{I}_x})^{*_{\Lambda[[G]]}} \cong \mathrm{H}^1(\mathcal{I}_x, \Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})$$

Hence,

$$\mathcal{L}_{F_\infty/F}(x, \eta_*\mathcal{T}^{*_\Lambda}(1))^{\circledast} = [\mathrm{id} - \mathfrak{F}_x^{-1} \circlearrowleft \mathrm{H}^1(\mathcal{I}_x, \Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})] = \mathcal{L}^{\circledast}_{F_\infty/F}(x, \eta_*\mathcal{T}).$$

If $\mathcal{T}$ has ramification prime to ℓ in x, then the differential in the complex

$$\mathcal{D}^0_{\hat{x}}(\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \to \mathcal{D}^1_{\hat{x}}(\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$$

is trivial, such that

$$\mathrm{H}^1(\mathcal{I}_x, \Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}) \cong \Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(-1).$$

Assume now that $\Lambda = \mathcal{O}_C$ and $F_\infty = F_{\mathrm{cyc}}$. Then

$$C \otimes_{\mathcal{O}_C} \left(\mathcal{T}^{*_{\mathcal{O}_C}}(1)^{\mathcal{I}_x}\right)^{*_{\mathcal{O}_C}} \cong C \otimes_{\mathcal{O}_C} \mathrm{H}^1(\mathcal{O}_C, \mathcal{T}).$$

If the base change of $\mathcal{T}$ to C is a semi-simple $\mathrm{Gal}(\bar{F}_x/F_x)$-representation, then

$$C \otimes_{\mathcal{O}_C} \mathrm{H}^1(\mathcal{O}_C, \mathcal{T}) \cong C \otimes_{\mathcal{O}_C} \mathcal{T}^{\mathcal{I}_x}(-1).$$

The elements

$$\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, \eta_*\mathcal{T}),$$
$$\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \eta_*\mathcal{T}^{*_{\mathcal{O}_C}}(1))^{\circledast},$$
$$[-\mathfrak{F}_x \circlearrowleft \mathrm{R}(\hat{x}, i^*\eta_*(\mathcal{O}_C[[\Gamma]]^\sharp \otimes_{\mathcal{O}_C} \mathcal{T}(-1)))]^{-1}\mathcal{L}_{F_\infty/F}(x, \eta_*\mathcal{T}(-1))^{-1}$$

are then all given by evaluating the reverse characteristic polynomial of $\mathfrak{F}_x^{-1}$ on $C \otimes_{\mathcal{O}_C} \mathrm{H}^1(\mathcal{I}_x, \mathcal{T})$ at the image γ_x of $\mathfrak{F}_x$ in Γ. $\square$

EXAMPLE 5.1.15. Assume that the image of Gal_F in $\mathrm{Gl}_2(\mathcal{O}_C)$ coincides with the image of $\mathrm{Gal}(\overline{F}_x/F_x)$ and is generated by

$$\tau \coloneqq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \varphi \coloneqq \begin{pmatrix} q_x^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

with τ generating the image of $\mathcal{I}_x$ and φ serving as lift of $\mathfrak{F}_x$. Write γ for the image of φ in $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. Then $\mathcal{T}$ has projective local cohomology in x, but the base change of $\mathcal{T}$ to C is not semi-simple. In this example,

$$\mathcal{L}^{\circledast}_{F_{\mathrm{cyc}}/F}(x, \eta_*\mathcal{T}) =$$
$$[1 - q_x\gamma \circlearrowleft \mathcal{O}_C[[\Gamma]]\,][\mathrm{id} - \varphi^{-1}\gamma \circlearrowleft \mathcal{O}_C[[\Gamma]]^2\,]^{-1}\left[\mathrm{id} - \left(\frac{\tau - 1}{\tau^{q_x} - 1}\right)\varphi^{-1}\gamma \circlearrowleft \mathcal{O}_C[[\Gamma]]^2\,\right]$$
$$= [1 - q_x^{-1}\gamma],$$

whereas

$$[-\mathfrak{F}_x \circlearrowleft \mathrm{R}(\hat{x}, i^*\eta_*(\mathcal{O}_C[[\Gamma]]^\sharp \otimes_{\mathcal{O}_C} \mathcal{T}(-1)))]^{-1}\mathcal{L}_{F_{\mathrm{cyc}}/F}(x, \eta_*\mathcal{T}(-1))^{-1} =$$
$$[-\gamma^{-1} \circlearrowleft \mathcal{O}_C[[\Gamma]]\,]^{-1}[1 - \gamma^{-1} \circlearrowleft \mathcal{O}_C[[\Gamma]]\,] = [1 - \gamma].$$

5.2. Main Conjectures for Galois Representations

From now on, we fix two open dense subschemes V and W of X such that $V \cup W = X$ and set

$$\Sigma \coloneqq X - W, \qquad \mathrm{T} \coloneqq X - V, \qquad U \coloneqq V \cap W.$$

We write

$$j\colon U \to V, \qquad k\colon U \to W$$

for the corresponding open immersions and

$$\eta\colon \operatorname{Spec} F \to U$$

for the inclusion of the generic point. We also fix a prime ℓ, an adic $\mathbb{Z}_\ell$-algebra Λ and a finitely ramified representation $\mathcal{T}$ of Gal_F over Λ.

PROPOSITION 5.2.1. *Let Λ be an adic $\mathbb{Z}_\ell$-algebra, $\mathcal{T}$ be a finitely ramified representation of* Gal_F *over Λ. Assume that $\mathcal{T}^{\mathcal{I}_x}$ is a finitely generated Λ-module for all $x \in U$. If $\ell = p$, assume that $V = X$. Then*

$$\mathrm{H}^s(V, j_!\eta_*\mathcal{T}) \cong \varprojlim_{J \in \mathfrak{I}_\Lambda} \mathrm{H}^s(V, j_!\eta_*\mathcal{T}/J\mathcal{T}).$$

for all $s \in \mathbb{Z}$.

PROOF. Let $(\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$ and $(\mathcal{C}_J)_{J\in\mathfrak{I}_\Lambda}$ denote the kernel and cokernel of the natural morphism of systems

$$(\eta_*\mathcal{T}_J)_{J\in\mathfrak{I}_\Lambda} \to (\eta_*\mathcal{T}/J\mathcal{T})_{J\in\mathfrak{I}_\Lambda}$$

of étale sheaves on U. The restriction of $(\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$ to the complement of

$$\Sigma \coloneqq \{x \in U \mid \mathcal{T}^{\mathcal{I}_x} \neq \mathcal{T}\}$$

in U is 0. For $x \in \Sigma$ the stalk of $\mathcal{K}_J$ in the geometric point $\hat{x}$ is a finite abelian group for each $J \in \mathfrak{I}_\Lambda$. From Proposition 5.1.3 we conclude

$$\mathcal{T}^{\mathcal{I}_x} = \varprojlim_{J\in\mathfrak{I}_\Lambda} (\eta_*\mathcal{T}_J)_{\hat{x}} = \varprojlim_{J\in\mathfrak{I}_\Lambda} (\eta_*\mathcal{T}/J\mathcal{T})_{\hat{x}}.$$

Hence, the projective limit of the system $((\mathcal{K}_J)_{\hat{x}})_{J\in\mathfrak{I}_\Lambda}$ is 0. It follows that the system must be Mittag-Leffler zero in the sense of [**Jan88**, Def. 1.10]: For each natural number n there exists a $m > n$ such that the transition maps

$$(\mathcal{K}_{\mathrm{Jac}(\Lambda)^m})_{\hat{x}} \to (\mathcal{K}_{\mathrm{Jac}(\Lambda)^n})_{\hat{x}}$$

is the zero map. We conclude that the system of sheaves $(\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$ are also Mittag-Leffler zero. The same remains true for $(j_!\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}$. Now [**Jan88**, Lem. 1.11] implies

$$\mathrm{H}^s(V,(j_!\mathcal{K}_J)_{J\in\mathfrak{I}_\Lambda}) = 0.$$

The same argumentation also shows

$$\mathrm{H}^s(V,(j_!\mathcal{C}_J)_{J\in\mathfrak{I}_\Lambda}) = 0.$$

Since the cohomology groups $\mathrm{H}^s(V,j_!\eta_*\mathcal{T}/J\mathcal{T})$ are finite if $\ell\neq p$ [**Mil06**, Rem. after Thm. II.3.1] or if $V = X$ [**Mil80**, Cor. VI.2.8] we conclude

$$\mathrm{H}^s(V,j_!\eta_*\mathcal{T}) = \varprojlim_{J\in\mathfrak{I}_\Lambda} \mathrm{H}^s(V,j_!\eta_*\mathcal{T}/J\mathcal{T}).$$

□

Fix an admissible extension F_∞/F with Galois group $G\cong H\rtimes\Gamma$.

Corollary 5.2.2. *Assume that $(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T})^{\mathcal{I}_x}$ is finitely generated for each closed point x in U. If $\ell = p$, assume $V = X$. Then*

$$\mathrm{H}^s(V,j_!\eta_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T})) = \varprojlim_{F\subset_f L\subset F_\infty} \mathrm{H}^s(V_L,j_{L!}\eta_{L*}\mathcal{T})$$

for each $s\in\mathbb{Z}$, where L runs through the finite Galois extensions of F inside F_∞.

Proof. By Proposition 5.2.1 we have

$$\mathrm{H}^s(V,j_!\eta_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T})) = \varprojlim_{F\subset_f L\subset F_\infty} \mathrm{H}^s(V,j_!\eta_*(\Lambda[\mathrm{Gal}(L/F)]^\sharp\otimes_\Lambda\mathcal{T})).$$

Let $f\colon V_L\to V$ denote the finite morphism of schemes corresponding to the finite extension L/F. Then

$$\mathrm{H}^s(V,j_!\eta_*(\Lambda[\mathrm{Gal}(L/F)]^\sharp\otimes_\Lambda\mathcal{T})) = \mathrm{H}^s(V,f_*j_{L!}\eta_{L*}\mathcal{T}) = \mathrm{H}^s(V_L,j_{L!}\eta_{L*}\mathcal{T})$$

by [**Mil80**, Cor. II.3.6]. □

Lemma 5.2.3. *Assume that $\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T}$ has projective stalks over U. If $\ell = p$, then we assume $V = X$. If $p = 0$ and $\ell = 2$, we assume that F has no real place.*

(1) *The complexes*

$$\mathrm{R}\Gamma_c(W,\mathrm{R}k_*\eta_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T}))\sim\mathrm{R}\Gamma(V,j_!\eta_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T}))$$

are objects of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$.

(2) *If $U\xrightarrow{j_1}U'\xrightarrow{j_2}V$ are open immersions such that F_∞/F has non-torsion ramification over the complement of U in U', then*

$$\mathrm{R}\Gamma(V,j_{2!}(j_1\circ\eta)_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T}))\to\mathrm{R}\Gamma(V,j_{U!}\eta_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T}))$$

is a weak equivalence in $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$.

The same is true for V replaced by $\overline{V}$ if $p>0$.

Proof. The first assertion follows from Proposition 5.1.5 and the fact that the derived section functors over V and $\overline{V}$ take objects of $\mathbf{PDG}^{\mathrm{cont}}(V,\Lambda[[G]])$ to objects of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$.

We prove the second assertion. By Proposition 5.1.5 we have

$$(j_1\circ\eta)_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T})_{\hat{x}} = 0$$

for each $x\in U'-U$. Hence, the canonical morphism

$$j_{1!}\eta_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T})\to(j_1\circ\eta)_*(\Lambda[[G]]^\sharp\otimes_\Lambda\mathcal{T})$$

is an isomorphism in $\mathbf{PDG}^{\mathrm{cont}}(U',\Lambda[[G]])$. The second assertion is an immediate consequence. □

REMARK 5.2.4. In particular, by using Lemma 5.2.3, we may exclude without loss of generality from U all points in which F_∞/F has non-torsion ramification. We will also neglect the remaining points in which F_∞/F does not have ramification prime to ℓ or $\mathcal{T}$ has no projective stalks over F_∞, but $\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}$ has projective stalks. These points may be considered as degenerate and it is not clear that their corresponding non-commutative Euler factors are well-behaved.

REMARK 5.2.5. The complexes $\mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ may be viewed as Selmer complexes in the sense of Nekovář [**Nek06**, §6], with unramified local conditions for each point x of U where $(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})$ is ramified, full local conditions in each point of $W - U$ and empty local conditions in each point not in W.

In the following, we choose an open dense subscheme $W' \subset W$ such that F_∞/F is unramified over $U' \coloneqq W' \cap U$. Set $\mathrm{T}' \coloneqq U - U'$. Write

$$\begin{aligned} \alpha&: W' \to W \\ \beta&: U' \to U \\ \eta'&: \operatorname{Spec} F \to U' \\ k'&: U' \to W' \\ \gamma&: \mathrm{T}' \to U \end{aligned}$$

for the inclusion maps.

THEOREM 5.2.6. *Assume that F is a function field of characteristic $p \neq \ell$. Further, assume that F_∞/F has ramification prime to ℓ and that $\mathcal{T}$ has projective stalks over F_∞ in all closed points of U. Then*

(1) *The complexes*

$$\mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \sim \mathrm{R}\,\Gamma(V, j_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$$

are in $\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]])$ and the endomorphisms

$$\mathrm{R}\,\Gamma_c(\overline{W}, \mathrm{R}\,k_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}} \mathrm{R}\,\Gamma_c(\overline{W}, \mathrm{R}\,k_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$$

$$\mathrm{R}\,\Gamma(\overline{V}, j_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \xrightarrow{\mathrm{id}-\mathfrak{F}_{\mathbb{F}}^{-1}} \mathrm{R}\,\Gamma(\overline{V}, j_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$$

are weak equivalences in $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$.

(2) *Set*

$$\begin{aligned} \mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}) &\coloneqq \mathcal{L}_{F_\infty/F}(W, \mathrm{R}\,k_*\eta_*\mathcal{T}) \\ &\coloneqq [\mathrm{id} - \mathfrak{F}_{\mathbb{F}} \circlearrowright \mathrm{R}\,\Gamma_c(\overline{W}, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))]^{-1}. \\ \mathcal{L}^{\circledast}_{F_\infty/F,\mathrm{T},\Sigma}(\mathcal{T}) &\coloneqq \mathcal{L}^{\circledast}_{F_\infty/F}(V, j_!\eta_*\mathcal{T}) \\ &\coloneqq [\mathrm{id} - \mathfrak{F}_{\mathbb{F}}^{-1} \circlearrowright \mathrm{R}\,\Gamma_c(\overline{V}, j_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))]. \end{aligned}$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$. Then

$$\begin{aligned} d\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}) &= -[\mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))], \\ d\mathcal{L}^{\circledast}_{F_\infty/F,\mathrm{T},\Sigma}(\mathcal{T}) &= [\mathrm{R}\,\Gamma(V, j_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))] \end{aligned}$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

PROOF. Consider the exact sequence

$$0 \to \mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*\beta_!\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \to \mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \to \mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*\gamma_*\gamma^*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \to 0.$$

Further, note that there are weak equivalences

$$\mathrm{R}\,\Gamma_c(W, \mathrm{R}\,k_*\beta_!\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \sim \mathrm{R}\,\Gamma_c(W', \mathrm{R}\,k'_*\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$$
$$\mathrm{R}\,\Gamma_c(W, k_*\gamma_*\gamma^*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \sim \mathrm{R}\,\Gamma(\Sigma', \gamma^*\eta'_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})).$$

Hence, the outer two complexes of the exact sequence are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ by Theorem 3.4.1 and by Lemma 5.1.11. We conclude that the complex in the middle is also in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. The rest is a consequence of the exact sequence (4.4.1) and the definition of d. □

THEOREM 5.2.7. *Assume that $\ell = p$, that F_∞/F has ramification prime to p and $\mathcal{T}$ has projective stalks over F_∞ in each closed point of U, and that F_∞/F and $\mathcal{T}$ have ramification prime to p in each point of* T. *Then*

(1) $\mathrm{R}\,\Gamma_c(W, k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ *is in* $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$ *and the endomorphism* $\mathrm{id} - \mathfrak{F}_{\mathbb{F}}$ *of* $\mathrm{R}\,\Gamma_c(\overline{W}, k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ *is a weak equivalence in* $w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$.
(2) *Set*

$$\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}) \coloneqq [\mathrm{id} - \mathfrak{F}_{\mathbb{F}} \circlearrowright \mathrm{R}\,\Gamma_c(\overline{W}, k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))]^{-1}$$
$$Q(\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}),1)\prod_{x\in\mathrm{T}}[\mathrm{id} - \mathfrak{F}_x q^{\deg(x)} \circlearrowright (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x}]$$

in $\mathrm{K}_1(\Lambda[[G]]_S)$. *Then*

$$d\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}) = -[\mathrm{R}\,\Gamma_c(W, k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))]$$

PROOF. If $\mathrm{T} = \varnothing$ and hence, $W = U$, then one proceeds exactly as in the proof of Theorem 5.2.6. For $\mathrm{T} \neq \varnothing$ it remains to notice that $\mathrm{id} - \mathfrak{F}_x q^{\deg(x)}$ is an automorphism of the finitely generated projective $\Lambda[[G]]$-module $(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})^{\mathcal{I}_x}$ such that its class lies in $\mathrm{K}_1(\Lambda[[G]]) \subset \mathrm{K}_1(\Lambda[[G]]_S)$ and hence, has trivial image under the boundary homomorphism d. □

If F is a totally real field, $\ell \neq 2$, and $\mathcal{T}$ is a finitely ramified representation of Gal_F over Λ, we say that $\mathcal{T}$ is *smooth at* ∞ if every complex conjugation in Gal_F operates trivially on $\mathcal{T}$. In particular, $\mathcal{T}$ is smooth at ∞ if and only if $\eta_*\mathcal{T}$ is smooth at ∞.

Let F_∞/F be a CM-admissible extension. We write as in Section 4.3

$$\mathcal{L}^{\varepsilon}_{F_\infty/F}(x, \eta_*\mathcal{T}(1+n)) \coloneqq \Psi_{e_\epsilon\Lambda[[G]]}(\mathcal{L}_{F_\infty/F}(x, \eta_*\mathcal{T}(1+n)))$$
$$\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(x, \eta_*\mathcal{T}(n)) \coloneqq \Psi_{e_\epsilon\Lambda[[G]]}(\mathcal{L}^{\circledast}_{F_\infty/F}(x, \eta_*\mathcal{T}(n)))$$

for any closed point $x \in U$, any integer n and $\varepsilon = +$ if n is even, $\varepsilon = -$ if n is odd. If F_∞/F has ramification prime to ℓ over U, ℓ is invertible on W, and $\mathcal{T}$ has projective stalks over F_∞ in all closed points of U, we set

$$\begin{aligned}\mathcal{L}^{\varepsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}(1+n)) &\coloneqq \mathcal{L}^{\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\eta_*\mathcal{T}(1+n))\\ &\coloneqq \mathcal{L}^{\varepsilon}_{F_\infty/F}(W', \mathrm{R}\,k'_*\eta'_*\mathcal{T}(1+n))\prod_{x\in\mathrm{T}'}\mathcal{L}^{\varepsilon}(x, \eta_*\mathcal{T}(1+n))\\ \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}(n)) &\coloneqq \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W, \mathrm{R}\,k_*\eta_*\mathcal{T}(n))\\ &\coloneqq \mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F}(W', \mathrm{R}\,k'_*\eta'_*\mathcal{T}(n))\prod_{x\in\mathrm{T}'}\mathcal{L}^{\circledast,\varepsilon}(x, \eta_*\mathcal{T}(n))\end{aligned}$$

THEOREM 5.2.8. *Assume that F_∞/F is a CM-admissible extension of a totally real field F and $\ell \neq 2$. Further, assume that F_∞/F has ramification prime to ℓ over U and that ℓ is invertible on W. Let $\mathcal{T}$ be a finitely ramified* Gal_F*-representation over Λ that is smooth at ∞ and has projective stalks over F_∞ in all closed points of U. Then*

(1) *The complexes*

$$e_\varepsilon \operatorname{R}\Gamma_{\mathrm{c}}(W, \operatorname{R}k_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(1+n))), \quad e_\varepsilon \operatorname{R}\Gamma(W, k_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(n)))$$

are in $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$.

(2) *We have*

$$d\mathcal{L}^{\varepsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}(1+n)) = -[\operatorname{R}\Gamma_{\mathrm{c}}(W, \operatorname{R}k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(1+n)))],$$
$$d\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}(n)) = [\operatorname{R}\Gamma(W, k_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(n)))]$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

Proof. For $\mathcal{L}^{\varepsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}(1+n))$ the argument is essentially the same as for Theorem 5.2.6. For $\mathcal{L}^{\circledast,\varepsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}(n))$ we use the exact sequence from Lemma 3.1.9. □

Note that in all three cases, one can also allow U to contain points x in which F_∞/F has non-torsion ramification. However, by Lemma 5.2.3, the corresponding non-commutative L-function $\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T})$ then agrees with the L-function $\mathcal{L}_{F_\infty/F,\Sigma\cup\Sigma_0,\mathrm{T}}(\mathcal{T})$ where the Euler factors in the set $\Sigma_0 \subset U$ of points in which F_∞/F has non-torsion ramification are removed.

Remark 5.2.9. Let Λ' be another adic $\mathbb{Z}_\ell$-algebra and let M be a Λ'-$\Lambda[[G]]$-bimodule which is finitely generated and projective as Λ'-module. Assume either that $\mathcal{T}$ has only ramification prime to ℓ over U or that $\mathcal{T}$ has projective local cohomology over F_∞ in all closed points of U or that M is projective as compact Λ^{op}-module. Then

$$\Psi_{M[[G]]^\delta}(\mathcal{L}^{\Box}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T})) = \mathcal{L}^{\Box}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{M} \otimes_\Lambda \mathcal{T})$$

for

$$\Box \in \{\varnothing, \circledast, \varepsilon, (\circledast, \varepsilon)\}$$

by Lemma 5.1.10. Furthermore, since $\Lambda[[G]]$ is projective as compact Λ^{op}-module for any profinite group G, the formulas (6), (8), and (9) of Corollary 4.2.9 remain valid.

5.3. Duality for Galois Representations

As before, we will write $\mathcal{H}om_{\mathbb{Z},U}(\mathcal{F}, \mathcal{G})$ for the sheaf of morphisms from $\mathcal{F}$ to $\mathcal{G}$ for any two étale sheaves $\mathcal{F}$, $\mathcal{G}$ of abelian groups on U. Write $\mathbb{G}_{\mathrm{m}U}$ for the étale sheaf corresponding to the multiplicative group on U. We set

$$\mathcal{D}_U(\mathcal{F}) = \operatorname{R}\mathcal{H}om_{\mathbb{Z},U}(\mathcal{F}, \mathbb{G}_{\mathrm{m}U}),$$

considered as an object in the derived category of étale sheaves of abelian groups on U. Further, we write

$$k' \colon W \to X$$

for the open immersion of W into X. Recall that $\mathcal{T}^\vee$ denotes the Pontryagin dual of $\mathcal{T}$.

Proposition 5.3.1. *Assume that* Λ *is a finite* $\mathbb{Z}_\ell$*-algebra with* ℓ *invertible on* W *and* $\mathcal{T}$ *be any finitely ramified* Gal_F*-representation over* Λ*. Then there exists a canonical isomorphism*

$$\mathcal{D}_X(k'_! \operatorname{R}k_*\eta_*\mathcal{T}) \cong \operatorname{R}k'_* k_!\eta_*\mathcal{T}^\vee(1)$$

in the derived category of complexes of étale sheaves of Λ^{op}*-modules on* X.

PROOF. By the adjunction formula for the pair $(k'_!, k'^*)$ we obtain an isomorphism

$$\mathcal{D}_X(k'_! \mathrm{R}\, k_* \eta_* \mathcal{T}) \cong \mathrm{R}\, k'_* \mathcal{D}_W(\mathrm{R}\, k_* \eta_* \mathcal{T})$$

in the derived category of complexes of étale sheaves of Λ-modules on X. Since ℓ is invertible on W, we have an isomorphism

$$\mathcal{D}_W(\mathrm{R}\, k_* \eta_* \mathcal{T}) \cong \mathrm{R}\, \mathcal{H}om_W(\mathrm{R}\, k_* \eta_* \mathcal{T}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$$

in the derived category of complexes of étale sheaves of Λ-modules on W. From the biduality theorem [**Del77**, Dualité, Thm. 1.4] and the adjunction formula for the pair $(k_!, k^*)$ we then obtain a natural isomorphism

$$k_! \mathcal{D}_U(\eta_* \mathcal{T}) \cong \mathcal{D}_W \mathcal{D}_W(k_! \mathcal{D}_U(\eta_* \mathcal{T})) \cong \mathcal{D}_W(\mathrm{R}\, k_* \mathcal{D}_U \mathcal{D}_U(\eta_* \mathcal{T})) \cong \mathcal{D}_W(\mathrm{R}\, k_* \eta_* \mathcal{T})$$

in the derived category of complexes of étale sheaves of Λ-modules on W. Finally we note that by [**Del77**, Dualité, Thm 1.3],

$$\mathcal{D}_U(\eta_* \mathcal{T}) \cong \eta_* \mathcal{T}^\vee(1)$$

if ℓ is invertible on U. □

COROLLARY 5.3.2. *Assume that Λ is a finite $\mathbb{Z}_\ell$-algebra with ℓ invertible on W. Let $\mathcal{T}$ be any finitely ramified Gal_F-representation over Λ.*

(1) *If F is a function field, there exists a canonical isomorphism*

$$\mathrm{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{R}\,\Gamma_c(\overline{W}, \mathrm{R}\, k_* \eta_* \mathcal{T}^\vee(1)), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \mathrm{R}\,\Gamma(\overline{W}, k_! \eta_* \mathcal{T})[2]$$

in the derived category of complexes of Λ-modules. It is compatible with the operations of $\mathfrak{F}_{\mathbb{F}}$ on the left-hand complex and $\mathfrak{F}_{\mathbb{F}}^{-1}$ on the righthand complex.

(2) *Assume that $\ell \neq 2$ or that F has no real places. Then there exists a canonical isomorphism*

$$\mathrm{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{R}\,\Gamma_c(W, \mathrm{R}\, k_* \eta_* \mathcal{T}^\vee(1)), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \mathrm{R}\,\Gamma(W, k_! \eta_* \mathcal{T})[3]$$

in the derived category of complexes of Λ-modules.

PROOF. Combine Prop 5.3.1 with Poincaré duality [**Del77**, Dualité, Thm. 2.2] and Artin-Verdier duality [**Mil06**, Prop. II.3.1], respectively. □

Let now Λ be a general adic $\mathbb{Z}_\ell$-algebra. In the following two corollaries, we consider a complex $P^\bullet = (P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$ as objects of the derived category of complexes of Λ-modules by passing to the projective limit

$$\varprojlim_{I \in \mathfrak{I}_\Lambda} P_I^\bullet.$$

We recall that the projective limit is an exact functor by the construction of $\mathbf{PDG}^{\mathrm{cont}}(\Lambda)$.

COROLLARY 5.3.3. *Assume that Λ is an adic $\mathbb{Z}_\ell$-algebra with ℓ invertible on U and that $\mathcal{T}$ has projective local cohomology over U. Then $\mathcal{T}^{*_\Lambda}(1)$ has projective local cohomology over U. Furthermore:*

(1) *If F is a function field, there exists a canonical isomorphism*

$$\mathrm{R}\,\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{R}\,\Gamma_c(\overline{W}, \mathrm{R}\, k_* \eta_* \mathcal{T}^{*_\Lambda}(1)), \Lambda^{\mathrm{op}}) \cong \mathrm{R}\,\Gamma(\overline{W}, k_! \eta_* \mathcal{T})[2]$$

in the derived category of complexes of Λ-modules. It is compatible with the operations of $\mathfrak{F}_{\mathbb{F}}$ on the left-hand complex and $\mathfrak{F}_{\mathbb{F}}^{-1}$ on the righthand complex.

(2) *Assume that $\ell \neq 2$ or that F has no real places. Then there exists a canonical isomorphism*

$$\mathrm{R}\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* \eta_* \mathcal{T}^{*_\Lambda}(1)), \Lambda^{\mathrm{op}}) \cong \mathrm{R}\Gamma(W, k_! \eta_* \mathcal{T})[3]$$

in the derived category of complexes of Λ-modules.

Proof. From our hypothesis and Lemma 3.2.1 it follows easily that

$$\mathrm{H}^s(\mathcal{I}_x, \mathcal{T}^{*_\Lambda}(1)) \cong \mathrm{H}^{1-s}(\mathcal{I}_x, \mathcal{T})^{*_\Lambda}$$

is finitely generated and projective for $s \in \{0, 1\}$.

For any finitely generated, projective Λ-module P, we have

$$(\Lambda^\vee \otimes_\Lambda P)^\vee \cong \operatorname{Hom}_\Lambda(P, \Lambda)$$

by the adjunction formula for Hom and $\otimes$ and by recalling that every homomorphism from P to Λ is automatically continuous for the compact topology. Hence,

$$\mathrm{R}\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P^\bullet, \Lambda^{\mathrm{op}}) \cong \mathrm{R}\operatorname{Hom}_{\mathbb{Z}}((\Lambda^{\mathrm{op}})^\vee \otimes^{\mathbb{L}}_{\Lambda^{\mathrm{op}}} P^\bullet, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

for every perfect complex of Λ-modules $P^\bullet$. Further,

$$(\Lambda^{\mathrm{op}})^\vee \otimes^{\mathbb{L}}_{\Lambda^{\mathrm{op}}} \mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* \mathcal{F}^\bullet) \cong \varinjlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* (\Lambda/I^{\mathrm{op}})^\vee \otimes_{\Lambda/I^{\mathrm{op}}} \mathcal{F}^\bullet_I)$$

for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(U, \Lambda^{\mathrm{op}})$. Arguing as in Lemma 5.1.10, we further see that the natural morphism

$$\begin{gathered}\varinjlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* (\Lambda/I^{\mathrm{op}})^\vee \otimes_{\Lambda/I^{\mathrm{op}}} (\eta_* \mathcal{T}^{*_\Lambda}(1))_I) \\ \downarrow \\ \varinjlim_{I \in \mathfrak{I}_\Lambda} \mathrm{R}\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* \eta_* ((\Lambda/I^{\mathrm{op}})^\vee \otimes_{\Lambda/I} (\mathcal{T}/I\mathcal{T})^{*_{\Lambda/I}}(1)))\end{gathered}$$

is an isomorphism in the derived category of complexes of Λ^{op}-modules. The same is true for W replaced by $\overline{W}$. We now apply Corollary 5.3.2 to the Gal_F-representation $\mathcal{T}/I\mathcal{T}$ over Λ/I for each I in $\mathfrak{I}_\Lambda$. □

Remark 5.3.4. Even for finite Λ, Corollary 5.3.3 is wrong without the hypothesis that $\mathcal{T}$ has projective local cohomology over U. If one merely assumes that $\mathcal{T}$ and $\mathcal{T}^{*_\Lambda}(1)$ have projective stalks over U and F is a function field, then the cone of the natural duality morphism

$$\mathrm{R}\Gamma(\overline{W}, k_! \eta_* \mathcal{T})[2] \to \mathrm{R}\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{R}\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\, k_* \eta_* \mathcal{T}^{*_\Lambda}(1)), \Lambda^{\mathrm{op}})$$

is given by the complex

$$C^\bullet\colon \qquad \bigoplus_{x \in U^0} \mathrm{H}^1(\mathcal{I}_x, \mathcal{T}) \to \bigoplus_{x \in U^0} (\mathcal{T}^{*_\Lambda}(1)^{\mathcal{I}_x})^{*_\Lambda}$$

sitting in degrees -1 and 0, with

$$\begin{aligned}\mathrm{H}^{-1}(C^\bullet) &\cong \bigoplus_{x \in U^0} \operatorname{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{H}^1(\mathcal{I}_x, \mathcal{T}^{*_\Lambda}(1)), \Lambda^{\mathrm{op}}) \\ \mathrm{H}^0(C^\bullet) &\cong \bigoplus_{x \in U^0} \operatorname{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{H}^1(\mathcal{I}_x, \mathcal{T}^{*_\Lambda}(1)), \Lambda^{\mathrm{op}}).\end{aligned}$$

Moreover, if $\mathcal{T}$ has projective stalks in x, the same does not need to be true for $\mathcal{T}^{*_\Lambda}(1)$, and vice versa. For example, the dual of the representation $\mathcal{T}$ from Remark 5.1.4 satisfies $(\mathcal{T}^{*_\Lambda}(1))^{\mathcal{I}_x} = 0$ for the given x.

Corollary 5.3.5. *Assume that Λ is an adic $\mathbb{Z}_\ell$-algebra with ℓ invertible on U. Let F_∞/F be an admissible extension with ramification prime to ℓ over U and let $\mathcal{T}$ be a finitely ramified representation of Gal_F over Λ that has projective local cohomology over F_∞ in all closed points of U. Then:*

(1) *If F is a function field, there exists a canonical isomorphism*

$$
{}^{\sharp}\mathrm{R}\,\mathrm{Hom}_{\Lambda^{\mathrm{op}}[[G]]}(\mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_* \eta_* (\Lambda^{\mathrm{op}}[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T}^{*_\Lambda}(1))), \Lambda^{\mathrm{op}}[[G]]) \cong \mathrm{R}\,\Gamma(\overline{W}, k_! \eta_* \Lambda[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T})[2]
$$

in the derived category of complexes of $\Lambda[[G]]$-modules. It is compatible with the operations of $\mathfrak{F}_{\mathbb{F}}$ on the left-hand complex and $\mathfrak{F}_{\mathbb{F}}^{-1}$ on the righthand complex.

(2) *Assume that $\ell \neq 2$ or that F has no real places. Then there exists a canonical isomorphism*

$$
{}^{\sharp}\mathrm{R}\,\mathrm{Hom}_{\Lambda^{\mathrm{op}}[[G]]}(\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_* \eta_* (\Lambda^{\mathrm{op}}[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T}^{*_\Lambda}(1))), \Lambda^{\mathrm{op}}[[G]]) \cong \mathrm{R}\,\Gamma(W, k_! \eta_* \Lambda[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T})[3]
$$

in the derived category of complexes of $\Lambda[[G]]$-modules.

PROOF. Note that $\Lambda[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T}$ has projective local cohomology in all closed points of U and that

$$
{}^{\sharp}(\Lambda[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T})^{*_{\Lambda[[G]]}} \cong \Lambda^{\mathrm{op}}[[G]]^{\sharp} \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda}.
$$

Then apply Corollary 5.3.3. □

We obtain the following functional equation for $\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T})$ if F is a function field of characteristic $p \neq \ell$. Recall the global ε-factor from Definition 4.4.2. To ease notation, we set

$$
\begin{aligned}
\varepsilon_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}) &\coloneqq \varepsilon(W, \mathrm{R}\,k_* \eta_* \Lambda[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T}) \\
&= [-\mathfrak{F}_{\mathbb{F}} \circlearrowleft \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_* \eta_* \Lambda[[G]]^{\sharp} \otimes_{\Lambda} \mathcal{T})] \in \mathrm{K}_1(\Lambda[[G]]).
\end{aligned}
$$

THEOREM 5.3.6. *Assume that F is a function field and that $\ell \neq p$. Let F_∞/F be an admissible extension and $\mathcal{T}$ be a finitely ramified Gal_F-representation over Λ. Assume that F_∞/F has ramification prime to ℓ and $\mathcal{T}$ has projective local cohomology over F_∞ in all closed points of U. Then*

$$
(\mathcal{L}_{F_\infty/F,\mathrm{T},\Sigma}(\mathcal{T}^{*_\Lambda}(1)))^{\circledast} = \mathcal{L}^{\circledast}_{F_\infty/F,\mathrm{T},\Sigma}(\mathcal{T}) = \varepsilon_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T})^{-1} \mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T})^{-1}
$$

PROOF. Choose a strictly perfect complex $P^\bullet$ of $\Lambda^{\mathrm{op}}[[G]]$-modules, an endomorphism $f\colon P^\bullet \to P^\bullet$, and a weak equivalence

$$
\alpha\colon P^\bullet \to \mathrm{R}\,\Gamma(\overline{W}, k_! \eta_* \Lambda^{\mathrm{op}}[[G]]^{\sharp} \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda}(1))
$$

such that the diagram

$$
\begin{tikzcd}
P^\bullet \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}-f"] & \mathrm{R}\,\Gamma(\overline{W}, k_! \eta_* \Lambda^{\mathrm{op}}[[G]]^{\sharp} \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda}(1)) \arrow[d, "\mathrm{id}-\mathfrak{F}_{\mathbb{F}}"] \\
P^\bullet \arrow[r, "\alpha"] & \mathrm{R}\,\Gamma(\overline{W}, k_! \eta_* \Lambda^{\mathrm{op}}[[G]]^{\sharp} \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda}(1))
\end{tikzcd}
$$

commutes in the derived category of complexes of $\Lambda^{\mathrm{op}}[[G]]$-modules. In particular, the diagram commutes up to homotopy in the Waldhausen category of perfect complexes of $\Lambda^{\mathrm{op}}[[G]]$-modules. By [**Wit08**, Lem. 3.1.6], this implies

$$
[\mathrm{id} - f \circlearrowleft P^\bullet]^{-1} = \mathcal{L}_{F_\infty/F,\mathrm{T},\Sigma}(\mathcal{T}^{*_\Lambda}(1))
$$

in $\mathrm{K}_1(\Lambda^{\mathrm{op}}[[G]], S)$. Applying Corollary 5.3.5 to the representation $\mathcal{T}$ over Λ, we obtain a commutative diagram

$$\begin{tikzcd} {}^{\sharp}(P^{\bullet})^{*_{\Lambda^{\mathrm{op}}[[G]]}} \arrow[r, "\beta"] \arrow[d, "\mathrm{id}-f^{\circledast}"] & \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_*\eta_*\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})[2] \arrow[d, "\mathrm{id}-\mathfrak{F}_{\mathbb{F}}^{-1}"] \\ {}^{\sharp}(P^{\bullet})^{*_{\Lambda[[G]]}} \arrow[r, "\beta"] & \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_*\eta_*\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})[2] \end{tikzcd}$$

in the derived category of complexes of $\Lambda[[G]]$-modules. Hence,

$$\begin{aligned} \mathcal{L}_{F_\infty/F,\mathrm{T},\Sigma}(\mathcal{T}^{*_\Lambda}(1))^{\circledast} &= [\mathrm{id}-f^* \circlearrowleft {}^{\sharp}(P^{\bullet})^{*_{\Lambda^{\mathrm{op}}[[G]]}}] \\ &= [\mathrm{id}-\mathfrak{F}_{\mathbb{F}}^{-1} \circlearrowleft \mathrm{R}\,\Gamma_{\mathrm{c}}(\overline{W}, \mathrm{R}\,k_*\eta_*\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})] \\ &= \mathcal{L}^{\circledast}_{F_\infty/F,\mathrm{T},\Sigma}(\mathcal{T}) \\ &= \varepsilon_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T})^{-1}\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T})^{-1} \end{aligned}$$

in $\mathrm{K}_1(\Lambda[[G]], S)$ □

REMARK 5.3.7. In the situation of Theorem 5.2.8, assume that $\mathcal{T}$ has projective local cohomology over F_∞ in all closed points of U. Then

$$\mathcal{L}^{\epsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}^{*_\Lambda}(1-n))^{\circledast} = \mathcal{L}^{\circledast,\epsilon}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathcal{T}(n))$$

holds almost by definition: Using Proposition 5.1.14 one reduces to the case that $\mathcal{T}$ is unramified over U. In this situation, one can refer to Corollary 4.3.3. Similarly, both this formula and Theorem 5.3.6 also hold if $\Lambda = \mathcal{O}_C$ is the valuation ring of a finite extension $C/\mathbb{Q}_\ell$, $F_\infty = F_{\mathrm{cyc}}$, and $\mathcal{T}$ is any finitely ramified Gal_F-representation.

5.4. Calculation of the Cohomology

In this section, we will give a description of the cohomology groups of the complex $\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T}))$.

LEMMA 5.4.1. *Let F be a function field and F_∞/F an admissible extension with ramification prime to ℓ over U. We assume either $\ell \neq p$ or $W = U$. We further assume that $\mathcal{T}$ has projective stalks over F_∞ in each closed point of U. Then*

(1) $\mathrm{H}^s_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})) = 0$ *for* $s \notin \{1,2,3\}$.

(2)
$$\mathrm{H}^1_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})) = \begin{cases} \mathcal{T}^{\mathrm{Gal}_{F_\infty}} & \text{if } W = X \text{ and } H \text{ is finite,} \\ 0 & \text{else.} \end{cases}$$

(3)
$$\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})) = \mathrm{H}^1_{\mathrm{c}}(W_{F_\infty}, \mathrm{R}\,k_{F_\infty *}\eta_{F_\infty *}(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})).$$

If $\ell \neq p$, then

$$\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})) = \mathrm{H}^1(W_{F_\infty}, k_{F_\infty !}\eta_{F_\infty *}\mathcal{T}^{\vee}(1))^{\vee}.$$

(4)
$$\mathrm{H}^3_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T})) = \begin{cases} \mathcal{T}(-1)_{\mathrm{Gal}_{F_\infty}} & \text{if } \ell \neq p \text{ and } W = U, \\ 0 & \text{else.} \end{cases}$$

PROOF. In the view of Proposition 5.2.1 we may assume that Λ is a finite ring. We will first consider the case that H is finite. As $\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*(\Lambda[[G]]^{\sharp}\otimes_{\Lambda}\mathcal{T}))$

has S-torsion cohomology by Theorem 5.2.6 and Theorem 5.2.6, we conclude from Proposition 2.8.1 and Remark 3.1.4 that

$$\begin{aligned}\mathrm{H}^s_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) &\cong \mathrm{H}^{s-1}_{\mathrm{c}}(W_{F_\infty}, \mathrm{R}\,k_{F_\infty *}\eta_{F_\infty *}\mathcal{T})\\ &\cong \mathrm{H}^{s-1}(V_{F_\infty}, j_{F_\infty !}\eta_{F_\infty *}\mathcal{T})\\ &\cong \mathrm{H}^{s-1}(V_{\overline{\mathbb{F}}F_\infty}, j_{F_\infty !}\eta_{F_\infty *}\mathcal{T})^{\mathrm{Gal}(\overline{\mathbb{F}}F_\infty/F_\infty)}.\end{aligned}$$

From the fact that the cohomology of an étale sheaf of Λ-modules on the curve $V_{\overline{\mathbb{F}}F_\infty}$ over the algebraically closed field $\overline{\mathbb{F}}$ is concentrated in degrees 0 up to 2 if $\ell \neq p$ and $V = X$ and up to 1 if $\ell = p$ [**Mil80**, Cor. VI.2.5] or $V \neq X$ [**Mil80**, Rem. V.2.4] we deduce Assertion (1) and the second case of Assertion (4). Assertion (2) for H finite follows since

$$\mathrm{H}^0(V_{F_\infty}, j_{F_\infty !}\eta_{F_\infty *}\mathcal{T}) = \begin{cases}\mathcal{T}^{\mathrm{Gal}_{F_\infty}} & \text{if } U = V\\ 0 & \text{else.}\end{cases}$$

We now assume $\ell \neq p$. Assertion (3) is a consequence of Corollary 5.3.2. Moreover, this corollary implies

$$\begin{aligned}\mathrm{H}^3_{\mathrm{c}}(U, \eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) &= \mathrm{H}^0(U_{F_\infty}, \eta_{F_\infty *}\mathcal{T}^\vee(1))^\vee\\ &= ((\mathcal{T}^\vee(1))^{\mathrm{Gal}_{F_\infty}})^\vee = \mathcal{T}(-1)_{\mathrm{Gal}_{F_\infty}}.\end{aligned}$$

This proves Assertion (4) in the case $\ell \neq p$.

Finally, we use Corollary 5.2.2 to deduce the assertions for general H. In the case of Assertion (2) it remains to notice that, since $\mathcal{T}$ is finite, there exists a finite extension L/F_{cyc} inside F_∞ with $\mathcal{T} = \mathcal{T}^{\mathrm{Gal}_L}$ and such that $\mathrm{Gal}(F_\infty/L)$ is pro-ℓ. Hence, the norm map $N_{L''/L'}\colon \mathcal{T} \to \mathcal{T}$ is multiplication by a power of ℓ for $L \subset_f L' \subset_f L'' \subset F_\infty$. We conclude that

$$\mathrm{H}^1(V, \eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) = \varprojlim_{F_{\mathrm{cyc}} \subset_f L \subset F_\infty} \mathcal{T}^{\mathrm{Gal}_L} = 0$$

if H is infinite. $\square$

LEMMA 5.4.2. *Let F be a totally real number field and F_∞/F be a CM-admissible extension. Assume that F_∞/F has ramification prime to $\ell \neq 2$ and that $\mathcal{T}$ has projective stalks over F_∞ in all closed points of U. Assume moreover that ℓ is invertible on W and that $\mathcal{T}$ is smooth at ∞. Fix an integer n. Choose $\varepsilon = +$ if n is even and $\varepsilon = -$ if n is odd. If Conjecture 3.3.4 is valid, then:*

(1) $\mathrm{H}^s_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(1+n))) = \mathrm{H}^s(W, k_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(n))) = 0$ *for* $s \notin \{1,2,3\}$.

(2)

$$e_\varepsilon\,\mathrm{H}^1_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(1+n))) = 0,$$

$$e_\varepsilon\,\mathrm{H}^1(W, k_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(n))) = \begin{cases}\mathcal{T}(n)^{\mathrm{Gal}_{F_\infty}} & \textit{if } U = W \textit{ and } H \textit{ is finite,}\\ 0 & \textit{else.}\end{cases}$$

(3)

$$\begin{aligned}e_\varepsilon\,\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(1+n))) &= e_\varepsilon\,\mathrm{H}^1_{\mathrm{c}}(W_{F_\infty}, \mathrm{R}\,k_{F_\infty *}\eta_{F_\infty *}\mathcal{T}(1+n))\\ &= e_\varepsilon(\mathrm{H}^1(W_{F_\infty}, k_{F_\infty !}\eta_{F_\infty *}\mathcal{T}^\vee(-n)))^\vee\\ e_\varepsilon\,\mathrm{H}^2(W, k_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(n))) &= e_\varepsilon\,\mathrm{H}^1(W_{F_\infty}, k_{F_\infty !}\eta_{F_\infty *}\mathcal{T}(n))\\ &= e_\varepsilon\mathrm{H}^1_{\mathrm{c}}(W_{F_\infty}, \mathrm{R}\,k_{F_\infty *}\eta_{F_\infty *}\mathcal{T}^\vee(1-n))^\vee\end{aligned}$$

(4)

$$\mathrm{H}^3_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(1+n))) = \begin{cases} \mathcal{T}(n)_{\mathrm{Gal}_{F_\infty}} & \textit{if } U = W, \\ 0 & \textit{else,} \end{cases}$$

$$\mathrm{H}^3(W, k_!\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(n))) = 0.$$

PROOF. Since $\ell \neq 2$ and $W \neq X$, we have

$$\mathrm{H}^s(W, \mathcal{G}) = 0$$

for every ℓ-torsion sheaf $\mathcal{G}$ on W and $s > 2$. If $W \neq U$, then $V \neq X$ and the same argument shows

$$\mathrm{H}^s_{\mathrm{c}}(W, \mathrm{R}\,k_*\mathcal{F}) \cong \mathrm{H}^s(V, j_!\mathcal{F}) = 0$$

for every ℓ-torsion sheaf $\mathcal{F}$ on U and $s > 2$. If $W = U$, then

$$\mathrm{H}^s_{\mathrm{c}}(U, \mathcal{F}) = 0$$

for $s > 3$. The rest follows exactly as in Lemma 5.4.1. □

As we will explain in Chapter 6, the following three corollaries may be viewed as a generalisation of [**GP12**, Thm. 3.10] and [**GP15**, Thm. 4.6], respectively.

COROLLARY 5.4.3. *Let F be a function field of characteristic $p \neq \ell$. Assume*

(1) *F_∞/F has ramification prime to ℓ over U,*
(2) *$\mathcal{T}$ has projective local cohomology over F_∞ in all closed points of U,*
(3) *either $W \neq X$ or $(\mathcal{T}^{*_\Lambda})_{\mathrm{Gal}_{F_\infty}} = 0$, and*
(4) *either $W \neq U$ or $\mathcal{T}(-1)_{\mathrm{Gal}_{F_\infty}} = 0$.*

Then $\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ is finitely generated and projective as $\Lambda[[H]]$-module and $\mathrm{H}^2(W, k_!\eta_*(\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1))$ is finitely generated and projective as $\Lambda^{\mathrm{op}}[[H]]$-module. Moreover, we have

$$\begin{aligned}
[\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))] &= [\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))] \\
&= [{}^\sharp\mathrm{H}^2(W, k_!\eta_*(\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1))^{*_{\Lambda^{\mathrm{op}}[[H]]}}] \\
&= -[\mathrm{H}^2(W, k_!\eta_*(\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1))]^\circledast \\
&= -[\mathrm{R}\,\Gamma(W, k_!\eta_*(\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1))]^\circledast
\end{aligned}$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

PROOF. Note that $\mathcal{T}^{\mathrm{Gal}_{F_\infty}} \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}((\mathcal{T}^{*_\Lambda})_{\mathrm{Gal}_{F_\infty}}, \Lambda^{\mathrm{op}})$. According to Theorem 5.2.6 we may find strictly perfect complexes $P^\bullet$ and $Q^\bullet$ of $\Lambda[[G]]$-modules and $\Lambda[[H]]$-modules, respectively, which are weakly equivalent to the complex $\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$. By Lemma 5.4.1 and assumptions (3) and (4), $\mathrm{H}^2(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ is the only cohomology group of these complexes that does not vanish. Hence,

$$[\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))] = [\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))]$$

holds by Remark 2.5.11. Moreover, we may assume that $P^\bullet$ and $Q^\bullet$ are concentrated in degrees less or equal to 2.

Let M be a simple $\Lambda[[G]]^{\mathrm{op}}$-module. Then M is also simple as module over $(\Lambda[[G]]/\operatorname{Jac}(\Lambda[[G]]))^{\mathrm{op}}$. By Schur's lemma the endomorphism ring of M is division ring k. Since k is clearly finite, it is a field. Hence, we may consider M as k-$\Lambda[[G]]$-bimodule, which is finitely generated and projective as k-module. Write $\mathcal{M}$ for the corresponding Gal_F-representation over k. Under assumptions (1) and (2), the natural map

$$\Psi_M\,\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})) \xrightarrow{\sim} \mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T}))$$

is a weak equivalence by Lemma 5.1.10. If $W \neq X$ and hence, $V \neq U$, we have

$$\mathrm{H}^0_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})) = \mathrm{H}^0(V, j_!\eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})) = 0.$$

If $W = X$ and $(\mathcal{T}^{*_\Lambda})_{\mathrm{Gal}_{F_\infty}} = 0$, then

$$\begin{aligned}
\mathrm{H}^0_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})) &= \mathrm{H}^0(U, \eta_*(\mathcal{M} \otimes_\Lambda \mathcal{T})) \\
&= (\mathcal{M} \otimes_\Lambda \mathcal{T})^{\mathrm{Gal}_{F_\infty}} \\
&= (((\mathcal{M} \otimes_\Lambda \mathcal{T})^{*_k})_{\mathrm{Gal}_{F_\infty}})^{*_k} \\
&= ((\mathcal{M}^{*_k} \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})_{\mathrm{Gal}_{F_\infty}})^{*_k} \\
&= (\mathcal{M}^{*_k} \otimes_{\Lambda^{\mathrm{op}}} (\mathcal{T}^{*_\Lambda})_{\mathrm{Gal}_{F_\infty}})^{*_k} \\
&= 0.
\end{aligned}$$

In particular, the flat dimension of $\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ over $\Lambda[[G]]$ is less or equal to 1 in both cases, such that we may assume that $P^\bullet$ is concentrated in degrees 1 and 2. We may then apply Lemma 2.5.1 to conclude that $\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ is projective over $\Lambda[[H]]$. The same reasoning applies to $\mathcal{T}^{*_\Lambda}(1)$.

We now apply Corollary 5.3.5 to the Gal_F-representation $\mathcal{T}$ over Λ and obtain

$$[\mathrm{R}\,\Gamma(W, k_!\eta_*(\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1))] = -[\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))]^{\circledast}.$$

Using Corollary 2.7.6, we conclude

$$\begin{aligned}
[\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\,k_*\eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))] &= -[\mathrm{H}^2(W, k_!\eta_*(\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1))]^{\circledast} \\
&= [{}^\sharp\mathrm{H}^2(W, k_!\eta_*(\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1))^{*_{\Lambda^{\mathrm{op}}[[H]]}}],
\end{aligned}$$

as desired. □

Corollary 5.4.4. *Let F be a function field of characteristic $p = \ell$. Assume*

(1) *F_∞/F has ramification prime to p in all closed points of U,*
(2) *$\mathcal{T}$ has ramification prime to p in all closed points of U, and*
(3) *either $U \neq X$ or $(\mathcal{T}^{*_\Lambda})_{\mathrm{Gal}_{F_\infty}} = 0$.*

Then $\mathrm{H}^2_{\mathrm{c}}(U, \eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))$ is finitely generated and projective as $\Lambda[[H]]$-module. Moreover, we have

$$[\mathrm{R}\,\Gamma_{\mathrm{c}}(U, \eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))] = [\mathrm{H}^2_{\mathrm{c}}(U, \eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}))]$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

Proof. Use Theorem 5.2.7 and proceed as in the first part of the proof of Corollary 5.4.3. □

Corollary 5.4.5. *Fix a prime $\ell \neq 2$ and an integer n. Choose $\varepsilon = +$ if n is even and $\varepsilon = -$ if n is odd. Let F_∞/F be a CM-admissible extension of a totally real number field F. Assume that*

(1) *F_∞/F has ramification prime to ℓ in all closed points of U,*
(2) *$\mathcal{T}$ has projective local cohomology in all closed points of U and is smooth at ∞,*
(3) *ℓ is invertible on W,*
(4) *either $W \neq U$ or $\mathcal{T}(n)_{\mathrm{Gal}_{F_\infty}} = 0$, and*
(5) *Conjecture 3.3.4 is valid.*

Then $e_\varepsilon \,\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\, k_* \eta_* (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})(1+n))$ *is finitely generated and projective as* $\Lambda[[H]]$*-module and* $e_\varepsilon \,\mathrm{H}^2(W, k_! \eta_* (\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(-n))$ *is finitely generated and projective as* $\Lambda^{\mathrm{op}}[[H]]$*-module. Moreover, we have*

$$\begin{aligned}
&[e_\varepsilon \,\mathrm{R}\,\Gamma_{\mathrm{c}}(W, \mathrm{R}\, k_* \eta_* (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T})(1+n))] \\
&\qquad\qquad = [e_\varepsilon \,\mathrm{H}^2_{\mathrm{c}}(W, \mathrm{R}\, k_* \eta_* (\Lambda[[G]]^\sharp \otimes_\Lambda \mathcal{T}(1+n)))] \\
&\qquad\qquad = [e_\varepsilon \,{}^\sharp\mathrm{H}^2(W, k_! \eta_* (\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1+n))^{*_{\Lambda^{\mathrm{op}}[[H]]}}] \\
&\qquad\qquad = -[e_\varepsilon \,\mathrm{H}^2(W, k_! \eta_* (\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(1+n))]^{\circledast} \\
&\qquad\qquad = -[e_\varepsilon \,\mathrm{R}\,\Gamma(W, k_! \eta_* (\Lambda^{\mathrm{op}}[[G]]^\sharp \otimes_{\Lambda^{\mathrm{op}}} \mathcal{T}^{*_\Lambda})(-n))]^{\circledast}
\end{aligned}$$

in $\mathrm{K}_0(\Lambda[[G]], S)$.

PROOF. This is completely analogous to Corollary 5.4.3. □

5.5. The Main Conjecture for Selmer Groups

In this section we will assume that R is a local, commutative, and regular adic $\mathbb{Z}_\ell$-algebra. Further, we assume that F has no real primes if $\ell = 2$ and that ℓ is different from the characteristic p of F if F is a function field. We fix an open, dense subscheme U of X with complement Σ and write

$$k\colon U \to X \leftarrow \Sigma\colon i$$

for the associated immersions. Let $\mathcal{T}$ be a finitely ramified representation of Gal_F over R. For $F \subset L \subset \overline{F}$ we may define a Selmer group for $\mathcal{T}$. If L/F is a finite extension, then

$$\mathrm{Sel}_\Sigma(L, \mathcal{T}^\vee(1)) \coloneqq \ker\left(\mathrm{H}^1(\mathrm{Gal}_L, \mathcal{T}^\vee(1)) \to \bigoplus_{x \in U_L^0} \mathrm{H}^1(\mathcal{I}_x, \mathcal{T}^\vee(1))\right)$$

Otherwise, one defines

$$\mathrm{Sel}_\Sigma(L, \mathcal{T}^\vee(1)) \coloneqq \varinjlim_{L'} \mathrm{Sel}_\Sigma(L', \mathcal{T}^\vee(1)),$$

where L'/F runs through the finite subextensions of L/F. If $W = X$, then

$$\mathrm{Sel}(L, \mathcal{T}^\vee(1)) \coloneqq \mathrm{Sel}_\varnothing(L, \mathcal{T}^\vee(1))$$

corresponds to the Selmer group as in [**Gre89**, §5] with trivial submodules for the primes above ℓ in the number field case. In the function field case, $\mathrm{Sel}(L, \mathcal{T}^\vee(1))$ is the correct analogue of the classical Selmer group. If $\Sigma \neq \varnothing$, then $\mathrm{Sel}_\Sigma(L, \mathcal{T}^\vee(1))$ is referred to as *imprimitive Selmer group* by Greenberg.

LEMMA 5.5.1. *For any extension* L/F *inside* $\overline{F}$*,*

$$\mathrm{Sel}_\Sigma(L, \mathcal{T}^\vee(1)) = \mathrm{H}^1(U_L, \eta_{L*} \mathcal{T}^\vee(1)).$$

PROOF. Without loss of generality we assume that $L = F$. According to [**AGV72a**, VII, Cor. 5.8] we have for every integer s

$$\mathrm{H}^s(\mathrm{Gal}_F, \mathcal{T}^\vee(1)) = \varinjlim_{U'} \mathrm{H}^s(U', \eta_{U'*} \mathcal{T}^\vee(1)).$$

Here, U' runs through the open dense subschemes of U and

$$\eta_{U'}\colon \mathrm{Spec}\, F \to U'$$

denotes the immersion of the generic point. For any such U', the Leray spectral sequence shows

$$\mathrm{H}^1(U, \eta_* \mathcal{T}^\vee(1)) \cong \ker\left(\mathrm{H}^1(U', \eta_{U'*} \mathcal{T}^\vee(1)) \to \bigoplus_{x \in U - U'} \mathrm{H}^0(x, i_x^* \mathrm{R}^1 \eta_* \mathcal{T}^\vee(1))\right)$$

with $i_x : x \to U$ the immersion of the closed point x. Recall that for any discrete Gal_F-module $\mathcal{M}$, one has $(i_x^* \eta_* \mathcal{M})_{\hat{x}} = \mathcal{M}^{\mathcal{I}_x}$. By considering an injective resolution of $\mathcal{T}^\vee(1)$ we conclude

$$(i_x^* \mathrm{R}^1 \eta_* \mathcal{T}^\vee(1))_{\hat{x}} = \mathrm{H}^1(\mathcal{I}_x, \mathcal{T}^\vee(1)).$$

The equality in the lemma follows after passing to the direct limit over U'. □

Fix an admissible ℓ-adic Lie extension F_∞/F, such that the Galois group G is an ℓ-adic Lie group. We set

$$\mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T}) \coloneqq \mathrm{Sel}_\Sigma(F_\infty, \mathcal{T}^\vee(1))^\vee.$$

Lemma 5.5.2. *Assume that ℓ is invertible on U. Then*

$$\mathrm{H}^2_\mathrm{c}(U, \eta_*(R[[G]]^\sharp \otimes_R \mathcal{T})) \cong \mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T})$$

Proof. Note that $\mathcal{T}[[G]]$ is a noetherian ring. Hence, $\mathcal{T}^{\mathcal{I}_x}$ is finitely generated for all closed points $x \in X$. According to Proposition 5.2.1 and Corollary 5.3.2, we have

$$\mathrm{H}^2_\mathrm{c}(U, \eta_*(R[[G]]^\sharp \otimes_R \mathcal{T})) \cong \varprojlim_{I \in \mathfrak{I}_{R[[G]]}} \mathrm{H}^2_\mathrm{c}(U, \eta_*(R[[G]]^\sharp / I \otimes_R \mathcal{T}))$$

$$\cong \varprojlim_{I \in \mathfrak{I}_{R[[G]]}} \mathrm{H}^1(U, \eta_*(R[[G]]^\sharp / I \otimes_R \mathcal{T}^\vee(1)))^\vee$$

By [**AGV72a**, VII, Prop. 3.3], the étale cohomology of U commutes with direct limits, such that

$$\begin{aligned}\varprojlim_{I \in \mathfrak{I}_{R[[G]]}} \mathrm{H}^1(U, \eta_*(R[[G]]^\sharp / I \otimes_R \mathcal{T}^\vee(1)))^\vee &\cong \mathrm{H}^1(U, \eta_*(R[[G]] \otimes_R \mathcal{T})^\vee)^\vee \\ &\cong \mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T})\end{aligned}$$

by Lemma 5.5.1. □

We may thus deduce the following reformulation of the non-commutative main conjecture in terms of the $R[[G]]$-module $\mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T})$.

Corollary 5.5.3. *Let F_∞/F be an admissible ℓ-adic Lie extension of a function field F of characteristic different from ℓ. Assume that G has no element of order ℓ. Let $\Sigma_0 \subset U$ denote the set of points over which F_∞/F has non-torsion ramification and assume that $\mathcal{T}$ is a finitely ramified Gal_F-representation over $\mathcal{T}$ that has projective stalks over F_∞ in all closed points of $U - \Sigma_0$. Then*

(1) *$\mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T})$ is in $\mathbf{N}_H(R[[G]])$.*
(2) *In $\mathrm{K}_0(R[[G]], S)$ we have*

$$d\mathcal{L}_{F_\infty/F,\Sigma\cup\Sigma_0,\varnothing}(\mathcal{T}) = -[\mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T})] + [\mathcal{T}(-1)_{\mathrm{Gal}_{F_\infty}}] + \begin{cases}[\mathcal{T}^{\mathrm{Gal}_{F_\infty}}] & \text{if } \Sigma = \varnothing \text{ and } H \text{ is finite,} \\ 0 & \text{else.}\end{cases}$$

Proof. Note that

$$\mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T}) = \mathcal{X}_{F_\infty/F,\Sigma\cup\Sigma_0}(\mathcal{T})$$

by Lemma 5.2.3. Moreover, since G has no element of order ℓ, F_∞/F has ramification prime to ℓ over $U - \Sigma_0$. The rest is a direct consequence of (2.10.1), Theorem 5.2.6, and Lemma 5.4.1. □

Without the extra interpretation of the cohomology group, we obtain from Theorem 5.2.7:

Corollary 5.5.4. *Let F be a function field of characteristic p and F_∞/F be an admissible p-adic Lie extension. Assume that G has no element of order p. Let $\Sigma_0 \subset U$ denote the set of points over which F_∞/F has non-torsion ramification and assume that $\mathcal{T}$ is a finitely ramified Gal_F-representation over $\mathcal{T}$ that has projective stalks over F_∞ in all closed points of $U - \Sigma_0$. Then*

(1) $\mathrm{H}^2_{\mathrm{c}}(U, \eta_*(R[[G]]^\sharp \otimes_R \mathcal{T}))$ *is in* $\mathbf{N}_H(R[[G]])$.
(2) *In* $\mathrm{K}_0(R[[G]], S)$ *we have*

$$d\mathcal{L}_{F_\infty/F,\Sigma\cup\Sigma_0,\varnothing}(\mathcal{T}) = -[\mathrm{H}^2_{\mathrm{c}}(U, \eta_*(R[[G]]^\sharp \otimes_R \mathcal{T}))] + \begin{cases} [\mathcal{T}^{\mathrm{Gal}_{F_\infty}}] & \text{if } \Sigma = \varnothing \text{ and } H \text{ is finite,} \\ 0 & \text{else.} \end{cases}$$

Remark 5.5.5.

(1) We recall from Example 5.1.8 that if R has global dimension less or equal to 2, for example $R = \mathbb{Z}_\ell$ or $R = \mathbb{Z}_\ell[[t]]$, then $\mathcal{T}$ has automatically projective stalks over F_∞ in every closed point of X. We may choose Σ to be empty in this case.
(2) If G satisfies the premisses of Proposition 2.10.2, then $[\mathcal{T}(-1)_{\mathrm{Gal}_{F_\infty}}] = 0$ in $\mathrm{K}_0(R[[G]], S)$.

As a special case of Corollary 5.5.3, we can deduce a non-commutative function field analogue of the most classical formulation of the Iwasawa main conjecture. We fix an admissible ℓ-adic Lie extension F_∞/F with Galois group $G = H \rtimes \Gamma$. Further, we will write Σ_0 for the closed subscheme of U where F_∞/F has non-torsion ramification.

Corollary 5.5.6. *Let F be a function field of characteristic different from ℓ. Assume that G does not contain any element of order ℓ. Let M be the maximal abelian ℓ-extension of F_∞ which is unramified outside Σ. Then*

(1) $\mathrm{Gal}(M/F_\infty)$ *is in* $\mathbf{N}_H(\mathbb{Z}_\ell[[G]])$ *and*

$$d\mathcal{L}_{F_\infty/F,\Sigma\cup\Sigma_0,\varnothing}(\mathbb{Z}_\ell(1)) = -[\mathrm{Gal}(M/F_\infty)] + [\mathbb{Z}_\ell] + \begin{cases} [\mathbb{Z}_\ell(1)] & \text{if } \Sigma = \varnothing,\ H \text{ is finite, and } \mu_\ell \subset F_\infty \\ 0 & \text{else.} \end{cases}$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], S)$

(2) *Let $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ be an Artin representation over the valuation ring $\mathcal{O}_C$ of a finite extension C of $\mathbb{Q}_\ell$ that factors through G. Then*

$$\Phi_\rho(\mathcal{L}_{F_\infty/F,\Sigma\cup\Sigma_0,\varnothing}(\mathbb{Z}_\ell(1))) = \mathcal{L}_{F_{\mathrm{cyc}}/F,\Sigma\cup\Sigma_0,\varnothing}(\rho(1)).$$

Proof. From Lemma 5.4.1 and from the equality

$$\mathrm{H}^1(U_{F_\infty}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathrm{Gal}(M/F_\infty)^\vee.$$

we deduce

$$\mathrm{H}^2_{\mathrm{c}}(U, \eta_*\mathbb{Z}_\ell[[G]]^\sharp(1)) = \mathcal{X}_{F_\infty/F,\Sigma}(\mathbb{Z}_\ell(1)) = \mathrm{Gal}(M/F_\infty).$$

We then apply Theorem 5.2.6 and Corollary 5.5.3. Finally, we remark that

$$\mathbb{Z}_\ell(1)^{\mathrm{Gal}_{F_\infty}} = 0$$

if F_∞ does not contain any ℓ-th root of unity. If F_∞ does contain an ℓ-th root of unity, then it also contains all ℓ^n-th roots of unity for any n, and therefore, $\mathbb{Z}_\ell(1)^{\mathrm{Gal}_{F_\infty}} = \mathbb{Z}_\ell(1)$ in this case. □

If G does contain elements of order ℓ, then Theorem 5.2.6 applied to $\mathbb{Z}_\ell(1)$ is still a sensible main conjecture if we assume that F_∞/F has ramification prime to ℓ over W; however, we can no longer replace the class of the complex

$$[\mathrm{R}\,\Gamma_c(U, \eta_*(\Lambda[[G]]^\sharp \otimes_\Lambda \mathbb{Z}_\ell(1)))] = -d\mathcal{L}_{F_\infty/F,\Sigma,\varnothing}(\mathbb{Z}_\ell(1))$$

by the classes of its cohomology modules. One may also apply Theorem 5.2.6 and Theorem 5.2.7 to $\mathbb{Z}_\ell$ resulting in a main conjecture for every ℓ. Main conjectures of this type have already been discussed in [**Bur11**].

Corollary 5.5.7. *Let F_∞/F be a CM-admissible ℓ-adic Lie extension of a totally real number field F. Assume that G has no element of order ℓ. Let $\Sigma_0 \subset U$ denote the set of points over which F_∞/F has non-torsion ramification and assume that $\mathcal{T}$ is a finitely ramified Gal_F-representation over $\mathcal{T}$ that has projective stalks over F_∞ in all closed points of $U - \Sigma_0$. Assume that $\mathcal{T}$ is smooth at ∞ and let $n \in \mathbb{Z}$. Choose $\varepsilon = +$ if n is even and $\varepsilon = -1$ if n is odd. If Conjecture 3.3.4 is valid, then*

(1) *$e_\varepsilon \mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T}(1+n))$ is in $\mathbf{N}_H(R[[G]])$.*
(2) *In $\mathrm{K}_0(R[[G]], S)$ we have*

$$d\mathcal{L}^\epsilon_{F_\infty/F,\Sigma\cup\Sigma_0,\varnothing}(\mathcal{T}(1+n)) = -[e_\varepsilon \mathcal{X}_{F_\infty/F,\Sigma}(\mathcal{T}(1+n))] + [\mathcal{T}(n)_{\mathrm{Gal}_{F_\infty}}]$$

Proof. Use Theorem 5.2.8 and Lemma 5.4.2. □

5.6. The Main Conjecture For Abelian Varieties

Assume that F is a function field of characteristic $\ell \neq p$. In this section we let A be an abelian variety over $\operatorname{Spec} F$. We continue to that U is an open dense subset of X with complement Σ (which may be empty). Our aim is to deduce a precise function field analogue of the Gl_2 main conjecture in [**CFK**$^+$**05**].

Let $\mathcal{O}_C$ be the valuation ring of a finite extension C of $\mathbb{Q}_\ell$ and ρ a finitely ramified representation of Gal_F over $\mathcal{O}_C$. The Σ-truncated L-function of A twisted by ρ is given by

$$L_\Sigma(A, \rho, t) \coloneqq \prod_{x \in U^0} \det(1 - \mathfrak{F}_x t^{\deg(x)} \circlearrowleft (\rho \otimes_{\mathbb{Z}_\ell} \mathrm{H}^1(A \times_{\operatorname{Spec} F} \operatorname{Spec} \overline{F}, \mathbb{Q}_\ell))^{\mathcal{I}_x})^{-1}.$$

If ρ is an Artin representation of Gal_F, then $L_\Sigma(A, \rho, q^{-s})$ is the Σ-truncated Hasse-Weil L-function of A twisted by ρ.

We will write $\check{A}$ for the dual abelian variety,

$$A(\overline{F})_n \coloneqq \ker A(\overline{F}) \xrightarrow{n} A(\overline{F})$$

for the group of n-torsion points and

$$\mathrm{T}_\ell A \coloneqq \varprojlim_k A(\overline{F})_{\ell^k}$$

for the ℓ-adic Tate module of A. It is well known that $\mathrm{T}_\ell A$ is a finitely ramified representation of Gal_F over $\mathbb{Z}_\ell$. Moreover, the argument of [**Sch82**, §1] shows that for any closed point $x \in X$

$$(\eta_*(\rho \otimes_{\mathbb{Z}_\ell} \mathrm{T}_\ell \check{A}(-1)))_x \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong (\rho \otimes_{\mathbb{Z}_\ell} \mathrm{H}^1(A \times_{\operatorname{Spec} F} \operatorname{Spec} \overline{F}, \mathbb{Q}_\ell))^{\mathcal{I}_x}$$

such that

$$L_\Sigma(A, \rho, q^{-1}t) = Z(U, \eta_* \mathrm{T}_\ell \check{A} \otimes_{\mathbb{Z}_\ell} \rho, t).$$

Recall that q denotes the number of elements in $\mathbb{F}$ and that $\gamma_{\mathbb{F}}$ is the image of the geometric Frobenius $\mathfrak{F}_{\mathbb{F}}$ in Γ.

As an immediate consequence of Theorem 5.2.6 we obtain:

COROLLARY 5.6.1. *Let F_∞/F be an admissible extension. Assume that F_∞/F has ramification prime to ℓ over U. Then:*

(1) *We have*

$$d\mathcal{L}_{F_\infty/F,\Sigma,\varnothing}(\mathrm{T}_\ell \check{A}) = -[\mathrm{R}\Gamma_c(U, \eta_*(\mathbb{Z}_\ell[[G]]^\sharp \otimes_{\mathbb{Z}_\ell} \mathrm{T}_\ell \check{A}))]$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], S)$.

(2) *Let $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ be a finitely ramified representation over the valuation ring $\mathcal{O}_C$ of a finite extension C of $\mathbb{Q}_\ell$ that factors through G. Then*

$$\Phi_\rho(\mathcal{L}_{F_\infty/F,\Sigma,\varnothing}(\mathrm{T}_\ell \check{A})) = L_\Sigma(A, \rho, q^{-1}\gamma_{\mathbb{F}}^{-1}).$$

For any extension L/F inside $\overline{F}$ we let

$$\mathrm{Sel}_\Sigma(L, A) \coloneqq \varinjlim_k \ker \mathrm{H}^1(\mathrm{Gal}_L, A(\overline{F})_{\ell^k}) \to \bigoplus_{x \in U_L^0} \mathrm{H}^1(\mathrm{Gal}_{F_x}, A(\overline{F}))$$

be the Σ-truncated Selmer group of A over L.

LEMMA 5.6.2. *For every admissible ℓ-adic Lie extension F_∞/F we have*

$$\mathrm{Sel}_\Sigma(F_\infty, A) \cong \mathrm{Sel}_\Sigma(F_\infty, \mathrm{T}_\ell(\check{A})^\vee(1))$$

PROOF. Let L be an extension of F and let L_x be the completion of L at $x \in U_L$. According to Greenberg's approximation theorem we have

$$\mathrm{H}^1(\mathrm{Gal}_{L_x}, A(\overline{F})) = \mathrm{H}^1(\mathrm{Gal}_{L_x}, A(\overline{L_x}))$$

[**Mil06**, Rem. I.3.10] for all finite extensions L/F. Since the points of the formal group of A form an open pro-p-subgroup of $A(L_x)$ we conclude from the Kummer sequence that

$$\mathrm{Sel}_\Sigma(L, A) = \varinjlim_k \ker \mathrm{H}^1(\mathrm{Gal}_L, A(\overline{F})_{\ell^k}) \to \bigoplus_{x \in U_L^0} \mathrm{H}^1(\mathrm{Gal}_{L_x}, A(\overline{F})_{\ell^k})$$

for all extensions L/F inside $\overline{F}$. If $F_{\mathrm{cyc}} \subset L$, then $\mathrm{Gal}(L_x^{\mathrm{nr}}/L_x)$ is a profinite group of order prime to ℓ and the Hochschild-Serre spectral sequence shows that

$$\mathrm{H}^1(\mathrm{Gal}_{L_x}, A(\overline{F})_{\ell^k}) \to \mathrm{H}^1(\mathcal{I}_x, A(\overline{F})_{\ell^k})$$

is an injection. Furthermore,

$$\mathrm{T}_\ell(\check{A})^\vee(1) = \varinjlim_k A(\overline{F})_{\ell^k}$$

[**Sch82**, §1] such that indeed $\mathrm{Sel}_\Sigma(F_\infty, A) \cong \mathrm{Sel}_\Sigma(F_\infty, \mathrm{T}_\ell(\check{A})^\vee(1))$. □

In particular, we deduce the following function field analogue of the Gl_2 main conjecture of [**CFK⁺05**] as a special case of Corollary 5.5.3.

COROLLARY 5.6.3. *Let F_∞/F be an admissible ℓ-adic Lie extension with Galois group G, and A an abelian variety over* $\operatorname{Spec} F$. *We assume that G does not contain any element of order ℓ and write Σ_0 for the set of points in W in which F_∞/F has non-torsion ramification. Then* $\mathrm{Sel}_\Sigma(F_\infty, A)^\vee$ *is in* $\mathbf{N}_H(\mathbb{Z}_\ell[[G]])$ *and*

$$\begin{aligned} d\mathcal{L}_{F_\infty/F,\Sigma\cup\Sigma_0,\varnothing}(\mathrm{T}_\ell(\check{A})) &= -[\mathrm{Sel}_\Sigma(F_\infty, A)^\vee] + [\mathrm{T}_\ell(\check{A})(-1)_{\mathrm{Gal}_{F_\infty}}] \\ &\quad + \begin{cases} [\mathrm{T}_\ell(\check{A})^{\mathrm{Gal}_{F_\infty}}] & \text{if } \Sigma = \varnothing \text{ and } H \text{ is finite,} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], S)$.

The terms $[\mathrm{T}_\ell(\check{A})(-1)_{\mathrm{Gal}_{F_\infty}}]$ and $[\mathrm{T}_\ell(\check{A})^{\mathrm{Gal}_{F_\infty}}]$ disappear in the following situation. Recall that by an old result of Grothendieck [**Oor73**, Thm. 1.1], an abelian variety over F is of CM-type over a $\overline{F}$ if and only if it is isogenous to an abelian variety over a finite field. Moreover, this is the case if the image of $\mathrm{Gal}_{F_{\mathrm{cyc}}}$ in the automorphism group of $\mathrm{T}_\ell(A)$ is finite [**Oor73**, Last step].

Proposition 5.6.4. *Let A be an abelian variety over* $\operatorname{Spec} F$ *of dimension $g \geq 1$ which is not of CM-type over $\overline{F}$. Let F_∞ be the extension of F obtained by adjoining the coordinates of all ℓ^n-torsion points of A. If $\ell > 8g^2 - 1$, then F_∞/F is an admissible ℓ-adic Lie extension,* $\mathrm{Gal}(F_\infty/F)$ *does not contain any element of order ℓ and*

$$d\mathcal{L}_{F_\infty/F,\Sigma,\varnothing}(\mathrm{T}_\ell(\check{A})) = -[\mathrm{Sel}_\Sigma(F_\infty, A)^\vee].$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]], S)$.

Proof. It is well known that the group $\mathrm{Gal}(F_\infty/F)$ is the image of Gal_F in $\mathrm{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\check{A}))$, that $\mathrm{T}_\ell(\check{A})$ is a free $\mathbb{Z}_\ell$-module of rank $2g$, and that Gal_F acts on the determinant of $\mathrm{T}_\ell(\check{A})$ via the cyclotomic character κ. This shows that F_∞/F is an admissible ℓ-adic Lie extension. Since $\ell - 1 > 2g$, the group $\mathrm{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\check{A}))$ does not contain any element of order ℓ. By a result of Zarhin [**Zar77**, §4], [**Zar14**, §6], the Lie algebra $L(G)$ of G is the direct product

$$L(G) = \mathfrak{g}^0 \times \mathfrak{c}$$

of a simi-simple Lie algebra $\mathfrak{g}^0$ of dimension less or equal to $4g^2 - 1$ over $\mathbb{Q}_\ell$ and a commutative Lie algebra $\mathfrak{c}$ of dimension 1. Since any finite extension of F has only one $\mathbb{Z}_\ell$-extension, $\mathfrak{g}^0$ necessarily coincides with $L(H)$. Since A is not of CM-type over $\overline{F}$, H is not finite and hence, $L(H)$ is non-trivial. In particular,

$$[\mathrm{T}_\ell(\check{A}(-1))] = 0$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]], S)$ by Corollary 2.10.3. □

Remark 5.6.5. With F_∞ as in Proposition 5.6.4, assume that $\ell > 2g - 1$, such that G has no elements of order ℓ.

(1) By the above result of Zarhin, one can always find a finite extension F'/F inside F_∞/F such that
$$\mathrm{Gal}(F_\infty/F') = \mathrm{Gal}(F_\infty/F'_{\mathrm{cyc}}) \times \mathrm{Gal}(F'_{\mathrm{cyc}}/F').$$
Hence, applying Proposition 2.10.2 to $N = \mathrm{Gal}(F'_{\mathrm{cyc}}/F')$, we conclude that
$$[\mathrm{T}_\ell(\check{A}(-1))] = 0$$
in $\mathrm{K}_0(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F')]], S)$.

(2) One may also try apply the criterion of [**FK06**, Prop. 4.3.17] to the representation $\mathrm{T}_\ell(\check{A}(-1))$. However, one of the requirements is that G has infinite intersection with the subgroup
$$\mathbb{Z}_\ell^\times \mathrm{id} \subset \mathrm{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\check{A})).$$
Different from the number field case, this condition is not always satisfied for abelian varieties over F. Zarhin constructs in [**Zar07**] for every odd $g > 1$ examples of abelian varieties of dimension g which are not of CM-type and such that G has finite intersection with $\mathbb{Z}_\ell^\times \mathrm{id}$ independent of the choice of ℓ.

(3) If $g = 1$, then one can always take $F' = F$. Indeed, $\mathrm{Gal}(F_\infty/F)$ must be open in $\mathrm{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\check{A})) = \mathrm{Gl}_2(\mathbb{Z}_\ell)$ and the intersection of $\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ with $\mathrm{Sl}_2(\mathbb{Z}_\ell)$ is open in $\mathrm{Sl}_2(\mathbb{Z}_\ell)$. Otherwise, $\mathrm{Gal}(F_\infty/F)$ would contain a commutative open subgroup by the above result of Zarhin, which is not possible since A is not of CM-type over $\overline{F}$ (This was also observed in the

thesis [**Sec06**], using a different argument). By the assumption on ℓ we may write

$$\mathrm{Gl}_2(\mathbb{Z}_\ell) = H' \times \mathbb{Z}_\ell$$

with H' not virtually solvable. So we may apply Proposition 2.10.2.

CHAPTER 6

Main Conjectures for Realisations of 1-Motives

In this chapter, we will clarify the relation of the main conjectures in Section 5.2 with the main conjecture for ℓ-adic realisations of Picard 1-motives over function fields considered in [**GP12**] and the main conjecture for ℓ-adic realisations of abstract 1-motives over number fields considered in [**GP15**].

In Section 6.1 we recall the notion of a Picard 1-motive and give a description of it in terms of étale cohomology. In Section 6.2, we consider the function field case and formulate a non-commutative generalisation of the main conjecture in [**GP12**] as a special case of the main conjecture for Galois representations considered in Section 5.2. Finally, in Section 6.3, we carry out the same program in the number field case.

6.1. Picard 1-Motives

We recall the notion of Picard 1-motives introduced by Deligne [**Del74**]. For this, we need some more notation. Let $\mathbb{G}_{\mathrm{m}Y}$ denote the group of units of a scheme Y, considered as a sheaf on the small étale site of Y. Let $i\colon Z \to Y$ be a closed immersion. Recalling that the stalk of $\mathbb{G}_{\mathrm{m}Y}$ in a geometric point of Y is given by the units of the strict henselisation of the local ring in this point [**Mil80**, Rem. II.2.9.(d)], we see that

$$\mathbb{G}_{\mathrm{m}Y} \to i_*\mathbb{G}_{\mathrm{m}Z}$$

is a surjection. We let $\mathbb{G}_{\mathrm{m}Y,Z}$ denote its kernel.

From now on, we assume that Y is a quasi-compact, excellent, noetherian, integral, normal scheme of dimension 1 with perfect residue fields at all closed points of Y and that Z is a finite subscheme of Y. We write K for the function field of Y. Let $\eta_*\mathbb{G}_{\mathrm{m}K}$ denote the étale sheaf of invertible rational functions on Y and

$$\mathcal{P}_Z \coloneqq \ker\left(\eta_*\mathbb{G}_{\mathrm{m}K} \to i_*i^*(\eta_*\mathbb{G}_{\mathrm{m}K}/\mathbb{G}_{\mathrm{m}Y,Z})\right)$$

its subsheaf of rational functions which are congruent to 1 modulo the effective divisor on Y corresponding to Z in the sense of [**Ser88**, Ch. III, §1]. For any locally closed subscheme Y' of Y we let $\mathcal{D}iv_{Y'}$ denote the étale sheaf on Y of divisors with support on Y'.

Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{G}_{\mathrm{m}Y,Z} \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}Y} \arrow[r] \arrow[d] & i_*\mathbb{G}_{\mathrm{m}Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{P}_Z \arrow[r] \arrow[d, "\mathrm{div}"] & \eta_*\mathbb{G}_{\mathrm{m}K} \arrow[r] \arrow[d] & i_*i^*(\eta_*\mathbb{G}_{\mathrm{m}K}/\mathbb{G}_{\mathrm{m}Y,Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{D}iv_{Y-Z} \arrow[r] \arrow[d] & \mathcal{D}iv_Y \arrow[r] \arrow[d] & \mathcal{D}iv_Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6.1.1}$$

of étale sheaves on Y. One checks easily by taking stalks that all the rows and columns are exact. By Hilbert 90 in the form of [**Mil80**, Prop. III.4.9], we have

$$\mathrm{H}^1(Y, i_*\mathbb{G}_{\mathrm{m}Z}) \cong \mathrm{H}^1(Z, \mathbb{G}_{\mathrm{m}Z}) = \mathrm{Pic}(Z) = 0.$$

Hence, the third column is exact even in the category of presheaves. Clearly, this is also true for the third row. The weak approximation theorem for K implies the exactness of the second row in the category of presheaves. Hence,

$$\mathrm{H}^1(Y, \mathcal{P}_Z) \subset \mathrm{H}^1(Y, \eta_*\mathbb{G}_{\mathrm{m}K})$$

and the group on the right-hand side is zero by Hilbert 90 and the Leray spectral sequence. In particular, we have for any open dense subscheme Y' of Y:

$$\mathrm{H}^1(Y', \mathbb{G}_{\mathrm{m}Y,Z}) = \mathcal{D}iv_{Y'-Z}(Y')/\{\mathrm{div}(f) \mid f \in \mathcal{P}_Z(Y')\}$$

The group

$$\mathrm{Pic}(Y,Z) \coloneqq \mathrm{H}^1(Y, \mathbb{G}_{\mathrm{m}Y,Z})$$

is usually called the Picard group of Y relative to the effective divisor corresponding to Z. If K is a global field, then it is also known as the ray class group of Y for the modulus Z.

We will now assume that Y is a smooth and proper curve over an algebraically closed field k of characteristic $p \geq 0$. Let $\mathcal{D}iv_Y^0$ denote the kernel of the degree map $\mathcal{D}iv_Y \to \mathbb{Z}$. Likewise, we write $\mathrm{Pic}^0(Y,Z)$ for the kernel of $\mathrm{Pic}(Y,Z) \to \mathbb{Z}$. It can be identified with k-valued points of the generalised Jacobian variety of Y with respect to Z [**Ser88**, Ch. V, Thm. 1].

Recall from [**Mil80**, Ex. III.1.9.(c)] that an étale sheaf $\mathcal{F}$ on Y is flabby if $\mathrm{H}^s(U, \mathcal{F}) = 0$ for all $s > 0$ and all étale schemes U of finite type over Y.

LEMMA 6.1.1. *Let k be a algebraically closed field, Y be a smooth and proper curve over k and $Z \subset X$ be a finite closed subscheme. The complex of étale sheaves*

$$\mathcal{P}_Z \to \mathcal{D}iv_{Y-Z}$$

is a flabby resolution of $\mathbb{G}_{\mathrm{m}Y,Z}$ on Y.

PROOF. Since Z is a scheme of finite type of dimension 0 over the algebraically closed field k, all étale sheaves on Z are flabby [**Mil80**, Rem. III.1.20.(b)]. Since i_* maps flabby sheaves to flabby sheaves by [**Mil80**, Lem. III.1.19], the sheaf $(i_*i^*(\eta_*\mathbb{G}_{\mathrm{m}K}/\mathbb{G}_{\mathrm{m}Y,Z}))$ is flabby. The sheaf $\eta_*\mathbb{G}_{\mathrm{m}K}$ is flabby by [**Mil80**, Ex. III.2.22.(d)]. As the second row of the diagram (6.1.1) is exact in the category of presheaves, $\mathcal{P}_Z$ must also be flabby. $\square$

Consider two closed subschemes Z_1 and Z_2 of Y with empty intersection. The Picard 1-motive for Z_1 and Z_2 is defined to be the complex of abelian groups

$$\mathcal{M}_{Z_1,Z_2}\colon \mathcal{Div}^0_{Z_1}(Y) \to \mathrm{Pic}^0(Y, Z_2)$$

concentrated in degrees 0 and 1 [**GP12**, Def. 2.3]. Its group of n-torsion points for a number $n > 0$ is given by

$$\mathcal{M}_{Z_1,Z_2}[n] \coloneqq \mathrm{H}^0(\mathcal{M}_{Z_1,Z_2} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n))$$

and its ℓ-adic Tate module for a prime number ℓ is given by

$$\mathrm{T}_\ell\, \mathcal{M}_{Z_1,Z_2} \coloneqq \varprojlim_{k>0} \mathcal{M}_{Z_1,Z_2}[\ell^k]$$

[**Del74**, §10.1.5].

LEMMA 6.1.2. *We have for all numbers $n > 0$*

$$\mathcal{M}_{Z_1,Z_2}[n] \cong \mathrm{H}^0(\mathrm{R}\Gamma(Y - Z_1, \mathbb{G}_{\mathrm{m}X,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n))$$

where $\mathrm{R}\Gamma(Y - Z_1, \mathbb{G}_{\mathrm{m}Y,Z_2})$ denotes the total derived section functor and $\otimes^{\mathbb{L}}_{\mathbb{Z}}$ denotes the total derived tensor product in the derived category of abelian groups.

PROOF. Consider the complexes

$$\begin{aligned} A^\bullet &\colon \mathcal{Div}_{Z_1}(Y) \to \mathrm{Pic}(Y, Z_2),\\ B^\bullet &\colon \mathcal{Div}_{Z_1}(Y) \oplus \mathcal{P}_{Z_2}(Y) \to \mathcal{Div}_{Y-Z_2}(Y),\end{aligned}$$

and

$$E^\bullet \coloneqq \begin{cases} \mathbb{Z}[-1] & \text{if } Z_1 = \varnothing,\\ 0 & \text{else,}\end{cases} \qquad F^\bullet \coloneqq \begin{cases} k^\times & \text{if } Z_2 = \varnothing,\\ 0 & \text{else.}\end{cases}$$

We obtain two obvious distinguished triangles

$$\mathcal{M}_{Z_1,Z_2} \to A^\bullet \to E^\bullet, \qquad F^\bullet \to B^\bullet \to A^\bullet.$$

Moreover, the obvious map from $B^\bullet$ to the complex

$$\mathcal{P}_{Z_2}(Y - Z_1) \to \mathcal{Div}_{Y-Z_2}(Y - Z_1)$$

is a quasi-isomorphism. The latter complex may be identified with the complex $\mathrm{R}\Gamma(Y - Z_1, \mathbb{G}_{\mathrm{m}Y,Z_2})$. For this, we note that

$$\mathcal{P}_{Z_2} \to \mathcal{Div}_{Y-Z_2}$$

is a flabby resolution of $\mathbb{G}_{\mathrm{m}Y,Z_2}$ by Lemma 6.1.1.

Since k is algebraically closed, the group $k^\times$ is divisible. Hence,

$$\mathrm{H}^0(F^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = \mathrm{H}^1(F^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = 0.$$

Since $\mathbb{Z}$ is free as $\mathbb{Z}$-module,

$$\mathrm{H}^{-1}(E^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = \mathrm{H}^0(E^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) = 0.$$

Hence,

$$\begin{aligned} \mathrm{H}^0(\mathrm{R}\Gamma(Y - Z_1, \mathbb{G}_{\mathrm{m}X,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n)) &\cong \mathrm{H}^0(B^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n))\\ &\cong \mathrm{H}^0(A^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n))\\ &\cong \mathcal{M}_{Z_1,Z_2}[n].\end{aligned}$$

□

Write $j_1\colon Y - Z_1 \to Y$, $j_2\colon Y - Z_2 \to Y$ for the open immersions of the complements of Z_1 and Z_2.

LEMMA 6.1.3. *If $p \nmid n$, then*

$$\mathrm{R}\,\Gamma(Y - Z_1, \mathbb{G}_{\mathrm{m}Y,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathrm{R}\,\Gamma(Y - Z_1, j_{2!}\mu_n)[1]$$

with μ_n the sheaf of n-th roots of unity on $Y - Z_2$. In particular,

$$\mathcal{M}_{Z_1,Z_2}[n] \cong \mathrm{H}^1(Y - Z_1, j_{2!}\mu_n) \cong \mathrm{H}^1(Y - Z_2, j_{1!}\mathbb{Z}/(n))^{\vee}$$

PROOF. The first statement follows from the Kummer sequences for $\mathbb{G}_{\mathrm{m}Y}$ and $\mathbb{G}_{\mathrm{m}Z_2}$ and the exactness of the sequence

$$0 \to j_{2!}\mu_n \to \mu_n \to i_2^*\mu_n \to 0$$

with $i_2\colon Z_2 \to Y$ denoting the closed immersion. The second statement follows from Lemma 6.1.2 and Corollary 5.3.2, which holds equally well over any algebraically closed field k of characteristic p. □

LEMMA 6.1.4. *Assume $p > 0$ and that Z_2 is reduced. For all numbers $r > 0$ the canonical morphism*

$$\mathrm{R}\,\Gamma(Y - Z_1, \mathbb{G}_{\mathrm{m}Y,Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^r) \to \mathrm{R}\,\Gamma(Y - Z_1, \mathbb{G}_{\mathrm{m}Y}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^r) \cong \mathrm{R}\,\Gamma(Y - Z_1, \nu_r^1)$$

is an isomorphism. Here, $\nu_r^1 \coloneqq \mathrm{W}_r\,\Omega^1_{Y,\log}$ is the logarithmic De Rham-Witt sheaf on Y. In particular,

$$\mathcal{M}_{Z_1,Z_2}[p^r] \cong \mathrm{H}^0(Y - Z_1, \nu_r^1) \cong \mathrm{H}^1_{\mathrm{c}}(Y - Z_1, \mathbb{Z}/(p^r))^{\vee}.$$

PROOF. Since we assume Z_2 to be reduced, we have

$$\mathrm{R}\,\Gamma(Z_2, \mathbb{G}_{\mathrm{m}Z_2}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^r) \cong 0.$$

This explains the first isomorphism in the first part of the statement. For the second isomorphism we may use [**Gei10**, Prop. 2.2] together with the identifications

$$\mathbb{Z}^c_Y \cong \mathbb{Z}_Y(1)[2] \cong \mathbb{G}_{\mathrm{m}Y}[1]$$

in the notation of *loc. cit.*. The duality statement

$$\mathrm{H}^0(Y - Z_1, \nu_r^1) \cong \mathrm{H}^1_{\mathrm{c}}(Y - Z_1, \mathbb{Z}/(p^r))^{\vee}$$

can be deduced from [**Gei10**, Thm. 4.1]:

$$\begin{aligned}
\mathrm{R}\,\Gamma(Y - Z_1, \nu_r^1) &\cong \mathrm{R}\,\Gamma(Y - Z_1, \mathbb{Z}^c_Y) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(p^r)[-1] \\
&\cong \mathrm{R}\,\mathrm{Hom}_{\mathbb{Z},Y-Z_1}(\mathbb{Z}/(p^r), \mathbb{Z}^c_Y) \\
&\cong \mathrm{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{R}\,\Gamma_{\mathrm{c}}(Y - Z_1, \mathbb{Z}/(p^r)), \mathbb{Z}) \\
&\cong \mathrm{R}\,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{R}\,\Gamma_{\mathrm{c}}(Y - Z_1, \mathbb{Z}/(p^r)), \mathbb{Q}/\mathbb{Z})[-1].
\end{aligned}$$

□

6.2. The Iwasawa Main Conjecture for Picard 1-Motives

We now return to our previous setting. Fix a prime number ℓ. Assume that F is a function field of characteristic p with field of constants $\mathbb{F}$. As before, we consider an admissible extension F_∞/F with Galois group $G = H \rtimes \Gamma$. We will assume that $H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ is finite. Let X be the proper smooth curve over $\mathbb{F}$ with function field F. Fix two open dense subschemes V and W of X such that $X = V \cup W$ and set $U \coloneqq V \cap W$. Further, we let

$$\Sigma \coloneqq X - W, \qquad \mathrm{T} \coloneqq X - V$$

denote the complements with their reduced closed subscheme structure. We write

$$\eta\colon \operatorname{Spec} F \to U$$

for the inclusion of the generic point and

$$k\colon U \to W, \qquad j\colon U \to V$$

for the open immersions of the subscheme U.

Set $\Upsilon \coloneqq \mathrm{Gal}(\overline{\mathbb{F}}F_\infty/F_\infty)$ and note that Υ is of order prime to ℓ. Furthermore, note that for a commutative adic $\mathbb{Z}_\ell$-algebra R, the Gal_F-representations R and (if $\ell \neq p$) $R(1)$ of Gal_F over R are unramified in all closed points of X. We recall from Definition 2.7.10 that for a $R[[G]]$-module M which is finitely generated and free as R-module, ${}^\sharp M^{*_R}$ denotes the R-dual considered as left $R[[G]]$-module.

Proposition 6.2.1. *Let F_∞/F be any admissible extension of a function field F of characteristic p. For an integer $r > 0$, we set $R \coloneqq \mathbb{Z}/(\ell^r)$.*

(1) *If $\ell \neq p$, then $\mathrm{H}^s(W, k_!\eta_* R[[G]]^\sharp)$ is a finitely generated free R-module for every $s \in \mathbb{Z}$ and*

$$\begin{aligned}(\mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty},\Sigma_{\overline{\mathbb{F}}F_\infty}}[\ell^r])^\Upsilon &\cong \mathrm{H}^2(V, j_!\eta_* R[[G]]^\sharp(1)) \\ &\cong {}^\sharp\mathrm{H}^2(W, k_!\eta_* R[[G]]^\sharp)^{*_R};\end{aligned}$$

(2) *if $\ell = p$, then $\mathrm{H}^s_{\mathrm{c}}(V, j_*\eta_* R[[G]]^\sharp)$ is a finitely generated free R-module for every $s \in \mathbb{Z}$ and*

$$(\mathcal{M}_{\Sigma_{\overline{\mathbb{F}}F_\infty},\mathrm{T}_{\overline{\mathbb{F}}F_\infty}}[p^r])^\Upsilon \cong {}^\sharp\mathrm{H}^2_{\mathrm{c}}(V, j_*\eta_* R[[G]]^\sharp)^{*_R}.$$

Proof. Since H is finite, we may find a finite extension F'/F inside F_∞/F such that $F_\infty = F'_{\mathrm{cyc}}$. In the case that $\ell = p$, we may further assume that $V = U$ and hence, $W = X$. By Corollary 5.2.2

$$\mathrm{H}^s(W, k_!\eta_* R[[G]]^\sharp) \cong \mathrm{H}^s(W_{F'}, k_{F'!}\eta_{F'*} R[[\mathrm{Gal}(F'_{\mathrm{cyc}}/F')]])$$

as R-modules. Hence, we may assume that $F_\infty = F_{\mathrm{cyc}}$. In particular, F_{cyc}/F is unramified over all of X. From Lemma 5.4.1 we conclude that $\mathrm{H}^s(W, k_!\eta_* R[[\Gamma]]^\sharp)$ is a finitely generated and free R-module for $s \neq 2$. Moreover, since R-duality and Pontryagin duality agrees for all finitely generated R-modules, we conclude

$${}^\sharp\mathrm{H}^2(W, k_!\eta_* R[[G]]^\sharp)^{*_R} \cong (\mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty},\Sigma_{\overline{\mathbb{F}}F_\infty}}[\ell^n])^\Upsilon$$

from Lemma 6.1.3 and Lemma 6.1.4. It remains to show the freeness for $s = 2$.

Assume that $W \neq U$. If $\ell = p$, we may apply Corollary 5.4.4. If $\ell \neq p$ and $V \neq U$, we may apply Corollary 5.4.3. We may proceed in the same way if $V = U$ and F_{cyc} does not contain μ_ℓ. In this case, the image of $\mathrm{Gal}_{F_{\mathrm{cyc}}}$ in the automorphism group of $R(-1)$ is a non-trivial group of order prime to ℓ, such that $R(-1)_{\mathrm{Gal}_{F_{\mathrm{cyc}}}} = 0$.

To settle the remaining case, assume that $\mu_\ell \subset F_{\mathrm{cyc}}$ and that the complement of U in V consists of a single point x that does not split in F_{cyc}/F. Consider the exact cohomology sequence

$$\to \mathrm{H}^s(V_{F_{\mathrm{cyc}}}, j_{F_{\mathrm{cyc}}!}\eta_{F_{\mathrm{cyc}}*} R(1)) \to \mathrm{H}^s(V_{F_{\mathrm{cyc}}}, j_{F_{\mathrm{cyc}}*}\eta_{F_{\mathrm{cyc}}*} R(1)) \to \mathrm{H}^s(x_{F_{\mathrm{cyc}}}, R(1)) \to$$

Since

$$\begin{aligned}\mathrm{H}^0(V_{F_{\mathrm{cyc}}}, j_{F_{\mathrm{cyc}}*}\eta_{F_{\mathrm{cyc}}*} R(1)) \cong \mathrm{H}^0(x_{F_{\mathrm{cyc}}}, R(1)) &\cong R(1), \\ \mathrm{H}^s(x_{F_{\mathrm{cyc}}}, R(1)) = 0 \qquad & \text{for } s > 0,\end{aligned}$$

we conclude that

$$\mathrm{H}^1(V_{F_{\mathrm{cyc}}}, j_{F_{\mathrm{cyc}}!}\eta_{F_{\mathrm{cyc}}*} R(1)) \cong \mathrm{H}^1(V_{F_{\mathrm{cyc}}}, j_{F_{\mathrm{cyc}}*}\eta_{F_{\mathrm{cyc}}*} R(1)).$$

Taking the Pontryagin dual, we conclude from Lemma 5.4.1 that

$$\mathrm{H}^2_{\mathrm{c}}(V, j_*\eta_* R[[\Gamma]]^\sharp) \cong \mathrm{H}^2(W, k_!\eta_* R[[\Gamma]]^\sharp)$$

is still finitely generated and free as R-module.

Similarly, for all ℓ, if the complement of U in W consists of a single point x that does not split in F_{cyc}/F, then

$$\begin{aligned}\mathrm{H}^2(W, k_*\eta_* R[[G]]^\sharp) &\cong \mathrm{H}^1(W_{F_{\mathrm{cyc}}}, k_{F_{\mathrm{cyc}}*}\eta_{F_{\mathrm{cyc}}*}R) \\ &\cong \mathrm{H}^1(W_{F_{\mathrm{cyc}}}, k_{F_{\mathrm{cyc}}!}\eta_{F_{\mathrm{cyc}}*}R) \\ &\cong \mathrm{H}^2(W, k_!\eta_* R[[G]]^\sharp)\end{aligned}$$

is finitely generated and free as R-module. This settles the case $W = U$. □

Corollary 6.2.2. *Let F_∞/F be any admissible extension of a function field F of characteristic p.*

(1) *If $\ell \neq p$, then $\mathrm{H}^s(W, k_!\eta_*\mathbb{Z}_\ell[[G]]^\sharp)$ is a finitely generated free $\mathbb{Z}_\ell$-module for every $s \in \mathbb{Z}$ and*

$$\begin{aligned}(\mathrm{T}_\ell \mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty}, \Sigma_{\overline{\mathbb{F}}F_\infty}})^\Upsilon &\cong \mathrm{H}^2(V, j_!\eta_*\mathbb{Z}_\ell[[G]]^\sharp(1)) \\ &\cong {}^\sharp\mathrm{H}^2(W, k_!\eta_*\mathbb{Z}_\ell[[G]]^\sharp)^{*_{\mathbb{Z}_\ell}};\end{aligned}$$

(2) *if $\ell = p$, then $\mathrm{H}^s_{\mathrm{c}}(V, j_*\eta_*\mathbb{Z}_\ell[[G]]^\sharp)$ is a finitely generated free $\mathbb{Z}_\ell$-module for every $s \in \mathbb{Z}$ and*

$$(\mathrm{T}_\ell \mathcal{M}_{\Sigma_{\overline{\mathbb{F}}F_\infty}, \mathrm{T}_{\overline{\mathbb{F}}F_\infty}})^\Upsilon \cong {}^\sharp\mathrm{H}^2_{\mathrm{c}}(V, j_*\eta_*\mathbb{Z}_\ell[[G]]^\sharp)^{*_{\mathbb{Z}_\ell}}.$$

Proof. Use Proposition 6.2.1 and pass to the inverse limit over $n \in \mathbb{N}$. □

Remark 6.2.3. Note that the image of Υ in $\mathrm{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell \mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty}, \Sigma_{\overline{\mathbb{F}}F_\infty}})$ is finite. Hence, we can always choose the admissible extension F_∞/F large enough such that

$$(\mathrm{T}_\ell \mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty}, \Sigma_{\overline{\mathbb{F}}F_\infty}})^\Upsilon = \mathrm{T}_\ell \mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty}, \Sigma_{\overline{\mathbb{F}}F_\infty}}.$$

The following two corollaries are a non-commutative generalisation of Greither's and Popescu's main conjecture for Picard-1-motives [**GP12**, Cor. 4.13].

Corollary 6.2.4. *Assume that $\ell \neq p$, that H is finite, that both Σ and T are non-empty, and that F_∞/F has ramification prime to ℓ over U. Then:*

(1) *The $\mathbb{Z}_\ell[[G]]$-module $(\mathrm{T}_\ell \mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty}, \Sigma_{\overline{\mathbb{F}}F_\infty}})^\Upsilon$ is finitely generated and projective over $\mathbb{Z}_\ell[[H]]$. In particular, it has a well-defined class in the Grothendieck group $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], S)$.*
(2) *We have*

$$d\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathbb{Z}_\ell(1)) = -\left[(\mathrm{T}_\ell \mathcal{M}_{\mathrm{T}_{\overline{\mathbb{F}}F_\infty}, \Sigma_{\overline{\mathbb{F}}F_\infty}})^\Upsilon\right]$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], S)$.
(3) *Let $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ be a finitely ramified representation over the valuation ring $\mathcal{O}_C$ of a finite extension C of $\mathbb{Q}_\ell$ that factors through G. Then*

$$\Phi_\rho(\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathbb{Z}_\ell(1))) = \mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\rho(1))$$

Proof. This follows from Theorem 5.2.6 with $\mathcal{T} = \mathbb{Z}_\ell(1)$ together with Corollary 5.4.3 and Corollary 6.2.2. □

Corollary 6.2.5. *Assume that H is finite and that Σ is not empty. If $\ell \neq p$ we also assume that T is not empty and that F_∞/F has ramification prime to ℓ over U. If $\ell = p$ we assume that F_∞/F has ramification prime to p over V. Then:*

(1) *The $\mathbb{Z}_\ell[[G]]$-module ${}^\sharp((\mathrm{T}_\ell \mathcal{M}_{\Sigma_{\overline{\mathbb{F}}F_\infty}, \mathrm{T}_{\overline{\mathbb{F}}F_\infty}})^\Upsilon)^{*_{\mathbb{Z}_\ell}}$ is finitely generated and projective over $\mathbb{Z}_\ell[[H]]$. In particular, it has a well-defined class in the Grothendieck group $\mathrm{K}_0(\mathbb{Z}_\ell[[G]], S)$.*

(2) *We have*

$$d\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathbb{Z}_\ell) = -\left[{}^\sharp((\mathrm{T}_\ell\,\mathcal{M}_{\Sigma_{\overline{\mathbb{F}}F_\infty},\mathrm{T}_{\overline{\mathbb{F}}F_\infty}})^\Upsilon)^{*\mathbb{Z}_\ell}\right]$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[G]],S)$.

(3) *Let* $\rho\colon \mathrm{Gal}_F \to \mathrm{Gl}_d(\mathcal{O}_C)$ *be a finitely ramified representation over the valuation ring* $\mathcal{O}_C$ *of a finite extension* C *of* $\mathbb{Q}_\ell$ *that factors through* G. *Then*

$$\Phi_\rho(\mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathbb{Z}_\ell)) = \mathcal{L}_{F_\infty/F,\Sigma,\mathrm{T}}(\rho).$$

Proof. This follows from Theorem 5.2.6 and Theorem 5.2.7 with $\mathcal{T} = \mathbb{Z}_\ell$ together with Corollary 5.4.3, Corollary 5.4.4, and Corollary 6.2.2. □

6.3. Realisations of Abstract 1-Motives

Assume that F is any number field and let $U \subset W$ be two open dense subschemes of $X = \operatorname{Spec}\mathcal{O}_F$. Write $k\colon U \to W$ for the corresponding open immersion. Fix a closed subscheme structure on the complement T of U in W and write $i\colon \mathrm{T} \to W$ for the closed immersion. Then

$$\mathrm{H}^1(W, \mathbb{G}_{\mathrm{m}W,\mathrm{T}}) \cong \operatorname{coker}(\mathcal{P}_{\mathrm{T}}(W) \xrightarrow{\mathrm{div}} \mathcal{D}iv_U(W))$$

is the ray class group of W with respect to the modulus T. If K/F is a possibly infinite algebraic extension of F, it follows from [**AGV72a**, VII, Cor. 5.8] that

$$\mathrm{H}^1(W_K, \mathbb{G}_{\mathrm{m}W_K,\mathrm{T}_K}) = \varinjlim_{K' \subset K} \operatorname{coker}\left(\mathcal{P}_{\mathrm{T}_{K'}}(W_{K'}) \to \mathcal{D}iv_{U_{K'}}(W_{K'})\right)$$

with

$$\varinjlim_{K' \subset K} \mathcal{D}iv_{U_{K'}}(W_{K'}) = \bigoplus_v \Gamma_v,$$

where v ranges over the places of K lying over the closed points of U and Γ_v denotes the value group of the associated, possibly non-discrete valuation.

Assume now that ℓ is invertible on W. We then obtain an exact 9-diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & j_!\mu_{\ell^n} \arrow[r] \arrow[d] & \mu_{\ell^n} \arrow[r] \arrow[d] & i_*i^*\mu_{\ell^n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{G}_{\mathrm{m}W,\mathrm{T}} \arrow[r] \arrow[d, "\ell^n"] & \mathbb{G}_{\mathrm{m}W} \arrow[r] \arrow[d, "\ell^n"] & i_*\mathbb{G}_{\mathrm{m}\mathrm{T}} \arrow[r] \arrow[d, "\ell^n"] & 0 \\
0 \arrow[r] & \mathbb{G}_{\mathrm{m}W,\mathrm{T}} \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}W} \arrow[r] \arrow[d] & i_*\mathbb{G}_{\mathrm{m}\mathrm{T}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and in combination with the diagram (6.1.1), an exact sequence

$$0 \to j_!\mu_{\ell^n} \to \mathcal{P}_{\mathrm{T}} \xrightarrow{(\ell^n,\mathrm{div})} \mathcal{P}_{\mathrm{T}} \oplus \mathcal{D}iv_U \xrightarrow{\left(\begin{smallmatrix}\mathrm{div} \\ -\ell^n\end{smallmatrix}\right)} \mathcal{D}iv_U \to 0.$$

We take global sections on W. Since $\mathrm{H}^1(W, \mathcal{P}_{\mathrm{T}}) = 0$ and since multiplication by ℓ^n is injective on $\mathcal{D}iv_U(W)$, we obtain

$$\mathrm{H}^1(W, j_!\mu_{\ell^n}) = \left\{ f \in \mathcal{P}_{\mathrm{T}}(W) \;\middle|\; \begin{array}{l} \mathrm{div}(f) = \ell^n D, \\ D \in \mathcal{D}iv_U(W) \end{array} \right\} / \{ g^{\ell^n} \mid g \in \mathcal{P}_{\mathrm{T}}(W) \}.$$

Note that this group does not depend on the subscheme structure of T. So, we might as well equip it with the reduced scheme structure.

We now assume in addition that F is totally real and fix a CM-admissible extension F_∞/F such that $F_\infty/F_{\mathrm{cyc}}$ is finite. Passing to the direct limit over all finite subextensions F'/F of F_∞/F, we obtain

$$\mathrm{H}^1(W_{F_\infty}, k_{F_\infty !}\mu_{\ell^n}) = \left\{ f \;\middle|\; \begin{array}{l} f \in \mathcal{P}_{\mathrm{T}_{F_\infty}}(W_{F_\infty}), \\ \operatorname{div}(f) = \ell^n D, \\ D \in \mathcal{D}iv_{U_{F_\infty}}(W_{F_\infty}) \end{array} \right\} / \{ g^{\ell^n} \mid g \in \mathcal{P}_{\mathrm{T}_{F_\infty}}(W_{F_\infty}) \}. \tag{6.3.1}$$

Write Σ for the complement of W in X and Σ' for the complement of W in $\operatorname{Spec} \mathcal{O}_F[\frac{1}{\ell}]$. The Iwasawa-theoretic 1-motive associated to $(F_\infty, \Sigma_{F_\infty}, \mathrm{T}_{F_\infty})$ is the complex of abelian groups

$$\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}: \qquad \mathcal{D}iv_{\Sigma'_{F_\infty}}(X_{F_\infty}) \xrightarrow{\delta} \mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

sitting in degrees 0 and 1 [**GP15**, §3.1]. Its group of ℓ^n-torsion points is defined to be

$$\begin{aligned} \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[\ell^n] &\coloneqq \left\{ (D, c) \;\middle|\; \begin{array}{l} D \in \mathcal{D}iv_{\Sigma'_{F_\infty}}(X_{F_\infty}), \\ c \in \mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell, \\ \delta(D) = \ell^n c \end{array} \right\} \otimes_{\mathbb{Z}} \mathbb{Z}/(\ell^n) \\ &= \mathrm{H}^0(\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/(\ell^n)) \end{aligned}$$

and its ℓ-adic Tate module is given by

$$\mathrm{T}_\ell \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}} \coloneqq \varprojlim_n \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[\ell^n]$$

[**GP15**, Def. 2.2, Def. 2.3].

Remark 6.3.1. The complex of abelian groups $\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}$ is an abstract 1-motive in the sense of [**GP15**, Def. 2.1] only if $\mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is divisible of finite corank. The proof of [**GP15**, Lem. 2.8] shows that this is true if and only if $\mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is divisible of finite corank. By [**NSW00**, Thm. 11.1.8] this is equivalent to the Galois group $\mathcal{X}_{\mathrm{nr}}(F_\infty)$ of the maximal abelian unramified pro-ℓ-extension of F_∞ being a finitely generated $\mathbb{Z}_\ell$-module. It suffices that $\mathcal{X}_{\mathrm{nr}}(F_\infty(\mu_\ell))$ is a finitely generated $\mathbb{Z}_\ell$-module. By [**Was97**, Thm 13.24] the latter statement is equivalent to $e_- \mathcal{X}_{\mathrm{nr}}(F_\infty(\mu_\ell))$ being finitely generated over $\mathbb{Z}_\ell$, which is in turn equivalent to the Galois group of the maximal abelian pro-ℓ-extension unramified outside the primes over ℓ of the maximal totally real subfield $F_\infty(\mu_\ell)^+$ being finitely generated over $\mathbb{Z}_\ell$ [**NSW00**, Cor. 11.4.4]. Hence, $\mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}$ is an abstract 1-motive under Conjecture 3.3.4.

Proposition 6.3.2. *There is a short exact sequence*

$$0 \to \mathrm{H}^0(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}/(\ell^n) \to \mathrm{H}^1(W_{F_\infty}, k_{F_\infty !}\mu_{\ell^n}) \to \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[\ell^n] \to 0.$$

In particular, there are isomorphisms

$$\begin{aligned} e_- \mathrm{H}^1(W_{F_\infty}, k_{F_\infty !}\mu_{\ell^n}) &\cong e_- \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[\ell^n], \\ e_- \mathrm{H}^1(W_{F_\infty}, k_{F_\infty !}(\mathbb{Z}_\ell)_{W_{F_\infty}}(1)) &\cong e_- \mathrm{T}_\ell \, \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}. \end{aligned}$$

Proof. This follows from (6.3.1) and [**GP15**, Prop. 3.2, Cor. 3.4]. Note that the proofs of these statements do not make use of the divisibility of the group $\mathrm{H}^1(X_{F_\infty}, \mathbb{G}_{\mathrm{m}\,X_{F_\infty}, \mathrm{T}_{F_\infty}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. ∎

COROLLARY 6.3.3. *Assume that F_∞/F has ramification prime to ℓ over U. Under Conjecture 3.3.4, there are isomorphisms*

$$e_- \mathrm{H}^2(W, k_! \eta_* \mathbb{Z}/(\ell^n)[[G]]^\sharp(1)) \cong e_- \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}[\ell^n],$$
$$e_- \mathrm{H}^2(W, k_! \eta_* \mathbb{Z}_\ell[[G]]^\sharp(1)) \cong e_- \mathrm{T}_\ell \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}.$$

In particular,

$$d\mathcal{L}^{\circledast,-}_{F_\infty/F,\Sigma,\mathrm{T}}(\mathbb{Z}_\ell(1)) = \left[e_- \mathrm{T}_\ell \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}\right]$$

in $\mathrm{K}_0(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]], S)$.

PROOF. Combine Proposition 6.3.2 with Lemma 5.4.2 and use Corollary 4.3.3. □

In particular, [**GP15**, Thm. 4.6] reduces to the special case $\mathcal{T} = \mathbb{Z}_\ell$ of Corollary 5.4.5. Moreover, if $\mathrm{Gal}(F_\infty/F)$ is commutative, we may identify the Fitting ideal and the characteristic ideal of $e_- \mathrm{T}_\ell \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}$ over $\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]$. The characteristic ideal may then be viewed as an element of

$$(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]_S)^\times/\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]]^\times \cong \mathrm{K}_0(\mathbb{Z}_\ell[[\mathrm{Gal}(F_\infty/F)]], S).$$

Under this identification, it corresponds to the class $\left[e_- \mathrm{T}_\ell \mathcal{M}^{F_\infty}_{\Sigma_{F_\infty}, \mathrm{T}_{F_\infty}}\right]^{-1}$. Furthermore, the interpolation property (11) in Corollary 4.3.3 shows that the element $\mathcal{L}^{\circledast,-}_{F_\infty/F}(W, k_!(\mathbb{Z}_\ell)_U(1))^{-1}$ agrees with the element $e_+ + \theta^{(\infty)}_{\Sigma,\mathrm{T}}$ with $\Sigma = X - W$, $\mathrm{T} = W - U$ in the notation of [**GP15**, Def. 5.16]. In particular, we recover the version of the equivariant main conjecture formulated in [**GP15**, Thm. 5.6] as a special case of Corollary 6.3.3. In the same way, one can also recover its non-commutative generalisation in [**Nic13**, Thm. 3.3].

APPENDIX A

Localisation in Polynomial Rings

Let R be any associative ring with 1 and let $R[t]$ be the polynomial ring over R in one indeterminate t that commutes with the elements of R. Write $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$ for the Waldhausen categories of strictly perfect and perfect complexes of $R[t]$-modules. Consider R as a R-$R[t]$-bimodule via the augmentation map

$$R[t] \to R, \qquad t \mapsto 0.$$

We then define full subcategories

$$\mathbf{SP}^{w_t}(R[t]) := \{P^\bullet \in \mathbf{SP}(R[t]) \mid R \otimes_{R[t]} P^\bullet \text{ is acyclic}\},$$
$$\mathbf{P}^{w_t}(R[t]) := \{P^\bullet \in \mathbf{P}(R[t]) \mid P^\bullet \text{ is quasi-isomorphic to a complex in } \mathbf{SP}^{w_t}(R[t])\}.$$

These categories are in fact Waldhausen subcategories of $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$, respectively, since they are closed under shifts and extensions [**Wit08**, 3.1.1]. We can then construct new Waldhausen categories $w_t\mathbf{SP}(R[t])$ and $w_t\mathbf{P}(R[t])$ with the same objects, morphisms, and cofibrations as $\mathbf{SP}(R[t])$ and $\mathbf{P}(R[t])$, but with weak equivalences being those morphisms with cone in $\mathbf{SP}^{w_t}(R[t])$ and $\mathbf{P}^{w_t}(R[t])$, respectively. By the Waldhausen approximation theorem [**TT90**, 1.9.1], the inclusion functor $w_t\mathbf{SP}(R[t]) \to w_t\mathbf{P}(R[t])$ induces isomorphisms

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \mathrm{K}_n(w_t\mathbf{P}(R[t]))$$

for all $n \geq 0$.

It might be reassuring to know that, if R is noetherian, we can identify these K-groups for $n \geq 1$ with the K-groups of a localisation of $R[t]$: Set

$$S_t := \{f(t) \in R[t] \mid f(0) \in R^\times\}$$

Proposition A.1. *Assume that R is a noetherian. Then S_t is a left (and right) denominator set in the sense of* [**GW04**, Ch. 10] *such that the localisation $R[t]_{S_t}$ exists and is noetherian. Its Jacobson radical $\mathrm{Jac}(R[t]_{S_t})$ is generated by the Jacobson radical $\mathrm{Jac}(R)$ of R and t. In particular, if R is semi-local, then so is $R[t]_{S_t}$.*

Moreover, the category $\mathbf{SP}^{w_t}(R[t])$ consists precisely of those complexes $P^\bullet$ in $\mathbf{SP}(R[t])$ with S_t-torsion cohomology. In particular,

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \mathrm{K}_n(R[t]_{S_t})$$

for $n \geq 1$.

Proof. Clearly, the set S_t consists of non-zero divisors, such that we only need to check the Ore condition:

$$\forall s \in S_t \colon \forall a \in R[t] \colon \exists x \in R[t] \colon \exists y \in S_t \colon xs = ya.$$

Moreover, we may assume that $s(0) = y(0) = 1$. Write

$$s = 1 - \sum_{i=1}^{\infty} s_i t^i, \quad a = \sum_{i=0}^{\infty} a_i t^i, \quad x = \sum_{i=0}^{\infty} x_i t^i, \quad y = 1 + \sum_{i=1}^{\infty} y_i t^i.$$

and assume that $s_i = a_{i-1} = 0$ for $i > n$. Comparing coefficients, we obtain the recurrence equation

$$(*) \qquad x_i = \sum_{j=0}^{i-1} x_j s_{i-j} + \sum_{j=1}^{i} y_j a_{i-j} + a_i = \sum_{j=1}^{i} y_j b_{i-j} + b_i$$

with

$$b_i \coloneqq \sum_{j=0}^{i-1} b_j s_{i-j} + a_i.$$

Write $B_i \coloneqq (b_{i-n+1}, \ldots, b_i) \in R^n$ with the convention that $b_i = 0$ for $i < 0$. Then for $i \geq n$

$$B_i = B_{i-1} S = B_{n-1} S^{i-n+1}$$

with

$$S \coloneqq \begin{pmatrix} 0 & \dots & \dots & s_n \\ 1 & \ddots & & s_{n-1} \\ 0 & \ddots & 0 & \vdots \\ \vdots & & 1 & s_1 \end{pmatrix}.$$

Since R was assumed to be noetherian, there exists a $m \geq n$ and $y_n, \ldots, y_m \in R$ such that

$$0 = \sum_{j=n}^{m} y_j B_{m-j} + B_m = \sum_{j=n}^{m} y_j B_{i-j} + B_i$$

for all $i \geq m$. Hence, we can find a solution $(x_i, y_i)_{i=0,1,2\ldots}$ of equation $(*)$ with $x_i = y_i = 0$ for $i > m$ and $y_i = 0$ for $i < n$. This shows that S_t is indeed a left denominator set such that $R[t]_{S_t}$ exists and is noetherian [**GW04**, Thm. 10.3, Cor. 10.16].

Let $N \subset R[t]$ be the semi-prime ideal of $R[t]$ generated by t and the Jacobson ideal $\operatorname{Jac}(R)$ of R. Then S_t is precisely the set of elements of $\Lambda[t]$ which are units modulo N. In particular, the localisation N_{S_t} is a semi-prime ideal of $R[t]_{S_t}$ such that

$$R[t]_{S_t}/N_{S_t} = R[t]/N = R/\operatorname{Jac}(R)$$

[**GW04**, Thm. 10.15, 10.18]. We conclude $\operatorname{Jac}(R[t]_{S_t}) \subset N_{S_t}$. For the other inclusion it suffices to note that for every $s \in S_t$ and every $n \in N$, the element $s + n$ is a unit modulo N.

The Nakayama lemma implies that for any noetherian ring R with Jacobson radical $\operatorname{Jac}(R)$, a strictly perfect complex of R-modules $P^\bullet$ is acyclic if and only if $R/\operatorname{Jac}(R) \otimes_R P^\bullet$ is acyclic. Hence, if $P^\bullet$ is a strictly perfect complex of $R[t]$-modules, then $R \otimes_{R[t]} P^\bullet$ is acyclic if and only if $R[t]_{S_t} \otimes_R P^\bullet$ is acyclic. This shows that $\mathbf{SP}^{w_t}(R[t])$ consists precisely of those complexes $P^\bullet$ in $\mathbf{SP}(R[t])$ with S_t-torsion cohomology. From the localisation theorem in [**WY92**] we conclude that the Waldhausen exact functor

$$w_t\mathbf{SP}(R[t]) \to \mathbf{SP}(R[t]_{S_t}), \qquad P^\bullet \mapsto R[t]_{S_t} \otimes_{R[t]} P^\bullet$$

induces isomorphisms

$$\mathrm{K}_n(w_t\mathbf{SP}(R[t])) \cong \begin{cases} \mathrm{K}_n(R[t]_{S_t}) & \text{if } n > 0, \\ \operatorname{im}\left(\mathrm{K}_0(R[t]) \to \mathrm{K}_0(R[t]_{S_t})\right) & \text{if } n = 0. \end{cases}$$

□

The set S_t fails to be a left denominator set if $R = \mathbb{F}_\ell\langle\langle x, y\rangle\rangle$ is the power series ring in two non-commuting indeterminates: $a(1 - xt) = by$ has no solution with $a \in R[t]$, $b \in S_t$. Note also that a commutative adic ring is always noetherian

[**War93**, Cor. 36.35]. In this case, S_t is the union of the complements of all maximal ideals of $\Lambda[t]$ containing t and the determinant provides an isomorphism

$$\mathrm{K}_1(w_t\mathbf{SP}(\Lambda[t])) \cong \mathrm{K}_1(\Lambda[t]_{S_t}) \xrightarrow[\cong]{\det} \Lambda[t]_{S_t}^{\times}.$$

For any adic $\mathbb{Z}_\ell$-algebra Λ and any $\gamma \in \Gamma \cong \mathbb{Z}_\ell$, we have a ring homomorphism

$$\mathrm{ev}_\gamma\colon \Lambda[t] \mapsto \Lambda[[\Gamma]], \quad f(t) \mapsto f(\gamma).$$

inducing homomorphisms $\mathrm{K}_n(\Lambda[t]) \to \mathrm{K}_n(\Lambda[[\Gamma]])$.

PROPOSITION A.2. *Assume that* $\gamma \neq 1$. *Then the ring homomorphism* ev_γ *induces homomorphisms*

$$\mathrm{ev}_\gamma\colon \mathrm{K}_n(w_t\mathbf{P}(\Lambda[[t]])) \cong \mathrm{K}_n(w_t\mathbf{SP}(\Lambda[t])) \to \mathrm{K}_n(\Lambda[[\Gamma]]_S)$$

for all $n \geq 0$.

PROOF. It suffices to show that for any complex $P^\bullet$ in $\mathbf{SP}^{w_t}(\Lambda[t])$, the complex

$$Q^\bullet \coloneqq \Lambda[[\Gamma]] \otimes_{\Lambda[t]} P^\bullet$$

is perfect as complex of Λ-modules. We can check this after factoring out the Jacobson radical of Λ [**Wit14**, Prop. 4.8]. Hence, we may assume that Λ is simisimple, i. e.

$$\Lambda = \prod_{i=1}^{m} M_{n_i}(k_i)$$

where $M_{n_i}(k_i)$ is the algebra of $n_i \times n_i$-matrices over a finite field k_i of characteristic ℓ. By the Morita theorem, the tensor product over Λ with the $\prod_i k_i$-Λ-bimodule

$$\prod_{i=1}^{m} k_i^{n_i}$$

induces equivalences of categories

$$\mathbf{SP}^{w_t}(\Lambda[t]) \to \mathbf{SP}^{w_t}\left(\prod_{i=1}^{n} k_i[t]\right),$$

$$\mathbf{SP}^{w_H}(\Lambda[[\Gamma]]) \to \mathbf{SP}^{w_H}(\prod_{i=1}^{n} k_i[[\Gamma]]),$$

with $H \subset \Gamma$ being the trivial subgroup. Hence, we are reduced to the case

$$\Lambda = \prod_{i=1}^{m} k_i.$$

In this case, the set $S \subset \Lambda[[\Gamma]]$ defined in (2.5.1) consists of all non-zero divisors of $\Lambda[[\Gamma]]$, i. e. all elements with non-trivial image in each component $k_i[[\Gamma]]$. Since $\Lambda[[\Gamma]]$ is commutative, this is trivially a left denominator set. Moreover, the complex $Q^\bullet$ is perfect as complex of Λ-modules precisely if its cohomology groups are S-torsion. On the other hand, as a trivial case of Proposition A.1, we know that S_t is a left denominator set and that the cohomology groups of $P^\bullet$ are S_t-torsion. Since $f(0)$ is a unit in Λ for each $f \in S_t$, the element $f(\gamma)$ has clearly non-trivial image in each component $k_i[[\Gamma]]$. Hence, ev_γ maps S_t to S and $Q^\bullet$ is indeed perfect as complex of Λ-modules. □

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