Remarks on the Fundamental Lemma for stable twisted Endoscopy of Classical Groups

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Contents

Introduction

The original aim of these notes is to prove a fundamental lemma for the stable lift from $H = \mathrm{Sp}_4$ to $\tilde{G} = \widetilde{\mathrm{PGL}}_5$ over a local non archimedean field F with residue characteristic $\neq 2$. Here $\widetilde{\mathrm{PGL}}_5 = \mathrm{PGL}_5 \rtimes \langle\Theta\rangle$ is generated by its normal subgroup PGL_5 of index 2 and the involution $\Theta : g \mapsto J \cdot {}^t g^{-1} \cdot J^{-1}$, where J is the antidiagonal matrix with entries $(1, -1, 1, -1, 1)$.

We will (Cor.5.10) prove that if the semisimple elements $\gamma\Theta \in \widetilde{\mathrm{PGL}}_5(F)$ and $\eta \in \mathrm{Sp}_4(F)$ match (see 1.11 for a definition of matching) then the corresponding stable orbital integrals (see 3.1) for the unit elements in the Hecke algebra match:

$$O^{st}_{\gamma\Theta}(1, \widetilde{\mathrm{PGL}}_5) = O^{st}_{\eta}(1, \mathrm{Sp}_4). \tag{\textbf{i}}$$

This theorem will have important applications in the theory of automorphic representations of the group GSp_4 over a number field and for the l–adic Galois-representations on the corresponding Shimura varieties [W1], [W2], [Wes].

In analyzing (**i**) using the Kazhdan-trick (lemma 3.5 below) we recognized that all essential computations had already been done by Flicker in [Fl2], where the corresponding fundamental lemma for the lift from $\mathrm{GSp}_4 \simeq \mathrm{GSpin}_5$ to $\widetilde{\mathrm{GL}}_4 \times \mathbb{G}_m$ has been proved. This phenomenon seems to be known to the experts [Hal2].

More generally one can discuss the fundamental lemma for the stable lift from H to a classical split group with outer automorphism $\tilde{G}$, where H is the stable endoscopic group for $\tilde{G}$. This fundamental lemma describes a relationship between ordinary stable orbital integrals on the endoscopic group H and Θ-twisted orbital integrals on $\tilde{G}$. We will discuss the following lifts in detail:

H	$\tilde{G}$
Sp_{2n}	$\mathrm{PGL}_{2n+1} \rtimes \langle g \mapsto J^t g^{-1} J^{-1}\rangle$
GSpin_{2n+1}	$(\mathrm{GL}_{2n} \times \mathbb{G}_m) \rtimes \langle (g, a) \mapsto (J^t g^{-1} J^{-1}, \det g \cdot a)\rangle$
Sp_{2n}	$\widetilde{SO_{2n+2}} \simeq \mathrm{O}_{2n+2}$.

In each case we will reduce the fundamental lemma using the Kazhdan trick and a lot of observations in linear algebra to a statement which we call the BC-conjecture and which seems to be proven only for $n = 1, 2$:

Conjecture: *If the regular topologically unipotent and algebraically semisimple elements $u \in SO_{2n+1}(\mathcal{O}_F)$ and $v \in Sp_{2n}(\mathcal{O}_F)$ are BC-matching (see 1.12) then*

$$O^{st}_u(1, \mathrm{SO}_{2n+1}) = O^{st}_v(1, \mathrm{Sp}_{2n}). \tag{BC_n}$$

Thus the (Θ-twisted) fundamental lemmas for the three series of endoscopy will be reduced to a fundamental lemma like statement for ordinary (i.e. untwisted) stable orbital integrals on the groups SO_{2n+1} of type B_n resp. Sp_{2n} of type C_n.

1 Stable endoscopy and matching

(1.1) Notations. In this paper we will denote by F a p-adic field with ring of integers $\mathcal{O}_F$, prime ideal $\mathfrak{p}$ and uniformizing element $\varpi = \varpi_F$. The residue field of characteristic p is denoted $\kappa = \kappa_F = \mathcal{O}_F/\mathfrak{p}$. By $\bar{F}$ we denote an algebraic closure of F. In the whole paper we will assume that $p \neq 2$. Only in this chapter R denotes an arbitrary integral domain.

(1.2) Split Groups with automorphism. Let G/R be a connected reductive split group scheme. We fix some "splitting" i.e. a tripel $(B, T, \{X_\alpha\}_{\alpha\in\Delta})$ where T denotes a maximal split-torus inside a rational Borel B, $\Delta = \Delta_G = \Delta(G, B, T) \subset \Phi(G, T) \subset X^*(T)$ the set of simple roots inside the system of roots and the X_α are a system (nailing) of isomorphisms between the additive group scheme $\mathbb{G}_a$ and the unipotent root subgroups B_α. Here $X^*(T) = Hom(T, \mathbb{G}_m)$ denotes the character module of T, while $X_*(T) = Hom(\mathbb{G}_m, T)$ will denote the cocharacter module of T. Let $\Theta \in Aut(G)$ be an automorphism of G which fixes the splitting, i.e. stabilizes B and T and permutes the X_α. We assume Θ to be of finite order l. We denote by

$$\tilde{G} = G \rtimes \langle\Theta\rangle$$

the (nonconnected) semidirect product of G with Θ. Θ acts on the (co)character module via $X_*(T) \ni \alpha^\vee \mapsto \Theta \circ \alpha^\vee$ resp. $X^*(T) \ni \alpha \mapsto \alpha \circ \Theta^{-1}$.

(1.3) The dual group. Let $\hat{G} = \hat{G}(\mathbb{C})$ be the dual group of G. By definition $\hat{G}$ has a tripel $(\hat{B}, \hat{T}, \{\hat{X}_{\hat{\alpha}}\})$ such that we have identifications $X^*(\hat{T}) = X_*(T)$, $X_*(\hat{T}) = X^*(T)$ which identifies the (simple) roots $\hat{\alpha} \in X^*(\hat{T})$ with the (simple) coroots $a^\vee \in X_*(T)$, and the (simple) coroots $\hat{\alpha}^\vee \in X_*(\hat{T})$ with the (simple) roots $\alpha \in X^*(T)$. There exists a unique automorphism $\hat{\Theta}$ of $\hat{G}$ which stabilizes $(\hat{B}, \hat{T}, \{\hat{X}_{\hat{\alpha}}\})$ and induces on $(X_*(\hat{T}), X^*(\hat{T}))$ the same automorphism as Θ on $(X^*(T), X_*(T))$.

(1.4) The Θ-invariant subgroup in $\hat{G}$. Let $\hat{H} = (\hat{G}^{\hat{\Theta}})^\circ$ be the connected component of the subgroup of $\hat{\Theta}$-fixed elements in $\hat{G}$. It is a reductive split group with triple $(\hat{B}_H, \hat{T}_H, \{X_\beta\}_{\beta\in\Delta_{\hat{H}}})$, where $\hat{B}_H = \hat{B}^{\hat{\Theta}}$, $\hat{T}_H = \hat{T}^{\hat{\Theta}}$ and the nailing will be explained soon. We have the inclusion of cocharacter modules $X_*(\hat{T}_H) = X_*(\hat{T})^{\hat{\Theta}} \subset X_*(\hat{T})$ and a projection for the character module

$$P_\Theta : X^*(\hat{T}) \twoheadrightarrow (X^*(\hat{T})_{\hat{\Theta}})_{free} = X^*(\hat{T}_H),$$

where $(X^*(\hat{T})_{\hat{\Theta}})_{free}$ denotes the maximal free quotient of the coinvariant module $X^*(\hat{T})_{\hat{\Theta}}$. For a $\mathbb{Z}[\Theta]$-module X we define a map

$$S_\Theta : \quad X \to X^\Theta, \qquad x \mapsto \sum_{i=0}^{ord_x(\Theta)-1} \Theta^i(x)$$

where $ord_x(\Theta) = \min\{i > 0 \mid \Theta^i(x) = x\}$ is length of the $\langle\Theta\rangle$-orbit of x.
For the roots Φ and coroots $\Phi^\vee$ of a given root datum $(X^*, X_*, \Phi, \Phi^\vee)$ we have to introduce a modified map S'_Θ by

$$S'_\Theta(\alpha) \quad = \quad c(\alpha) \cdot S_\Theta(\alpha) \qquad \text{where} \qquad c(\alpha) \quad = \quad \frac{2}{\langle \alpha^\vee, S_\Theta(\alpha)\rangle}$$

resp. by the formula where the roles of α and $\alpha^\vee$ are exchanged. For all simple root systems with automorphisms which are not of type A_{2n} we have $\langle\alpha^\vee, \Theta^i(\alpha)\rangle = 0$ for $i = 1, \ldots, ord_\alpha(\Theta) - 1$ which implies $c(\alpha) = 1$ i.e. $S'_\Theta(\alpha) = S_\Theta(\alpha)$. We furthermore introduce the subset of short-middle roots and the dual concept of long-middle coroots:

$$\begin{aligned}
\Phi(\hat{G}, \hat{T})^{sm} \quad &= \quad \left\{\alpha \in \Phi(\hat{G}, \hat{T}) \,\middle|\, \frac{1}{2} \cdot P_\Theta(\alpha) \notin P_\Theta(\Phi(\hat{G}, \hat{T}))\right\} \\
\Phi^\vee(\hat{G}, \hat{T})^{lm} \quad &= \quad \left\{\alpha^\vee \,\middle|\, \alpha \in \Phi(\hat{G}, \hat{T})^{sm}\right\}
\end{aligned}$$

Proposition 1.5. *With the above notations we have*

$$\begin{aligned}
&\textbf{(i)} & \Phi(\hat{H}, \hat{T}_H) \quad &= \quad P_\Theta(\Phi(\hat{G}, \hat{T})^{sm}) & &\textit{for the roots} \\
&\textbf{(ii)} & \Phi^\vee(\hat{H}, \hat{T}_H) \quad &= \quad S'_\Theta(\Phi^\vee(\hat{G}, \hat{T})^{lm}) & &\textit{for the coroots} \\
& & \Delta^\vee_{\hat{H}} = \Delta^\vee(\hat{H}, \hat{B}_H, \hat{T}_H) \quad &= \quad S'_\Theta(\Delta^\vee_{\hat{G}}) & &\textit{for the simple coroots} \\
& & \Delta_{\hat{H}} = \Delta(\hat{H}, \hat{B}_H, \hat{T}_H) \quad &= \quad P_\Theta(\Delta_{\hat{G}}) & &\textit{for the simple roots}
\end{aligned}$$

Proof: This follows from [St, 8.1] . □

Definition 1.6 (stable Θ-endoscopic group)**.** *In the above situation a connected reductive split group scheme H/R will be called a stable Θ-endoscopic group for (G, Θ) resp. $\tilde{G}$ if its dual group together with the splitting is isomorphic to the above $(\hat{H}, \hat{B}_H, \hat{T}_H, \{X_\beta\}_{\beta\in\Delta_{\hat{H}}})$.*

Remarks: Since H is unique up to isomorphism (up to unique isomorphism if we consider H together with a splitting) we can call H the stable Θ-endoscopic group for (G, Θ). For a maximal split-torus $T_H \subset H$ we have:

$$\begin{aligned}
&\textbf{(iii)} & X_*(T_H) \quad &= \quad X_*(T)_\Theta & &\text{for the cocharacter-module} \\
& & X^*(T_H) \quad &= \quad X^*(T)^\Theta & &\text{for the character-module}
\end{aligned}$$

(1.7) To get examples we use the following **notations**:
$diag(a_1, \ldots, a_n) \in \mathrm{GL}_n$ denotes the diagonal matrix $(\delta_{i,j} \cdot a_i)_{ij}$ and
$antidiag(a_1, \ldots, a_n) \in \mathrm{GL}_n$ the antidiagonal matrix $(\delta_{i,n+1-j} \cdot a_i)_{ij}$ with a_1 in the upper right corner. We introduce the following matrix

$$J = J_n = (\delta_{i,n+1-j}(-1)^{i-1})_{1\le i,j\le n} = antidiag(1, -1, \ldots, (-1)^{n-1}) \in \mathrm{GL}_n(R).$$

and its modification $J'_{2n} = antidiag(1,-1,1,\dots,(-1)^{n-1},(-1)^{n-1},\dots,1,-1,1)$. Since ${}^tJ_n = (-1)^{n-1}\cdot J_n$ and J'_{2n} is symmetric we can define the standard symplectic resp. orthogonal groups

$$\mathrm{Sp}_{2n} = \mathrm{Sp}(J_{2n}), \qquad \mathrm{SO}_{2n+1} = \mathrm{SO}(J_{2n+1}), \qquad \mathrm{SO}_{2n} = \mathrm{SO}(J'_{2n}).$$

We consider the groups $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n$ with the splittings consisting of the diagonal torus, the Borel consisting of upper triangular matrices and the standard nailing. We remark that the following map defines an involution of $\mathrm{GL}_n, \mathrm{SL}_n$ and PGL_n:

$$\Theta = \Theta_n : g \mapsto J_n \cdot {}^tg^{-1} \cdot J_n^{-1}.$$

Example 1.8 ($A_{2n} \leftrightarrow C_n$)**.** The pair $(G,\Theta) = (\mathrm{PGL}_{2n+1}, \Theta_{2n+1})$ has dual $\hat{G} = \mathrm{SL}_{2n+1}(\mathbb{C})$, $\hat{\Theta} = \Theta_{2n+1}$ and the stable endoscopic group $H = \mathrm{Sp}_{2n}$, since $\hat{H} = \mathrm{SO}_{2n+1}(\mathbb{C}) = \hat{G}^{\hat{\eta}}$.

Example 1.9 ($A_{2n-1} \leftrightarrow B_n$)**.** The group $G = \mathrm{GL}_{2n} \times \mathbb{G}_m$ has the involution

$$\Theta : (g,a) \mapsto (\Theta_{2n}(g), \det(g)\cdot a).$$

The dual $\hat{\Theta} \in Aut(\hat{G})$ satisfies $\hat{\Theta}(g,b) = (\Theta_{2n}(g)\cdot b, b)$, so that we get as the stable endoscopic group $H = \mathrm{GSpin}_{2n+1}$, since it has as dual $\hat{H} = \mathrm{GSp}_{2n}(\mathbb{C}) = \hat{G}^{\hat{\eta}}$. Recall that GSpin_{2n+1} fits into an exact sequence

$$1 \to \mathrm{Spin}_{2n+1} \to \mathrm{GSpin}_{2n+1} \xrightarrow{\mu} \mathbb{G}_m \to 1,$$

where the derived group Spin_{2n+1}, the kernel of the "multiplier" map μ, is a connected, simply connected split group.

Example 1.10 ($D_{n+1} \leftrightarrow C_n$)**.** Let $s \in \mathrm{O}_{2n+2}$ denote the reflection which interchanges the standard basis vectors e_{n+1} and e_{n+2} and fixes all other basis elements e_i. Then the pair $(G,\Theta) = (\mathrm{SO}_{2n+2}, int(s))$ has dual $\hat{G} = \mathrm{SO}_{2n+2}(\mathbb{C})$, $\hat{\Theta} = int(\hat{s})$, where $\hat{s}$ is of the same shape as s. The stable endoscopic group is $H = \mathrm{Sp}_{2n}$, since its dual satisfies $\hat{H} = \mathrm{SO}_{2n+1}(\mathbb{C}) = (\hat{G}^{\hat{\eta}})^\circ$.

(1.11) Matching elements Since each semisimple Θ-conjugacy resp. conjugacy class in $G(\bar{F})$ resp. $H(\bar{F})$ meets $T(\bar{F})$ resp. $T_H(\bar{F})$ we have canonical bijections

$$\begin{array}{rlcl} i_G: & G(\bar{F})_{ss}/\Theta - conj & \simeq & T(\bar{F})_\Theta/(W_G)^\Theta \\ i_H: & H(\bar{F})_{ss}/conj & \simeq & T_H(\bar{F})/W_H \end{array}$$

where $W_G = Norm(T,G)/T$ and $W_H = Norm(T_H,H)/T_H$ denote the Weyl-groups. We further have an isomorphism

$$N_{KS}: \quad T(\bar{F})_\Theta \simeq (X_*(T)\otimes \bar{F}^\times)_\Theta \simeq X_*(T)_\Theta \otimes \bar{F}^\times = X_*(T_H)\otimes \bar{F}^\times \simeq T_H(\bar{F})$$

and observe $W_H \simeq (W_G)^\Theta$. Therefore we may define:
Two (Θ-)semisimple elements $\gamma\Theta \in G(F)\Theta$ and $h \in H(F)$ are called *matching* if their (Θ-)stable conjugacy classes in $G(\bar{F})$ resp. $H(\bar{F})$ correspond to each other via the isomorphism $i_H^{-1} \circ N_{KS} \circ i_G$.

(1.12) BC**-matching**: We have an isomorphism between the standard diagonal tori:

$$\begin{gathered} i_{BC} : T_{\mathrm{SO}_{2n+1}} \to T_{\mathrm{Sp}_{2n}}, \\ diag(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \mapsto diag(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}). \end{gathered}$$

We observe that i_{BC} induces an isomorphism of Weylgroups:

$$W_{\mathrm{SO}_{2n+1}} \quad \simeq \quad S_n \ltimes \{\pm 1\}^n \quad \simeq W_{\mathrm{Sp}_{2n}}$$

Two semisimple elements $h \in \mathrm{SO}_{2n+1}(F)$ and $b \in \mathrm{Sp}_{2n}(F)$ are called *BC-matching* if their stable conjugacy classes correspond to each other under the isomorphism $i_{\mathrm{Sp}_{2n}}^{-1} \circ i_{BC} \circ i_{\mathrm{SO}_{2n+1}}$.

Example 1.13. In example 1.8 above the norm map $N_{KS} : T \to T_\Theta \simeq T_H$ is given by

(iv) $$\begin{aligned} \gamma \quad = \quad & diag(t_1, t_2, \dots, t_n, t_{n+1}, t_{n+2}, \dots, t_{2n+1}) \qquad \in T \subset \mathrm{PGL}_{2n+1} \\ \mapsto h = \; & diag(t_1/t_{2n+1}, t_2/t_{2n}, \dots, t_n/t_{n+2}, t_{n+2}/t_n, \dots, t_{2n+1}/t_1) \in T_H \subset \mathrm{Sp}_{2n}. \end{aligned}$$

Example 1.14. In example 1.9 above we consider additionally the projection $pr_{ad} : \mathrm{GSpin}_{2n+1} \to \mathrm{SO}_{2n+1} = \mathrm{Spin}_{2n+1}/\{\pm 1\}$. Then the composite map $pr_{ad} \circ N_{KS} : T \to T_{ad} \subset \mathrm{SO}_{2n+1}$ is given by

(v) $$\begin{aligned} \gamma \quad = \quad & (diag(t_1, t_2, \dots, t_n, t_{n+1}, \dots, t_{2n}), t_0) \qquad \in T \subset \mathrm{GL}_{2n} \times \mathbb{G}_m \\ \mapsto h = \; & diag(t_1/t_{2n}, \dots, t_n/t_{n+1}, 1, t_{n+1}/t_n, \dots, t_{2n}/t_1) \in T_{ad} \subset \mathrm{SO}_{2n+1}. \end{aligned}$$

Furthermore we have the following relation between the multiplier map μ and matching: If $(h, a) \in \mathrm{GL}_{2n}(F) \times F^\times$ and $\eta \in \mathrm{GSpin}_{2n+1}(F)$ match then:

$$\mu(\eta) \quad = \quad \det(h) \cdot a^2.$$

Example 1.15. In example 1.10 above the norm map $N_{KS} : T \to T_\Theta \simeq T_H$ is given by

(vi) $$\begin{aligned} \gamma \quad = \quad & diag(t_1, t_2, \dots, t_n, t_{n+1}, t_{n+1}^{-1}, \dots, t_1^{-1}) \qquad \in T \subset \mathrm{SO}_{2n+2} \\ & \mapsto h = diag(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \in T_H \subset \mathrm{Sp}_{2n}. \end{aligned}$$

In this section let F be a p-adic field and $\tilde{G}/\mathcal{O}_F$ a (not necessarily connected) reductive group scheme with connected component $G = \left(\tilde{G}\right)^\circ$.

Definition 1.16. *An element $g \in \tilde{G}(F)$ is called*

strongly compact, if g lies in a compact subgroup of G.

topologically unipotent, if $\lim_{n\to\infty} g^{p^n} = 1$.

residually semisimple, if g is of a finite order, which is prime to p.

For an element g it is equivalent to be topologically unipotent and to lie in a pro–p–subgroup of G.

Lemma 1.17 (topological Jordan decomposition)**.** *Every strongly compact $g \in \tilde{G}(F)$ has a unique decomposition*

$$g = g_u \cdot g_s = g_s \cdot g_u \ ,$$

where $g_u \in \tilde{G}(F)$ is topologically unipotent and $g_s \in \tilde{G}(F)$ is residually semisimple. If furthermore $|\pi_0(\tilde{G})|$ is prime to p, then we have $g_u \in (\tilde{G})^\circ(F)$.

□

Lemma 1.18 (Properties of the topological Jordan decomposition)**.**

(1) *We have $g_u \in G^{g_s}$ and $G^g = Cent(g_u, G^{g_s})$.*
(2) *Residually semisimple elements are semisimple.*
(3) *Let m be prime to p and u be topologically unipotent. Then there exists a unique topologically unipotent u_1 such that $u_1^m = u$.*
(4) *The topological Jordan decomposition is functorial.*
(5) *If H is a closed subgroup of $\tilde{G}(F)$ and $g \in H$, then also g_s and g_u are in H.*

The functoriality implies the following statements:

(1) Let $\rho : \tilde{G} \to \tilde{G}'$ be a morphism of (not necessarily connected) reductive groups, defined over a finite extension of F. Then we have $\rho(g)_s = \rho(g_s)$ and $\rho(g)_u = \rho(g_u)$.
(2) If $g \in \tilde{G}(\mathcal{O}_F)$, then the image of the topological Jordan decomposition under the reduction map is the Jordan decomposition in $\tilde{G}(\mathbb{F}_q)$.

(1.19) Our strategy to prove the fundamental lemma in the case of classical split groups now goes as follows: By the Kazhdan lemma 3.5 the (twisted) stable orbital integral of $g\theta \in G(F)\theta$ in the group $\tilde{G}$ can be replaced by the ordinary stable orbital integral of the topologically unipotent part u in the group $G^{s\theta}$, the centralizer of the residually semisimple part $s\theta$. Similarly the stable orbital integral of some $\gamma \in H(F)$, which matches with g, can be computed as the stable orbital integral of the topologically unipotent part v in the group H^σ, where the residually semisimple part σ of γ matches with $s\theta$. Now $G^{s\theta}$ resp. H^σ is isogenous to a product $G_+^{s\theta} \times G_*^{s\theta}$ resp. $H_+^\sigma \times H_*^\sigma$, such that $G_+^{s\theta}$ has simple factors of type B or C, H_+^σ has factors of the other of these two types of groups and $G_*^{s\theta}$ is isogenous to H_*^{σ}. Decomposing $u = (u_+, u_*)$ and $v = (v_+, v_*)$ we get that u_* and v_* coincide up to stable conjugation und up to powering, so they have matching stable orbital integrals. The fundamental lemma for $g\theta$ and γ will now follow if we can assume that the stable orbital integrals of u_+ and v_+ match. After we have proved the BC-matching of which u_+ and v_+, this follows from the BC conjecture 3.3, stated already in the introduction.

2 Classification of Θ-conjugacy classes

(2.1) If (G, Θ) is as in the examples 1.8 or 1.9, the problem of determining the Θ-conjugacy classes of elements $s\Theta \in \widetilde{G}(F)$ is equivalent to determine the classes of $h = sJ$ under the transformations $h \mapsto g \cdot h \cdot {}^tg$. Namely we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{GL}_{2n} & \xrightarrow{h \mapsto hJ^{-1}\Theta} & \widetilde{G}L_{2n} \\ {\scriptstyle h \mapsto g\cdot h\cdot {}^tg}\downarrow & & \downarrow{\scriptstyle x \mapsto g\cdot x\cdot g^{-1}} \\ \mathrm{GL}_{2n} & \xrightarrow[h \mapsto hJ^{-1}\Theta]{} & \widetilde{G}L_{2n} \end{array}$$

If we decompose $h = q + p$ in its symplectic part p and its symmetric part q we thus have to consider the problem of simultaneous normal forms for a symplectic and a symmetric bilinear form. To obtain results for orbital integrals we have to deal with this problem also over the ring of integers $\mathcal{O}_F$. The problem can be attacked if we assume $s\Theta$ to be semisimple (resp. residually semisimple if we work over $\mathcal{O}_F$).

(2.2) Notations: In the following R denotes either a field of characteristic 0 or the ring of integers $\mathcal{O}_F$ of a local p-adic field F, where $p \neq 2$. We denote by $\mathfrak{m}$ the maximal ideal of R (i.e. $\mathfrak{m} = (0)$ if R is a field) and by $\kappa = R/\mathfrak{m}$ the residue field in the case $R = \mathcal{O}_F$.

Let M denote a free R-module of finite rank r with basis $(b_i)_{1 \le i \le r}$. A bilinear form $q : M \times M \to R$ is called *unimodular* if $\Delta(q) := \det(q(b_i, b_j)) \in R^\times$. This definition is obviously independent of the chosen basis (b_i) since $\Delta(q)$ is an invariant in $R/(R^\times)^2$. For $h \in \mathrm{GL}_n(R)$ we have the following bilinear forms b_h and b'_h on the module $M = R^n$ of column vectors: $b_h(m_1, m_2) = {}^tm_1 \cdot h \cdot m_2$ and $b'_h(m_1, m_2) = {}^tm_1 \cdot {}^th \cdot m_2$.

An element $g \in \mathrm{GL}(M)$ is called *R-semisimple* iff

- g is semisimple in the case R is a field,
- g is residually semisimple (i.e. has finite order prime to $char(\kappa)$) in the case $R = \mathcal{O}_F$.

For $h \in \mathrm{GL}_n(R)$ we call $N(h) = h \cdot {}^th^{-1}$ the *(right) norm* of h and $N_l(h) = {}^th^{-1} \cdot h$ the *left norm* of h. $N(h)$ and $N_l(h)$ are conjugate by h in $\mathrm{GL}_n(R)$. Then h is called *R-Θ-semisimple* if $N(h)$ (or equivalently $N_l(h)$) is R-semisimple.

We remark that h is R-Θ-semisimple if and only if $h \cdot J^{-1} \cdot \Theta$ is semisimple respectively residually semisimple as an element of $\mathrm{GL}_n(R) \rtimes \langle \Theta \rangle$.

Lemma 2.3. *If $p : M \times M \to R$ is a unimodular symplectic form, then there exists a basis $(e_1, \dots, e_g, f_g, \dots, f_1)$ of M, such that p has standard form with respect to this basis, i.e. $p(e_i, e_j) = p(f_i, f_j) = 0$ and $p(e_i, f_j) = \delta_{ij}$.*

Proof: The standard procedure to get a symplectic basis of M applies for unimodular forms. □

Lemma 2.4. *If $q : M \times M \to R$ is a unimodular symmetric bilinear form and $R = \mathcal{O}_F$, then there exists a basis $(e_i)_{1 \le i \le r}$ of M such that $q(e_i, e_j) = \delta_{ij}$ for $(i, j) \ne (r, r)$ and $q(e_r, e_r)$ is some given element in the class of $\Delta(q)$ in $R^\times/(R^\times)^2$.*

Proof: Consider the reductions $\kappa = R/\mathfrak{m}$, $\bar{M} = M/\mathfrak{m}M$ and $\bar{q} : \bar{M} \times \bar{M} \to \kappa$. Since quadratic forms over finite fields are classified by their discriminants, the analogous statement for $\bar{q}$ holds. By lifting a basis from $\bar{M}$ to M we can therefore assume that $q(b_i, b_j) \cong \delta_{ij} \mod \mathfrak{m}$ for $(i, j) \ne (r, r)$. But now we can apply the Gram-Schmidt-Orthogonalization procedure (observe that elements congruent to 1 modulo $\mathfrak{m}$ are squares since $p \ne 2$) to obtain the claim. □

Lemma 2.5.

(a) If $g \in GL(M)$ is R-semisimple then there exists a finite étale galois extension R'/R such that $M' = M \otimes_R R'$ decomposes into the direct sum of eigenspaces: $M' = \bigoplus_\lambda M'_\lambda$, where g acts on M'_λ as the scalar λ.

(b) If $g = N_l(h)$ for an R-Θ-semisimple $h \in GL_n(R)$ (see 2.2) then $b_h(M'_\lambda, M'_\mu) = 0 = b'_h(M'_\lambda, M'_\mu)$ unless $\lambda\mu = 1$.

(c) The restrictions of the forms b_h and b'_h to M'_1 and M'_{-1} are unimodular. For $\lambda \ne \pm 1$ also the restrictions of $b_h, b'_h, b_h + b'_h$ and $b_h - b'_h$ to the modules $N'_\lambda = M'_\lambda \oplus M'_{\lambda^{-1}}$ are unimodular.

Proof: (a) The minimal polynomial $\chi(X)$ of g decomposes in pairwise different linear factors $\chi(X) = \prod_{i=1}^r (X - \lambda_i)$ over some extension ring of R. The ring $R' = R[\lambda_i]_{1 \le i \le r}$ is finite étale and galois over R, since the λ_i are roots of unity of order prime to $char(\kappa)$ in the case $R = \mathcal{O}_F$. By the same reason we have

$$\text{(i)} \qquad \lambda_i - \lambda_j \in (R')^\times \qquad \text{for } i \ne j$$

in both cases for R. We remark for later use that this statement remains correct if we add ± 1 to the set of the λ_i (if they are not already among them). Therefore

$\chi_i(X) = \prod_{j\neq i}((X-\lambda_j)\cdot(\lambda_i-\lambda_j)^{-1}) \in R'[X]$. We have $\sum_{i=1}^r \chi_i(X) = 1$ since the left hand side is a polynomial of degree $r-1$ which has the value 1 at r different places. Therefore M' is the sum of the subspaces $M'_{\lambda_i} = \chi_i(g)(M)$. Since $(g-\lambda_i)\cdot\chi_i(g)$ equals $\chi(g)\cdot\prod_{j\neq i}(\lambda_i-\lambda_j)^{-1} = 0$, the spaces M'_{λ_i} are eigenspaces for g and the sum $M' = \Sigma_{i=1}^r M'_{\lambda_i}$ is direct.

(b) For $m \in M'_\lambda$ and $n \in M'_\mu$ we have $m = \lambda^{-1}\cdot gm$ and $n = \mu\cdot g^{-1}n$. The claims follow immediately from the relations $b_h(m,n) = \lambda^{-1}\cdot b'_h(m,n)$ and

$$b_h(m,n) = \mu\cdot b'_h(m,n). \qquad \textbf{(ii)}$$

(c) In view of the orthogonality relations (b) and the unimodularity of h and th the claims for the restrictions of b_h and b'_h follow immediately. By the formula (**ii**) above we have for $m \in M_\lambda, n \in M_{\lambda^{-1}}$:

$$(b_h \pm b'_h)(m,n) = (1\pm\lambda)b_h(m,n)$$
$$(b_h \pm b'_h)(n,m) = (1\pm\lambda^{-1})b_h(n,m).$$

Since $1\pm\lambda, 1\pm\lambda^{-1} \in (R')^\times$ by the remark following (**i**) above, the claim follows also for the restrictions of $b_h \pm b'_h$. □

Lemma 2.6. *For an R-Θ-semisimple $h \in GL_n(R)$ with decomposition $h = p+q$, where p is skew-symmetric and q symmetric, we have a direct sum decomposition for $M = R^n$*

$$M = M_+ \oplus M_- \oplus M_0,$$

where $M_+ = \ker p, M_- = \ker q$ and $M_0 = (M_+)^{\perp_q} \cap (M_-)^{\perp_p}$ is the intersection of the orthogonal complement of M_+ with the symplectic orthogonal complement of M_-. The restrictions

$$q_+ = q \mid M_+\times M_+, \qquad p_- = p \mid M_-\times M_-,$$
$$q_0 = q \mid M_0\times M_0, \qquad p_0 = p \mid M_0\times M_0$$

are unimodular.

Proof: We identify the matrices $p, q \in \mathrm{Mat}_n(R)$ with the forms b_p, b_q. We take an extension R'/R as in Lemma 2.5(a) and compute

$$M'_{\pm 1} = \{m \in M' \mid {}^th^{-1}\cdot h\cdot m = \pm m\} = \{m \in M' \mid hm = \pm{}^thm\} = \ker(h \mp {}^th).$$

This means $M'_1 = \ker(p \mid M')$ and $M'_{-1} = \ker(q \mid M')$ and implies $M'_1 = M_+\otimes_R R', M'_{-1} = M_-\otimes_R R'$. Since unimodularity can be checked after the extension R'/R and b_h restricts to q_+ resp. p_- on M_+ resp. M_-, we conclude from Lemma 2.5(c) that q_+ and p_- are unimodular. Then it is clear that we have the claimed decomposition in (orthogonal and symplectic orthogonal) direct summands. By Lemma 2.5(b) we get $M_0\otimes_R R' = \bigoplus_{\lambda\neq\pm1} M'_\lambda$. By Lemma 2.5(c) again we conclude that the restrictions of $b_h + b'_h = 2q$ and $b_h - b'_h = 2p$ to this module are unimodular. So p_0 and q_0 are unimodular. □

Lemma 2.7 (Cayley transformation)**.** *Let $p \in GL_n(R)$ be a skew-symmetric matrix. Let $Sym_n(R)_{p-ess}$ denote the set of symmetric matrices q such that $q \pm p \in GL_n(R)$ and $Sp(p, R)_{ess}$ the set of symplectic transformations b such that $b - 1 \in GL_n(R)$. Then the following holds:*

(a) We have a bijection

$$C : Sym_n(R)_{p-ess} \to Sp(p, R)_{ess}, \quad q \mapsto (q-p)^{-1} \cdot (q+p) = N_l(p+q).$$

The inverse map is $C^{-1} : b \mapsto p \cdot (b+1) \cdot (b-1)^{-1}$.

(b) C induces a bijection between those elements q of $Sym_n(R)_{p-ess}$, for which $p+q$ is R-Θ-semisimple, and the R-semisimple elements of $Sp(p, R)_{ess}$.

(c) The map C satisfies $C({}^t g \cdot q \cdot g) = g^{-1} \cdot C(q) \cdot g$ for $g \in Sp(p, R)$.

Proof: (a) For $q \in Sym_n(R)_{p-ess}$ we put $h = p + q$ and $b = {}^t h^{-1} \cdot h$. We have

$$\begin{array}{llcll} & {}^t b \cdot h \cdot b & = & {}^t h \cdot h^{-1} \cdot h \cdot {}^t h^{-1} \cdot h \quad = \quad {}^t h \cdot {}^t h^{-1} \cdot h & \text{i.e.} \\ \textbf{(iii)} & {}^t b \cdot h \cdot b & = & h & \text{and by transposing} \\ \textbf{(iv)} & {}^t b \cdot {}^t h \cdot b & = & {}^t h. & \end{array}$$

Subtracting the last two equations we get ${}^t b \cdot p \cdot b = p$, i.e. $b \in \mathrm{Sp}(p, R)$. Furthermore $b - 1 = (q-p)^{-1} \cdot ((p+q) - (q-p)) = (q-p)^{-1} \cdot 2p \in \mathrm{GL}_n(R)$ by the assumptions. The map C is therefore defined.
Conversely we get for $b \in \mathrm{Sp}(p, R)_{ess}$ and $q = p \cdot (b+1) \cdot (b-1)^{-1}$ the equivalences:

$$\begin{aligned} q = {}^t q &\Leftrightarrow p \cdot (b+1) \cdot (b-1)^{-1} = {}^t(b-1)^{-1} \cdot {}^t(b+1) \cdot (-p) \\ &\Leftrightarrow ({}^t b - 1) p (b+1) = ({}^t b + 1) p (1-b) \\ &\Leftrightarrow {}^t bpb + {}^t bp - pb - p = -{}^t bpb + {}^t bp - pb + p \\ &\Leftrightarrow {}^t bpb = p \Leftrightarrow b \in \mathrm{Sp}(p, R). \end{aligned}$$

Furthermore $q \pm p = p \cdot ((b+1) \pm (b-1)) \cdot (b-1)^{-1} \in \mathrm{GL}_n(R)$ since $(b-1)^{-1}, 2b, 2, p \in \mathrm{GL}_n(R)$. Therefore the map C^{-1} is also well defined. An easy calculation (as in the case of the usual Cayley transform) shows that the maps C and C^{-1} are inverse to another in their domain of definition.

(b) Since $C(q) = N_l(p+q) = (p+q)^{-1} \cdot N(p+q) \cdot (p+q)$ this follows from the definition of R-Θ-semisimplicity.

(c) We have $C({}^t g \cdot q \cdot g) = ({}^t gqg - p)^{-1}({}^t gqg + p) = g^{-1}(q-p)^t g^{-1} \cdot {}^t g(q+p)g = g^{-1} \cdot (q-p)^{-1} \cdot (q+p) \cdot g = g^{-1} \cdot C(q) \cdot g$ for $g \in Sp(p, R)$. □

Lemma 2.8. *If p is a unimodular symplectic form on a free R-module N and $b \in Sp(p, R)$ is R-semisimple then there exists a b-invariant and with respect to p orthogonal direct sum decomposition $N = N_1 \oplus N_*$ such that b acts as identity on N_1 and $b|N_* \in Sp(p_*, R)_{ess}$, where p_* is the restriction of p to N_*.*

Proof: We argue as before: By lemma 2.5(a) we have for some finite étale ring extension R'/R a decomposition of $N' = N \otimes_R R'$ into eigenspaces of b: $N' = \bigoplus N'_\lambda$, where b acts as the scalar λ on N'_λ. As in lemma 2.5(b) we can see, that $p(N'_\lambda, N'_\mu) = 0$ unless $\lambda \cdot \mu = 1$. This implies that p is unimodular on N'_1 and therefore on N_1, thus N is the direct sum of N_1 and the p-orthogonal complement N_* of N_1. Since b is a symplectic transformation, it leaves N_* invariant. By the orthogonality relations for the N_λ we have $N_* \otimes_R R' = \bigoplus_{\lambda \neq 1} N'_\lambda$. Since $\lambda - 1 \in (R')^\times$ for $\lambda \neq 1$ the endomorphism $b - 1$ of N_* induces an automorphism of $N_* \otimes_R R'$ and is therefore itself an automorphism of N_*, □

Lemma 2.9. *Let $h = p + q \in GL_n(R)$ be R-Θ-semisimple. Let $G^{h,\Theta}(R) = \{g \in GL_n(R) | {}^t g \cdot h \cdot g = h\}$. Then the following holds:*

(a) With the notations of lemma 2.6 and of lemma 2.7 we have

$$\begin{aligned} G^{h,\Theta}(R) &= O(q_+, R) \times Sp(p_-, R) \times (Sp(p_0, R) \cap O(q_0, R)) \\ &\cong O(q_+, R) \times (Sp(p_- \oplus p_0, R) \cap O(q_- \oplus q_0, R)) \\ &\cong O(q_+, R) \times Cent\,(C(q_- \oplus q_0), Sp(p_- \oplus p_0, R)) \,. \end{aligned}$$

(b) In the situation and with the notations of lemma 2.5 we have moreover

$$\begin{aligned} (Sp(p_0, R') \cap O(q_0, R')) &= \left\{ (\phi_\lambda) \in \prod_{\lambda \neq \pm 1} GL(M'_\lambda) \,\middle|\, \phi_{\lambda^{-1}} = {}^t\phi_\lambda \text{ for all } \lambda \right\} \\ &\cong \prod_{\lambda \in \mathcal{L}} GL(M'_\lambda) \end{aligned}$$

where $\phi_{\lambda^{-1}} = {}^t\phi_\lambda$ means that $b_h(\phi_{\lambda^{-1}} m_{\lambda^{-1}}, \phi_\lambda m_\lambda) = b_h(m_{\lambda^{-1}}, m_\lambda)$ for all $m_{\lambda^{-1}} \in M'_{\lambda^{-1}}, m_\lambda \in M'_\lambda$ and where $\mathcal{L}$ denotes a subset of the set of all $\lambda \neq \pm 1$, which takes from every pair $\{\lambda, \lambda^{-1}\}$ exactly one member.

(c) $(Sp(p_- \oplus p_0) \cap O(q_- \oplus q_0)) \cong Cent\,(C(q_- \oplus q_0), Sp(p_- \oplus p_0))$ is a connected reductive smooth group scheme $/R$ with connected special fiber, which becomes split over the finite étale extension R'/R.

(d) We have in the situation of 4.1

$$Cent(\mathcal{N}(h), Sp_{2n}) \cong Sp_{2(n-g)} \times Cent\,(C(q_- \oplus q_0), Sp(p_- \oplus p_0))$$

where $2g$ is the rank of $M_- \oplus M_0$.

(e) To obtain the intersections of $G^{h,\Theta}(R)$ with $SL_n(R)$ one has only to replace $O(q_+, R)$ by $SO(q_+, R)$ on the right hand sides of (a).

Proof: (a) Since every $g \in G^{h,\Theta}(R)$ stabilizes the decomposition of lemma 2.6 one immediately gets the first two isomorphisms. The last one follows from lemma 2.7(c).

(b) Every $g \in G^{h,\Theta}(R)$ centralizes $N_l(h)$ and therefore has to respect the decomposition of $M_0 \otimes_R R'$ in eigenspaces of $N_l(h)$. The first description of $\mathrm{Sp}(p_0, R') \cap \mathrm{O}(q_0, R')$ follows now from 2.5(b). Since b_h is unimodular on $M'_{\lambda^{-1}} \oplus M'_\lambda$ it induces an identification of $M'_{\lambda^{-1}}$ with the dual space of M'_λ. This means that ϕ_λ can vary through the whole $\mathrm{GL}(M'_\lambda)$, while $\phi_{\lambda^{-1}}$ is then uniquely determined as the inverse of its adjoint.

We remark that the condition $\phi_\lambda = {}^t\phi_{\lambda^{-1}}$ is equivalent to the condition $\phi_{\lambda^{-1}} = {}^t\phi_\lambda$ and gives no extra restrictions. This is clear since we have $b_h(m_\lambda, m_{\lambda^{-1}}) = b'_h(m_{\lambda^{-1}}, m_\lambda) = \lambda \cdot b_h(m_{\lambda^{-1}}, m_\lambda)$ for $m_{\lambda^{-1}} \in M'_{\lambda^{-1}}, m_\lambda \in M'_\lambda$ by (**ii**), so the two possible identifications of $M'_{\lambda^{-1}}$ with the dual of M'_λ differ by a scalar and create the same adjoint. The last isomorphism follows.

(c) This follows from (a) and (b).

(d) This follows from the definition of $\mathcal{N}$ by the remark, that an element of $Cent(b, \mathrm{Sp}_{2n}(R))$ has to respect the decomposition of lemma 2.8.

(e) is clear, since symplectic transformations have determinant 1. □

Lemma 2.10. *Let G/R where $R = \mathcal{O}_F$ be a connected reductive group with connected special fiber $G \times_{\mathcal{O}_F} \kappa$ and $b \in G(R)$ be R-semisimple.*
If $b' = h_F^{-1} \cdot b \cdot h_F \in G(R)$ for some $h_F \in G(\bar{F})$ then there exists $h_R \in G(R)$ with $b' = h_R^{-1} \cdot b \cdot h_R$.

Proof: This follows from [K3, Prop. 7.1.]. □

Lemma 2.11. *Let $R = \mathcal{O}_F$ and $h \in GL_n(R)$ be R-Θ-semisimple and $h' = {}^t g_F \cdot h \cdot g_F \in GL_n(R)$ for some $g_F \in GL_n(\bar{F})$. Then we have:*

- *(a) If additionally $\det(g_F) \in F^\times$ there exists $g_R \in GL_n(R)$ with $h' = {}^t g_R \cdot h \cdot g_R$.*
- *(b) If we only assume $g_F \in GL_n(\bar{F})$ and if n is odd there exist $g_R \in GL_n(R)$ and $\epsilon \in \mathcal{O}_F^\times$ such that $h' = \epsilon \cdot {}^t g_R \cdot h \cdot g_R$.*
- *(c) We get the statement of (a) if we additionally assume that the discriminants of q_+ and q'_+ coincide in $R^\times/(R^\times)^2$.*
- *(d) Under the additional conditions $h, h' \in SL_n(\mathcal{O}_F)$, $g_F \in SL_n(\bar{F})$ and n odd we can find $g_R \in SL_n(R)$ with $h' = {}^t g_R \cdot h \cdot g_R$.*

Proof: We use the objects occurring in lemma 2.6 for h and denote the corresponding objects for h' by a $'$. We have $rank(M_+) = \dim(M_+ \otimes_R F) = \dim(M'_+ \otimes_R F) = rank(M'_+)$. By transforming h and h' with elements of $\mathrm{GL}_n(R)$ we can therefore assume (using lemma 2.3) that

$$\textbf{(v)} \qquad M_+ = M'_+ = R^m, \quad M_0 \oplus M_- = M'_0 \oplus M'_-, \quad p_* := p_0 \oplus p_- = p'_0 \oplus p'_-.$$

The assumption and lemma 2.7(c) (applied in the case $R = \bar{F}$) now imply that the elements $C(0 \oplus q_0)$ and $C(0 \oplus q_0')$ of $\mathrm{Sp}(p_*, R)$ are conjugate by an element of $\mathrm{Sp}(p_*, \bar{F})$. By lemma 2.10 they are conjugate by an element $g_* \in Sp(p_*, R)$, hence we get from lemma 2.7(c) the equality $q_0' = {}^t g_* \cdot q_0 \cdot g_*$ and therefore $p_0' + p_-' + q_0' = {}^t g_*(p_0 + p_- + q_0) g_*$ in $M_0 \oplus M_-$. We have $\det(q_+') = \det(h') \cdot \det(p_0' + p_-' + q_0')^{-1} = \det(g_F)^2 \cdot \det(h) \cdot \det(p_0 + p_- + q_0)^{-1} = \det(g_F)^2 \cdot \det(q_+)$ (observe $\det g_* = 1$).

If case (a) we conclude using $R^\times \cap (F^\times)^2 = (R^\times)^2$ and lemma 2.4, that q_+' and q_+ are transformed via an element $g_+ \in \mathrm{GL}_n(M_+)$, a statement which has been an additional assumption in (c) in view of lemma 2.4. We put g_* and g_+ together to $g_R \in \mathrm{GL}_n(R)$ which does the required job in cases (a) and (c).

We prove (b) for $\epsilon = \det(q_+')/\det(q_+)$: We have $h" := \epsilon^{-1} h' = {}^t g_F' \cdot h \cdot g_F'$ for $g_F' = \sqrt{\epsilon^{-1}} \cdot g_F \in \mathrm{GL}_n(\bar{F})$. If $2r+1$ is the rank of M_+' we have $\det(q"_+) = \det(q_+') \cdot \epsilon^{2r+1} = \det(q_+) \cdot \epsilon^{2r}$. Thus the additional assumption of (c) is fulfilled and we get $g_R \in \mathrm{GL}_n(R)$ with $h" = {}^t g_R \cdot h \cdot g_R$.

To prove (d) observe at first that we can assume the matrices transforming h and h' into the standard form (**v**) being in $\mathrm{SL}_n(R)$ since one can modify them by elements of $\mathrm{GL}(M_+)$ and since $\mathrm{rank}(M_+) \geq 1$. From $\det(g_F) = 1$ we furthermore get $\det(q_+') = \det(q_+)$ and therefore $\det(g_+) = \pm 1$. Since we can replace g_+ by $-g_+$ if necessary and $\mathrm{rank}(M_+)$ is odd we can achieve $\det(g_+) = 1$ and therefore $\det(g_R) = 1$. □

3 Orbital integrals

(3.1) Orbital integrals. For elements $\gamma \in \tilde{G}(F)$, $f \in \mathcal{C}_c^\infty(\tilde{G}(F))$ we define the orbital integral by:

$$O_\gamma(f, \tilde{G}(F)) = \int_{G(F)/G(F)^\gamma} f(x\gamma x^{-1}) dx/dx^\gamma$$

where $G(F)^\gamma$ denotes the centralizer of γ in $G(F)$ and where we have chosen Haar measures dx resp. dx^γ on $G(F)$ resp. $G(F)^\gamma$ such that

$$vol_{dx}(G(\mathcal{O}_F)) = 1 \quad \text{and} \quad vol_{dx^\gamma}((G^\gamma)^\circ(\mathcal{O}_F)) = 1.$$

If 1_K denotes the characteristic function of a compact open subset $K \subset G(F)$, we will use the following abbreviation:

$$O_\gamma(1, \tilde{G}) = O_\gamma(1_{\tilde{G}(\mathcal{O}_F)}, \tilde{G}(F))$$

We further introduce stable orbital integrals

$$\begin{aligned} O_\gamma^{st}(f, \tilde{G}(F)) &= \sum_{\gamma' \sim \gamma} O_{\gamma'}(f, \tilde{G}(F)) \qquad \text{respectively} \\ O_\gamma^{st}(1, \tilde{G}) &= \sum_{\gamma' \sim \gamma} O_{\gamma'}(1_{\tilde{G}(\mathcal{O}_F)}, \tilde{G}(F)) \end{aligned}$$

where γ' runs through a set of representatives for the conjugacy classes inside the stable conjugacy class of γ.

(3.2) Recall the construction of the **quotient measure** dg/dh on G/H for totally disconnected locally compact groups $H \subset G$, where H is unimodular (e.g. H is the set of F-valued points of a reductive group). One defines

$$vol(K\gamma H/H) \quad = \quad \int_{G/H} 1_{K\gamma H/H}(g)dg/dh \quad := \quad \frac{vol_{dg}(K)}{vol_{dh}(\gamma^{-1}K\gamma \cap H)},$$

where $K \subset G$ is any open compact subgroup, and extends this by linearity to the space of all locally constant compactly supported functions on G/H. It is well known and easy to check that this definition does not depend the choice of K.

The crucial statement we need in the following is the following type of a fundamental lemma:

Conjecture 3.3. *If the regular algebraically semisimple and topologically unipotent elements $u \in SO_{2n+1}(F)$ and $v \in Sp_{2n}(F)$ are BC-matching (see 1.12) then*

$$(BC_n) \qquad O_u^{st}(1, SO_{2n+1}) = O_v^{st}(1, Sp_{2n}).$$

The (easy) case (BC_1) is proved in [Fl1, Stable case I in Proof of Theorem]. The case (BC_2) is essentially proved in [Fl2, Part II], as will be explained in 5.9.

Warning: While (BC_1) is an immediate consequence of the exceptional isogeny $i_2 : \mathrm{Sp}_2 = \mathrm{SL}_2 \twoheadrightarrow \mathrm{PGL}_2 = \mathrm{SO}_3$ and the fact, that γ^2 and $i_2(\gamma)$ are BC-matching for $\gamma \in \mathrm{SL}_2(F)$, the statement ($\mathrm{BC}_2$) is much deeper, since the exceptional isogeny $i_4 : \mathrm{Sp}_4 \twoheadrightarrow \mathrm{SO}_5$ does not satisfy the analogous matching property.

Remark 3.4. It follows immediately from the construction in 1.12 that we have a bijection between F-rational conjugacy classes in $\mathrm{SO}_{2n+1}(\bar{F})$ and in $\mathrm{Sp}_{2n}(\bar{F})$. By the theorem of Steinberg each F-rational conjugacy class in $\mathrm{Sp}_{2n}(\bar{F})$ contains a rational element, since Sp_{2n} is quasisplit and simply connected. But the same statement holds for F-rational topologically unipotent conjugacy classes in $\mathrm{SO}_{2n+1}(\bar{F})$ as well: If $u \in \mathrm{SO}_{2n+1}(\bar{F})$ is topologically unipotent and represents an F-rational conjugacy class, consider its two preimages v_1 and $v_2 = -v_1$ in Spin_{2n+1}. Since $p \neq 2$ we have $\lim_{n\to\infty} v_2^{p^n} = -\lim_{n\to\infty} v_1^{p^n}$, so that exactly one of the elements v_1, v_2 is topologically unipotent, say v_1. Since the Galois group respects the property to be topologically unipotent, the conjugacy class of v_1 is F-rational and therefore contains an F-rational element v' by the theorem of Steinberg. The image of v' in $\mathrm{SO}_{2n+1}(F)$ is the desired F-rational representative of the conjugacy class of u.

Thus to every topologically unipotent element in $v \in \mathrm{Sp}_{2n}(F)$ is associated at least one BC-matching $u \in \mathrm{SO}_{2n+1}(F)$ and vice versa.

Lemma 3.5 (Kazhdan-Lemma).

(a) *For $\tilde{G} = G \rtimes \langle\Theta\rangle$ as in 1.2 let us assume that the following statement holds:*

(∗) *If $s_1\Theta$ and $s_2\Theta$ for $s_1, s_2 \in G(\mathcal{O}_F)$ are residually semisimple and conjugate by an element of $G(F)$ then they are also conjugate by an element of $G(\mathcal{O}_F)$.*

If $\gamma\Theta = u{\cdot}s\Theta = s\Theta{\cdot}u$ is a topological Jordan decomposition, where $\gamma \in G(\mathcal{O}_F)$, u is topologically unipotent and $s\Theta$ residually semisimple, we have

$$O_{\gamma\Theta}(1, \tilde{G}) \quad = \quad \frac{1}{[G^{s\Theta}(\mathcal{O}_F) : (G^{s\Theta})^\circ(\mathcal{O}_F)]} \cdot O_u(1, G^{s\Theta})$$

(b) *Let $H/\mathcal{O}_F$ be connected reductive with connected special fiber. For $h \in H(\mathcal{O}_F)$ with topological Jordan decomposition $h = v \cdot b = b \cdot v$, where v is topologically unipotent and b residually semisimple, we have*

$$O_h(1, H) \quad = \quad \frac{1}{[H^b(\mathcal{O}_F) : (H^b)^\circ(\mathcal{O}_F)]} \cdot O_v(1, H^b)$$

Proof: (a) We first prove:

(**) We have $g\gamma\Theta g^{-1} \in G(\mathcal{O}_F)\Theta$ if and only if g is of the form $g = k \cdot x$ where $k \in G(\mathcal{O}_F)$ and $x \in G^{s\Theta}(F)$ satisfies $xux^{-1} \in G^{s\Theta}(\mathcal{O}_F)$.

The direction "$\Leftarrow$" is easy: Under the hypothesis we have $g\gamma\Theta g^{-1} = kxus\Theta x^{-1}k^{-1} = k(xux^{-1})(s\Theta)k^{-1} \in G(\mathcal{O}_F)$. For the converse direction "$\Rightarrow$" let us assume that $g\gamma\Theta g^{-1} \in G(\mathcal{O}_F)\Theta$. The topological Jordan decomposition is $g\gamma\Theta g^{-1} = (gug^{-1}) \cdot (gs\Theta g^{-1})$. Since $\langle G(\mathcal{O}_F), \Theta\rangle$ is a closed subgroup of $\widetilde{G}(F)$ we conclude from 1.18(4) that $gs\Theta g^{-1} \in G(\mathcal{O}_F)\Theta$ and $gug^{-1} \in G(\mathcal{O}_F)$. By the first inclusion and assumption (∗) we get an element $k \in G(\mathcal{O}_F)$ such that $g(s\Theta)g^{-1} = k(s\Theta)k^{-1}$, which implies $x = k^{-1} \cdot g \in G^{s\Theta}(F)$, where $G^{s\Theta}$ is the centralizer of $s\Theta$ in G. Using $g = kx$ the inclusion $gug^{-1} \in G(\mathcal{O}_F)$ is now equivalent to $xux^{-1} \in G(\mathcal{O}_F)$, which proves (∗∗).

The end of the proof is now a straightforward application of (**), the definition of the quotient measure and the formula

$$g^{-1} \cdot G(\mathcal{O}_F) \cdot g \cap G^{\gamma\Theta}(F) = g^{-1} \cdot G^{s\Theta}(\mathcal{O}_F) \cdot g \cap G^{\gamma\Theta}(F)$$

for $g \in G^{s\Theta}(F)$, which follows from $G^{s\Theta}(F) \cap G(\mathcal{O}_F) = G^{s\Theta}(\mathcal{O}_F)$.

(b) is the special case $\tilde{G} = G$, $\Theta = 1$, the assumption (∗) being satisfied by 2.10. □

The following lemmas will be useful in later chapters.

Lemma 3.6. *If $N \in \mathbb{N}$ is prime to p then we have for a reductive group $G/\mathcal{O}_F$ and $\gamma \in G(F)$*

$$O_{\gamma^N}(1, G) = O_\gamma(1, G).$$

For the proof notice that $g \cdot \gamma \cdot g^{-1}$ lies in the closure of $(g \cdot \gamma^N \cdot g^{-1})^{\mathbb{Z}}$ if $N \in \mathbb{Z}_p^\times$. $\square$

Lemma 3.7. *If $G/\mathcal{O}_F$ is of the form $G = G_1 \times Z$ with a reductive group G_1 and a finite group $Z \simeq Z(\mathcal{O}_F)$ then we have for $\gamma \in G_1(F) \subset G(F)$ the following identity of orbital integrals:*

$$O_\gamma(1, G) \quad = \quad O_\gamma(1, G_1).$$

$\square$

Lemma 3.8. *Let $1 \to T \to G \to H \to 1$ be an exact sequence of algebraic groups over $\mathcal{O}_F$ where T is a split torus. Then we have for $\gamma \in G(F)$ with image $\eta \in H(F)$:*

$$O_\gamma^{st}(1, G) = O_\eta^{st}(1, H).$$

Proof: We use the fact that the image of $(G^\gamma)^\circ$ in H is $(H^\eta)^\circ$. By Hilbert 90 we get exact sequences $1 \to T(F) \to G(F) \to H(F) \to 1$ and $1 \to T(F) \to (G^\gamma)^\circ(F) \to (H^\eta)^\circ(F) \to 1$, so that we have an isomorphism $G(F)/(G^\gamma)^\circ(F) \simeq H(F)/(H^\eta)^\circ(F)$. Since $(H^\eta)^\circ$ has finite index in H^η we can compute $O_\eta(1, H)$ as $\int_{H(F)/(H^\eta)^\circ(F)} 1_{H(\mathcal{O}_F)}(h\eta h^{-1}) dh/dh^\eta$. Similarly

$$O_\gamma(1, G) = \int_{G(F)/(G^\gamma)^\circ(F)} 1_{G(\mathcal{O}_F)}(g\gamma g^{-1}) dg/dg^\gamma.$$

Now the quotient measures on $G(F)/(G^\gamma)^\circ(F)$ and $H(F)/(H^\eta)^\circ(F)$ coincide since $G(\mathcal{O}_F) \twoheadrightarrow H(\mathcal{O}_F)$ and $(G^\gamma)^\circ(\mathcal{O}_F) \twoheadrightarrow (H^\eta)^\circ(\mathcal{O}_F)$, and we conclude $O_\gamma(1, G) = O_\eta(1, H)$.

It remains to check that the set St_γ of conjugacy classes inside the stable conjugacy class of γ maps bijectively to the corresponding set St_η associated to η. But in the following commutative diagram of abelian groups with exact rows and columns the map ι must be an isomorphism, since $H^1(F, T) = 1$:

$$\begin{tikzcd}
 & & H^1(F,T) \arrow[r, equal] \arrow[d] & H^1(F,T) \arrow[d] \\
1 \arrow[r] & St_\gamma \arrow[r] \arrow[d, "\iota"'] & H^1_{ab}(F, G^\gamma) \arrow[r] \arrow[d] & H^1_{ab}(F, G) \arrow[d] \\
1 \arrow[r] & St_\eta \arrow[r] & H^1_{ab}(F, H^\gamma) \arrow[r] \arrow[d] & H^1_{ab}(F, H) \arrow[d] \\
 & & H^2(F,T) \arrow[r, equal] & H^2(F,T).
\end{tikzcd}$$

Here $H^1_{ab}(F, .)$ denotes the abelianized cohomology of [Boro] which coincides for nonarchimedean F as a pointed set with the usual cohomology. $\square$

4 Comparison between $\mathbf{PGL}_{2n+1}$ and $\mathbf{Sp}_{2n}$

Recall (see 2.2) that R is either a field of characteristic 0 or the ring of integers $\mathcal{O}_F$ of a local non archimedean field F with residue characteristic $\neq 2$.

(4.1) The explicit norm map $\mathcal{N}$. Our final goal being the comparison of Θ-twisted stable orbital integrals on PGL_{2n+1} with stable orbital integrals on Sp_{2n}, we will represent elements of PGL_{2n+1} by elements of the groups GL_{2n+1} resp. SL_{2n+1}. Let $\mathrm{GL}_n(R)_{R\Theta ss}/traf$ resp. $\mathrm{SL}_n(R)_{R\Theta ss}/traf$ be the set of transformation classes of R-Θ-semisimple (see 2.2) elements of $h \in \mathrm{GL}_n(R)$ resp. $h \in \mathrm{SL}_n(R)$ under the transformations $h \mapsto {}^tghg$ for $g \in GL_n(R)$ resp $g \in SL_n(R)$. Similarly let $Sp_{2g}(R)_{Rss}/conj$ be the set of conjugacy classes of R-semisimple elements in $Sp_{2g}(R)$. We define a norm map

$$\mathcal{N} : \mathrm{GL}_{2n+1}(R)_{R\Theta ss}/traf \longrightarrow \mathrm{Sp}_{2n}(R)_{Rss}/conj$$

as follows: If $h = p + q \in \mathrm{GL}_{2n+1}(R)$ represents a class of the left hand side, we decompose $M = R^{2n+1} = M_+ \oplus M_- \oplus M_0$ as in lemma 2.6. We consider M_+ as the degenerate part of M with respect to p and denote the non degenerate part by $M_* := M_- \oplus M_0$. Since $p_* = p_- \oplus p_0$ is a unimodular form on M_* we can find a basis $(e_1, \ldots, e_g, f_g, \ldots, f_1)$ of M_* such that p_* has standard form with respect to this basis by lemma 2.3. Let P_* resp. Q_* be the matrix describing the (skew-) symmetric bilinear form p_* resp. $q_- \oplus q_0$ with respect to this basis (q_- is the zero form). Thus $P_* = J_{2g}$ and $\mathrm{Sp}(P_*) = \mathrm{Sp}_{2g}$. Now $\mathcal{N}(h)$ or more precisely the image of the class of h under the norm map $\mathcal{N}$ is defined to be the $\mathrm{Sp}_{2n}(R)$-conjugacy class of $1_{2(n-g)} \times C(Q_*) \in \mathrm{Sp}_{2(n-g)}(R) \times \mathrm{Sp}_{2g}(R) \subset \mathrm{Sp}_{2n}(R)$, where we use the Cayley-transform-map C from lemma 2.7.

Remark 4.2. In the situation where the decomposition $M = R^{2n+1} = M_+ \oplus M_*$ is of the form $M = R^{2(n-g)+1} \oplus R^{2g}$ the matrix h splits into the blocks $h_+ \in \mathrm{GL}_{2(n-g)+1}(R)$ and $h_* \in \mathrm{GL}_{2g}(R)$ so that $N_l(h_*) = {}^th_*^{-1} \cdot h_*$ is a symplectic transformation with respect to the alternating part p_* of h_*. Then $C(Q_*) \in \mathrm{Sp}_{2g}(R)$ is the conjugate of $N_l(h_*)$ by a matrix, which transforms p_* into the standard form J_{2g}.

Proposition 4.3. *Let R be as above. Then the following statements hold:*

(a) The map $\mathcal{N} : GL_{2n+1}(R)_{R\Theta ss}/traf \longrightarrow Sp_{2n}(R)_{Rss}/conj$ is well defined and surjective. In the case $R = \mathcal{O}_F$ its fibers are of order $2 = \#(R^\times/(R^\times)^2)$ and describe the two different classes of unimodular quadratic forms on M_+.

(b) The restriction $\mathcal{N}_{SL}$ of $\mathcal{N}$ to $SL_{2n+1}(R)_{R\Theta ss}/traf$ is surjective as well. It is bijective if R is an algebraically closed field or if $R = \mathcal{O}_F$.

(c) If h represents a class in $GL_{2n+1}(R)_{R\Theta ss}/traf$ then the image of $h \cdot J^{-1}\Theta$ in $PGL_{2n+1}(R) \rtimes \langle\Theta\rangle$ matches with $\mathcal{N}(h)$ in the sense of Θ-endoscopy.

Proof: (a) and (b) The choices made in constructing $\mathcal{N}(h)$ only allow Q_* to be replaced by some ${}^tg \cdot Q_* \cdot g$ for $g \in \mathrm{Sp}(P_*, R)$. By lemma 2.7(c) this does not change the conjugacy class of $\mathcal{N}(h)$. Therefore the map $\mathcal{N}$ is well defined. To prove surjectivity first observe that each class in $\mathrm{Sp}_{2n}(R)_{Rss}/conj$ has a representative of the form $(1_{2(n-g)}, b)$ with $b \in \mathrm{Sp}_{2g}(R)_{ess}$ by lemma 2.8 with a unique $g \leq n$. The $\mathrm{Sp}_{2g}(R)$-conjugacy-class of b is unique. The bijectivity of the Cayley-transform map and property 2.7(c) then imply that there is a $Q_* \in Sym_{2g}(R)$, which is unique up to transformations with elements of $\mathrm{Sp}_{2g}(R) = \mathrm{Sp}(P_*, R)$, such that $b = C(Q_*)$. Now we consider the unimodular bilinear form $h_* = P_* + Q_*$ on R^{2g} and some unimodular symmetric bilinear form q_+ on $R^{2(n-g)+1}$. The form $q_+ \oplus h_*$ on R^{2n+1} is then unimodular and R-Θ-semisimple. Since we can choose q_+ in such a way that $\det(q_+ \oplus h'_*) = 1$ we get the surjectivity statements of (a) and (b). Since the transformation class of h'_* is unique by the considerations above and since $h = q_+ \oplus h_*$ we conclude that the fibers of $\mathcal{N}$ correspond to the transformation classes of unimodular quadratic forms on M_+. The remaining statements of (a) and (b) now follow from lemma 2.4.

(c) By the definition of matching (1.11) we can work over $R = \bar{F}$ and therefore may assume that $\gamma = h \cdot J_{2n+1}^{-1}$ has diagonal form $\gamma = diag(t_1, \ldots, t_{2n+1})$. After applying a permutation in $W_{\mathrm{SO}_{2n+1}}$ we may assume

(i) $\quad t_i \neq t_{2n+2-i}$ for $i \leq g$ and $t_i = t_{2n+2-i}$ for $g+1 \leq i \leq 2n+1-g$.

We have:

$$\begin{aligned} h &= antidiag(t_1, -t_2, t_3, \ldots, t_{2n+1}) \\ h \pm {}^th &= antidiag(t_1 \pm t_{2n+1}, -t_2 \mp t_{2n}, t_3 \pm t_{2n-1}, \ldots, t_{2n+1} \pm t_1) \\ {}^th^{-1} \cdot h &= diag(t_{2n+1}/t_1, t_{2n}/t_2, \ldots, t_{n+2}/t_n, 1, t_n/t_{n+2}, \ldots, t_1/t_{2n+1}) \end{aligned}$$

This means that $M_+ \simeq R^{2(n-g)+1}$ is spanned by the standard basis elements $e_{g+1}, \ldots, e_{2n+1-g}$ of R^{2n+1}, and $M_* = M_- \oplus M_0$ by $e_1, \ldots, e_g, e_{2n+2-g}, \ldots, e_{2n+1}$. Since $h - {}^th$ is an antidiagonal matrix, its non degenerate part can be transformed by a diagonal matrix d into the standard form J_{2g}. Now we use remark 4.2 to get the following representative for $\mathcal{N}(h)$, observing that conjugation by d does not change a diagonal matrix:

$$diag(t_{2n+1}/t_1, t_{2n}/t_2, \ldots, t_{n+2}/t_n, t_n/t_{n+2}, \ldots, t_1/t_{2n+1}),$$

which may be conjugated by an element of the Weylgroup into the form

$$diag(t_1/t_{2n+1}, t_2/t_{2n}, \ldots, t_n/t_{n+2}, t_{n+2}/t_n, \ldots, t_{2n+1}/t_1).$$

The claim now follows from example 1.13. □

Corollary 4.4. *For every semisimple $\bar{\gamma}\Theta \in \widetilde{PGL}_{2n+1}(F)$ there exists a semisimple $\eta \in Sp_{2n}(F)$ matching with η in the sense of 1.11 and vice versa.*

Proof: If $\gamma \in \mathrm{GL}_{2n+1}(F)$ represents a given $\bar\gamma$ one applies part (c) of the proposition to $h = \gamma \cdot J_{2n+1}$. If η is given one applies (b) and (c). $\square$

Proposition 4.5. *Let $Z = Cent(GL_{2n+1}) \simeq \mathbb{G}_m$ denote the center of GL_{2n+1}. Let $\bar\gamma \in PGL_{2n+1}(F)$ be represented by $\gamma \in GL_{2n+1}(F)$. Since $2n+1$ is odd we can achieve that* $\det(\gamma)$ *has even valuation. Then*

$$\textbf{(ii)} \qquad O_{\bar\gamma\Theta}(1, \widetilde{PGL}_{2n+1}) \quad = \quad 2 \cdot O_{\gamma\Theta}(1, \widetilde{GL}_{2n+1}).$$

If moreover $\gamma\Theta$ is strongly compact with topological Jordan decomposition $\gamma\Theta = u \cdot (s\Theta) = (s\Theta) \cdot u$ we have $u \in SL_{2n+1}(F)$ and get

$$\textbf{(iii)} \qquad O_{\bar\gamma\Theta}(1, \widetilde{PGL}_{2n+1}) \quad = \quad O_u(1, SL^{s\Theta}_{2n+1})$$

Proof: The relation $\bar g \cdot \bar\gamma\Theta \cdot \bar g^{-1} \in \widetilde{\mathrm{PGL}_{2n+1}}(\mathcal{O}_F)$ means $g \cdot \gamma \cdot \Theta(g)^{-1} = \zeta \cdot k$ with $\zeta \in Z(F) \simeq F^*$ and $k \in \mathrm{GL}_{2n+1}(\mathcal{O}_F)$. Since $\det(\Theta(g)) = \det(g)^{-1}$ the relation implies taking determinants

$$\textbf{(iv)} \qquad \det(g)^2 \cdot \det(\gamma) \cdot \zeta^{-2n-1} \in \mathcal{O}_F^*.$$

This implies that ζ has even valuation $2m$ for $m \in \mathbb{Z}$, since the valuation of $\det(\gamma)$ was assumed to be even. If we replace g by $g' = \zeta_{\mathcal{O}} \cdot \varpi^{-m} \cdot g$ for $\zeta_{\mathcal{O}} \in \mathcal{O}_F^*$ we get $g' \cdot \gamma \cdot \Theta(g')^{-1} \in \mathrm{GL}_{2n+1}(\mathcal{O}_F)$. Conversely the equation (**iv**) implies that every $g' \in g \cdot Z(F)$ with this property must be of the stated form.

Next observe that the condition $\bar g \in \mathrm{PGL}^{\gamma\Theta}_{2n+1}(F)$ means that we have for some representative $g \in \mathrm{GL}_{2n+1}(F)$ of $\bar g$ and some $\zeta \in Z(F) \simeq F^*$ the relation $g\gamma\Theta(g)^{-1} = \zeta\gamma$. This implies the determinant equation: $\det(g)^2 = \zeta^{2n+1}$. Putting $\rho = \det(g)/\zeta^n \in Z(F)$ this implies $\zeta = \rho^2$ and $\det(g) = \rho^{2n+1}$. If we replace g by $\rho^{-1} \cdot g$ we get $g\gamma\Theta(g)^{-1} = \gamma$ and $\det(g) = 1$. The only other element in $g \cdot Z(F)$ having the first property is $-g$, but $\det(-g) = -1$. This means that we have isomorphisms

$$\begin{array}{ccll} \mathrm{GL}_{2n+1}(F)^{\gamma\Theta} & \xrightarrow{\sim} & \mathrm{SL}_{2n+1}(F)^{\gamma\Theta} \times \{\pm 1\} & \quad \text{and} \\ \mathrm{SL}_{2n+1}(F)^{\gamma\Theta} & \xrightarrow{\sim} & \mathrm{PGL}^{\bar\gamma\Theta}_{2n+1}(F). & \end{array}$$

Since the normalized Haar measure on $\mathrm{PGL}_{2n+1}(F)$ is the quotient of the normalized Haar measure on $\mathrm{GL}_{2n+1}(F)$ by the normalized Haar measure on $Z(F)$ (i.e. $vol(Z(\mathcal{O}_F)) = 1$) and since the normalized measure on $\mathrm{GL}_{2n+1}(F)^{\gamma\Theta}$ restricts to the normalized Haar measure on $\mathrm{PGL}^{\bar\gamma\Theta}_{2n+1}(F) \simeq \mathrm{SL}^{\gamma\Theta}_{2n+1}(F)$, the above considerations imply the relation (**ii**).

If $\gamma\Theta$ is strongly compact we can assume that $\gamma \in \mathrm{GL}_{2n+1}(\mathcal{O}_F)$ and apply lemma 3.5 to get

$$O_{\bar\gamma\Theta}(1, \widetilde{\mathrm{PGL}}_{2n+1}) \quad = \quad O_u(1, \mathrm{GL}^{s\Theta}_{2n+1})$$

observing that $[\mathrm{GL}^{s\Theta}_{2n+1}(\mathcal{O}_F) : (\mathrm{GL}^{s\Theta}_{2n+1})^\circ(\mathcal{O}_F)] = 2$. But since $\mathrm{GL}_{2n+1}(F)^{s\Theta} \simeq \mathrm{SL}_{2n+1}(F)^{s\Theta} \times \{\pm 1\}$ we can apply lemma 3.7 to conclude (**iii**). $\square$

Lemma 4.6. *Let $h = sJ \in GL_{2n+1}(\mathcal{O}_F)$ be R-Θ-semisimple and $b = (1_{2(n-g)}, b_*) \in Sp_{2n}(\mathcal{O}_F)$ a representing element of $\mathcal{N}(h)$ with $b_* \in Sp_{2g}(\mathcal{O}_F)_{ess}$. Since M_+ is of odd rank $2(n-g)+1$ we can identify (M_+, q_+) with $(\mathcal{O}_F^{2(n-g)+1}, \epsilon q_{sp})$ for some $\epsilon \in \mathcal{O}_F^\times$ and the standard splitform q_{sp}. Assume that we have BC-matching algebraically semisimple and topologically unipotent elements*

$$u_+ \in SO_{2(n-g)+1}(F) \simeq SO(q_+) \quad \text{and} \quad v_+ \in Sp_{2(n-g)}(F) \simeq \ker(b-1)(F) \cap Sp_{2n}(F)$$

and an additional algebraically semisimple and topologically unipotent element

$$u_* \in SO(q_*)(F) \cap Sp(p_*)(F) \simeq Cent(b_*, Sp_{2g}(F)).$$

Then the elements $\gamma\Theta = s\Theta \cdot (u_+, u_*) = (u_+, u_*) \cdot s\Theta \in PGL_{2n+1}(F)\Theta$ and $\eta := (v_+^2, u_*^2) \cdot b = b \cdot (v_+^2, u_*^2) \in Sp_{2n}(F)$ match.

Proof: As in the proof of lemma 4.1(c) we work in the case $F = \bar{F}$ and assume that γ resp. η lie in the diagonal tori. The same holds for the residually semisimple parts s resp. b and the topologically unipotent parts $u = (u_+, u_*)$ and $v = (v_+^2, u_*^2)$. As the matching of $s\Theta$ and b is already proved in 4.1(c) we only have to examine the topologically unipotent elements. We can make the assumption (**i**) and write

$$\begin{aligned} u_+ &= diag(w_{g+1}, \ldots, w_n, 1, w_n^{-1}, \ldots, w_{g+1}^{-1}) \in \mathrm{SO}_{2(n-g)+1}(\bar{F}) \\ u_* &= diag(w_1, \ldots, w_g, w_g^{-1}, \ldots, w_1^{-1}) \in Cent(b_*, \mathrm{Sp}_{2g}(\bar{F})) \end{aligned}$$

By the definition of BC-matching we can assume

$$v_+ = diag(w_{g+1}, \ldots, w_n, w_n^{-1}, \ldots, w_{g+1}^{-1}) \in \mathrm{Sp}_{2(n-g)}(\bar{F})$$

Now we get from the description of M_+ and M_* in the proof of lemma 4.1(c) taking everything together:

$$u = (w_1, \ldots, w_n, 1, w_n^{-1}, \ldots, w_1^{-1}) \qquad \text{resp.} \qquad v = (w_1^2, \ldots, w_n^2, w_n^{-2}, \ldots, w_1^{-2})$$

and the claim follows again from example 1.13. □

The statement of the following theorem is the **fundamental lemma** for semisimple elements in the stable endoscopic situation $(\mathrm{Sp}_{2n}, \widetilde{\mathrm{PGL}}_{2n+1})$. Recall that the fundamental lemma also predicts the vanishing of orbital integrals for those rational elements, which match with no rational elements on the other side. But in view of corollary 4.4 this case does not occur.

Theorem 4.7. *If the semisimple elements $\bar{\gamma}\Theta \in \widetilde{PGL}_{2n+1}(F)$ and $\eta \in Sp_{2n}(F)$ match in the sense of 1.11 and if conjecture (BC_m) is true for all $m \leq n$ then we have*

$$(\mathbf{v}) \qquad O^{st}_{\bar{\gamma}\Theta}(1, \widetilde{PGL}_{2n+1}) = O^{st}_{\eta}(1, Sp_{2n}).$$

Proof: Step 1 **(Reductions):** In the first step we will prove that the nonvanishing of one side of (**v**) implies the nonvanishing of the other side and that we can reduce to the situation where:
$\gamma \in \mathrm{GL}_{2n+1}(\mathcal{O}_F)$, $\eta \in \mathrm{Sp}_{2n}(\mathcal{O}_F)$
and the topological Jordan decompositions are of the form

$$\gamma\Theta = (u_+, u_*) \cdot s\Theta \quad \text{and } \eta = (v_+^2, v_*^2) \cdot b \qquad \text{such that}$$

b lies in $\mathcal{N}(h)$ where $h = s \cdot J_{2n+1}$,
u_+ and v_+ are BC-matching,
u_* can be identified with v_* under an isomorphism $Cent(b_*, \mathrm{Sp}_{2g}(\mathcal{O}_F)) \simeq Aut(h_*)$

So let us assume that the right hand side of (**v**) does not vanish. Then there exists $\eta' \in \mathrm{Sp}_{2n}(F)$ stably conjugate to η which has a nonvanishing orbital integral, i.e. can be conjugated into $\mathrm{Sp}_{2n}(\mathcal{O}_F)$. We can assume that $\eta' \in \mathrm{Sp}_{2n}(\mathcal{O}_F)$ and that its topological Jordan decomposition satisfies $\eta' = b' \cdot v' = v' \cdot b$ with residually semisimple $b' = (1_{2(n-g)}, b_*) \in \mathrm{Sp}_{2(n-g)}(\mathcal{O}_F) \times \mathrm{Sp}_{2g}(\mathcal{O}_F)$ and topologically unipotent v'. We write v' in the form $v' = ((v'_+)^2, (u'_*)^2)$ with $v'_+ \in \mathrm{Sp}_{2(n-g)}(\mathcal{O}_F)$ and $u'_* \in Cent(b_*, \mathrm{Sp}_{2g}(\mathcal{O}_F))$ using 1.18(3) and the general assumption $p \neq 2$. Thus we have nonvanishing $O_{h'}(1, \mathrm{Sp}_{2n})$ and get from the Kazhdan-lemma 3.5 and lemma 3.6:

$$\textbf{(vi)} \qquad \begin{aligned} O_{\eta'}(1, \mathrm{Sp}_{2n}) &= O_{v'}(1, Cent(b', \mathrm{Sp}_{2n})) \\ &= O_{(v'_+)^2}(1, \mathrm{Sp}_{2(n-g)}) \cdot O_{(u'_*)^2}(1, Cent(b_*, \mathrm{Sp}_{2g})) \\ &= O_{v'_+}(1, \mathrm{Sp}_{2(n-g)}) \cdot O_{u'_*}(1, Cent(b_*, \mathrm{Sp}_{2g})). \end{aligned}$$

Hence the stable orbital integral $O^{st}_{v'_+}(1, \mathrm{Sp}_{2(n-g)})$ (being the sum of integrals of nonnegative functions) is strictly positive.

By remark 3.4 there exists a BC-matching between v'_+ and some $u'_+ \in \mathrm{SO}_{2(n-g)+1}(F)$. Then the equation ($\mathrm{BC}_{n-g}$) implies that there exists $u_+ \in \mathrm{SO}_{2(n-g)+1}(F)$ with strictly positive orbital integral and BC-matching with v'_+, i.e. we can assume $u_+ \in \mathrm{SO}_{2(n-g)+1}(\mathcal{O}_F)$.

Let $h = sJ \in \mathrm{GL}_{2n+1}(\mathcal{O}_F)_{R\Theta ss}$ be a residually semisimple element with $\mathcal{N}(h) = b'$ and define the element $\gamma'\Theta = (u_+, u'_*) \cdot s\Theta = s\Theta \cdot (u_+, u'_*) \in \widetilde{\mathrm{GL}_{2n+1}}(\mathcal{O}_F)$. Here we identify the $Cent(s\Theta, \mathrm{GL}_{2n+1} \simeq G^{h,\Theta} \simeq \mathrm{O}(q_+, R) \times Cent\,(C(q_*), \mathrm{Sp}(p_*)) \simeq \mathrm{SO}_{2(n-g)+1} \times Cent(b_*, \mathrm{Sp}_{2g}$, so that $\widetilde{(u_+, u'_*)}$ can be viewed as an element of the left hand side. The element $\overline{\gamma'}\Theta \in \widetilde{\mathrm{PGL}_{2n+1}}(\mathcal{O}_F)$ matches with η' (and therefore also with η) by lemma 4.6 and therefore lies in the stable conjugacy class of $\gamma\Theta$.

If the left hand side of (**v**) does not vanish, it is immediate that there exists $\gamma'\Theta \in \widetilde{\mathrm{GL}_{2n+1}}(\mathcal{O}_F)$ in the stable conjugacy class of $\gamma\Theta$. By reversing the above arguments we see that there exists $\eta' \in \mathrm{Sp}_{2n}(\mathcal{O}_F)$ in the stable class of η. So excluding the tautological case that (**v**) means $0 = 0$ we may assume without loss of generality that $\gamma \in \mathrm{GL}_{2n+1}(\mathcal{O}_F)$ and $\eta \in \mathrm{Sp}_{2n}(\mathcal{O}_F)$. We may furthermore assume that $\gamma\Theta = (u_+, u_*) \cdot s\Theta$ and $\eta = (v_+^2, u_*^2) \cdot b$ are the topological Jordan decompositions with BC-matching u_+ and v_+ and matching residually semisimple $s\Theta$ and b.

Step 2 **(Calculation of the symplectic orbital integral):** If $\eta' \in \mathrm{Sp}_{2n}(F)$ is stable conjugate to η then the residually semisimple parts b' and b are stable conjugate as well. If η' has nonvanishing orbital integral then η' and therefore also b' can be conjugated into $\mathrm{Sp}_{2n}(\mathcal{O}_F)$, i.e. we can assume $b' \in \mathrm{Sp}_{2n}(\mathcal{O}_F)$. By the Kottwitz lemma 2.10 b' and b are conjugate over $\mathrm{Sp}_{2n}(\mathcal{O}_F)$ i.e. we can assume $b' = b$. This means that we obtain all relevant conjugacy classes in the stable conjugacy class of η if we let v'_+ vary through a set of representatives for the conjugacy classes inside the stable conjugacy class of v_+ in $\mathrm{Sp}_{2(n-g)}(F)$ and u'_* through a set of representatives for the conjugacy classes inside the stable conjugacy class of u_* in $Cent(b_*, \mathrm{Sp}_{2g})$. Then the corresponding η' are of the form

$$\eta' = b \cdot ((v'_+)^2, (u'_*)^2).$$

We get using (**vi**) and lemma 3.6:

$$\textbf{(vii)} \quad \begin{aligned} O^{st}_\eta(1, \mathrm{Sp}_{2n}) &= \sum_{v'_+ \sim v_+} O_{(v'_+)^2}(1, \mathrm{Sp}_{2(n-g)}) \cdot \sum_{u'_* \sim u_*} O_{(u'_*)^2}(1, Cent(b_*, \mathrm{Sp}_{2g})) \\ &= \sum_{v'_+ \sim v_+} O_{v'_+}(1, \mathrm{Sp}_{2(n-g)}) \cdot \sum_{u'_* \sim u_*} O_{u'_*}(1, Cent(b_*, \mathrm{Sp}_{2g})). \end{aligned}$$

Step 3 **(Calculation of the Θ-twisted orbital integral):** We can repeat this argument in the Θ-twisted situation, since by lemma 2.11(b) the class of the residually semisimple part $\bar{s}\Theta$ of $\bar{\gamma}\Theta$ is the only $\mathrm{PGL}_{2n+1}(F)$-conjugacy class inside the stable class of $\bar{s}\Theta$, which meets $\mathrm{PGL}_{2n+1}(\mathcal{O}_F)$. If we denote by u'_+ a set of representatives for the $\mathrm{SO}_{2(n-g)+1}(F)$-conjugacy classes in the stable class of $u_+ \in \mathrm{SO}_{2(n-g)+1}(\mathcal{O}_F)$ we therefore get using proposition 4.5

$$\textbf{(viii)} \quad \begin{aligned} O^{st}_{\bar{\gamma}\Theta}(1, \widetilde{\mathrm{PGL}}_{2n+1}) &= \sum_{(u'_+, u'_*) \sim (u_+, u_*)} O_{(u'_+, u'_*)}(1, \mathrm{SL}^{s\Theta}_{2n+1}) \\ &= \sum_{u'_+ \sim u_+} O_{u'_+}(1, \mathrm{SO}_{2(n-g)+1}) \cdot \sum_{u'_* \sim u_*} O_{u'_*}(1, Cent(b_*, \mathrm{Sp}_{2g})). \end{aligned}$$

Step 4 **(End of the proof):** Since v_+ and u_+ are BC-matching it only remains to apply (BC_{n-g}) in order to identify

$$\sum_{v'_+ \sim v_+} O_{v'_+}(1, \mathrm{Sp}_{2(n-g)}) \quad \text{with} \quad \sum_{u'_+ \sim u_+} O_{u'_+}(1, \mathrm{SO}_{2(n-g)+1}).$$

Thus the right hand sides of (**vii**) and (**viii**) coincide, and the proof of the theorem is finished.

□

5 Comparison between $\mathbf{GL}_{2n} \times \mathbf{GL}_1$ and $\mathbf{GSpin}_{2n+1}$

Lemma 5.1 (Cayley transformation again). *For a symmetric matrix $q \in GL_n(R)$ the following holds:*

(a) We have a bijection

$$\tilde{C} : Alt_n(R)_{q-ess} \to O(q,R)_{ess}, \quad p \mapsto (p-q)^{-1} \cdot (q+p) = -N_l(p+q)$$

between the set $Alt_n(R)_{q-ess}$ of skew-symmetric matrices p such that $p \pm q \in GL_n(R)$ and the set $O(q,R)_{ess}$ of orthogonal transformations b such that $b-1 \in GL_n(R)$. The inverse map is $\tilde{C}^{-1} : b \mapsto q \cdot (b+1) \cdot (b-1)^{-1}$.

(b) $\tilde{C}$ induces a bijection between those elements q of $Alt_n(R)_{q-ess}$, for which $p+q$ is R-Θ-semisimple, and the R-semisimple elements of $O(q,R)_{ess}$.

(c) The map $\tilde{C}$ satisfies $\tilde{C}({}^tg \cdot p \cdot g) = g^{-1} \cdot \tilde{C}(p) \cdot g$ for $g \in O(q,R)$.

(d) We have $\det(b) = (-1)^n$ for $b \in O(q,R)_{ess}$.

Proof: (a) For $p \in Alt_n(R)_{q-ess}$ we put $h = p+q$ and $b = (p-q)^{-1} \cdot (q+p) = -{}^th^{-1} \cdot h$ Adding the formulas (**iii**) and (**iv**) in the proof of lemma 2.7 we get $(-{}^tb) \cdot q \cdot (-b) = q$, i.e. $b \in \mathrm{O}(q,R)$.
Furthermore $b - 1 = (p-q)^{-1} \cdot ((p+q) - (p-q)) = (p-q)^{-1} \cdot 2q \in \mathrm{GL}_n(R)$ by the assumptions. The map $\tilde{C}$ is therefore defined.
Conversely we get for $b \in \mathrm{O}(q,R)_{ess}$ and $p = q \cdot (b+1) \cdot (b-1)^{-1}$ the equivalences:

$$\begin{aligned} p = -{}^tp &\Leftrightarrow q \cdot (b+1) \cdot (b-1)^{-1} = {}^t(b-1)^{-1} \cdot {}^t(b+1) \cdot (-q) \\ &\Leftrightarrow ({}^tb - 1)q(b+1) = ({}^tb+1)q(1-b) \\ &\Leftrightarrow {}^tbqb + {}^tbq - qb - q = -{}^tbqb + {}^tbq - qb + q \\ &\Leftrightarrow {}^tbqb = q \Leftrightarrow b \in \mathrm{O}(q,R). \end{aligned}$$

Furthermore $p \pm q = q \cdot ((b+1) \pm (b-1)) \cdot (b-1)^{-1} \in \mathrm{GL}_n(R)$ since $(b-1)^{-1}, 2b, 2, q \in \mathrm{GL}_n(R)$. Therefore the map $\tilde{C}^{-1}$ is also well defined. An easy calculation using the relation $(b+1) \cdot (b-1)^{-1} = (b-1)^{-1} \cdot (b+1)$ shows that the maps $\tilde{C}$ and $\tilde{C}^{-1}$ are inverse to another in their domain of definition.

(b) and (c) follow as in the proof of lemma 2.7.

(d) is clear since every $b \in \mathrm{O}_n(R)$ with $\det(b) = (-1)^{n-1}$ has 1 as an eigenvalue. (Alternatively we can use (a) and the computation $\det(-{}^th^{-1} \cdot h) = (-1)^n$.) □

Lemma 5.2. *If q is a unimodular symmetric bilinear form on a free R-module N and $b \in O(q,R)$ is R-semisimple then there exists a b-invariant q-orthogonal direct sum decomposition $N = N_1 \oplus N_*$ such that b acts as identity on N_1 and $b|N_* \in O(q_*,R)_{ess}$, where q_* is the restriction of q to N_*.*

Proof: The proof of lemma 2.8 can be adapted with obvious modifications. $\square$

(5.3) The explicit norm map $\mathcal{N}$. Let $(\mathrm{GL}_{2n}(R) \times R^\times)_{R\Theta ss}/traf$ be the set of transformation classes of R-Θ-semisimple elements $(h, a) \in \mathrm{GL}_{2n}(R) \times R^\times$ under the transformations $(h, a) \mapsto ({}^tghg, \det g^{-1} \cdot a)$ for $g \in GL_{2n}(R), a \in R^\times$. Similarly let $\mathrm{SO}_{2n+1}(R)_{Rss}/conj$ be the set of conjugacy classes of R-semisimple elements in $\mathrm{SO}_{2n+1}(R)$. We define a norm map

$$\mathcal{N} : (\mathrm{GL}_{2n}(R) \times R^\times)_{R\Theta ss}/traf \longrightarrow \mathrm{SO}_{2n+1}(R)_{Rss}/conj$$

as follows: If $(h, a) \in \mathrm{GL}_{2n}(R) \times R^\times$ represents a class of the left hand side and if $h = p+q$ is the decomposition in the symmetric part q and the skew-symmetric part p, we decompose $M = R^{2n} = M_+ \oplus M_- \oplus M_0$ as in lemma 2.6. The form $q_* = q_+ \oplus q_0$ on $M_* = M_+ \oplus M_0$ is unimodular. Since the ranks of M and M_- are even we have $M_* \simeq R^{2r}$ for some $r \in \mathbb{N}_0$. Let p'_* and q'_* be the $2r \times 2r$-matrices which describe p_* and q_* with respect to the standard basis ob R^{2r}. Let $\tilde{q}_-$ be a symmetric bilinear form on $\tilde{M}_- := R^{2(n-r)+1}$ such that $\Delta(q'_*) \cdot \Delta(\tilde{q}_-) \in (R^\times)^2$. By lemma 2.4 we have an isomorphism of quadratic spaces

$$i : (M_*, q_*) \oplus (\tilde{M}_-, \tilde{q}_-) \quad \xrightarrow{\sim} \quad (R^{2n+1}, J_{2n+1})$$

(observe $\det(J_{2n+1}) = 1$) which induces an injection

$$j : \mathrm{O}(M_*, q_*) \times \mathrm{O}(\tilde{M}_-, \tilde{q}_-) \quad \hookrightarrow \quad \mathrm{O}\left(M_* \oplus \tilde{M}_-,\ q_* \oplus \tilde{q}_-\right) \quad \xrightarrow{\sim} \quad \mathrm{O}_{2n+1}.$$

This injection is canonical (i.e. independent of the chosen isomorphism i) on the set of conjugacy classes.

Now $\mathcal{N}(h)$, the image of the class of h under $\mathcal{N}$, is defined to be the $\mathrm{O}_{2n+1}(R)$-conjugacy class of $j(\tilde{C}(p'_*), 1_{2(n-r)+1}) \in \mathrm{O}_{2n+1}(R)$, where we use the Cayley-transform-map $\tilde{C}$ with respect to q'_* from lemma 5.1. We observe that $\det(\tilde{C}(p'_*)) = 1$ by lemma 5.1(d) and therefore $\mathcal{N}(h)$ lies in $\mathrm{SO}_{2n+1}(R)$. Since the centralizer of $j(\tilde{C}(p'_*), 1_{2(n-r)+1})$ in $\mathrm{O}_{2n+1}(R)$ contains $\{1_{2r}\} \times \mathrm{O}_{2(n-r)+1}(R)$ i.e. elements of determinant -1, the $\mathrm{O}_{2n+1}(R)$-conjugacy class is in fact a $\mathrm{SO}_{2n+1}(R)$-conjugacy class.

Lemma 5.4. *In the notations of 5.3 the spinor norm of* $\mathcal{N}(h)$ *is the class of* $\det(h)$ mod $(R^\times)^2$.

Proof: It is sufficient to consider the case $R = F$, since we have an injection $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \hookrightarrow F^\times/(F^\times)^2$. If σ denotes the spinor norm of $\mathcal{N}(h)$ we have by a theorem of Zassenhaus (comp. [Zas]) in the version of [Mas]

$$\sigma \quad \equiv \quad \det\left(id - \tilde{C}(p'_*)\right) \cdot \Delta(q'_*) \mod (R^\times)^2.$$

But $id - \tilde{C}(p'_*) = (q'_* - p'_*)^{-1} \cdot 2 \cdot q'_*$ so that we get $\sigma \equiv \det(q'_* - p'_*)^{-1} \cdot 2^{2r} \equiv \det(q'_* - p'_*)$ mod $(R^\times)^2$. Furthermore $\det(q'_* - p'_*) = \det {}^t(q'_* - p'_*) = \det(p'_* + q'_*)$. Since the discriminant of p_- is a square we finally get $\sigma \equiv \det(p'_* + q'_*) \cdot \det(p_-) \equiv \det(h)$ mod $(R^\times)^2$. $\square$

Proposition 5.5.

(a) The map $\mathcal{N} : (GL_{2n}(R)\times R^\times)_{R\Theta ss}/traf \longrightarrow SO_{2n+1}(R)_{Rss}/conj$ is well defined and surjective. Two classes lie in the same fiber iff they have representatives of the form (h, a_1) and (h, a_2).

(b) If (h, a) represents a class in $(GL_{2n}(R) \times R^\times)_{R\Theta ss}/traf$ then $(h \cdot J^{-1}, a)\Theta \in (GL_{2n}(R) \times R^\times) \rtimes \langle\Theta\rangle$ matches in the sense of Θ-endoscopy with some element $\eta \in GSpin_{2n+1}(R)$, which maps to $\mathcal{N}(h)$ under the projection $pr_{ad} : GSpin_{2n+1} \to SO_{2n+1}$.

Proof: (a) If we replace p'_* by some ${}^tg \cdot p'_* \cdot g$ for $g \in \mathrm{O}(q'_*, R)$, this does not change the conjugacy class of $\mathcal{N}(h)$ by lemma 5.1(c). Since the effect of the other choices has already been considered, the map $\mathcal{N}$ is well defined.

To prove surjectivity first observe that each class $b \in \mathrm{SO}_{2n+1}(R)_{Rss}/conj$ can be represented after some transformation of J_{2n+1} in the form $(b', 1_{2(n-r)+1})$ with $b' \in \mathrm{SO}(q'_*, R)_{ess}$ by lemma 5.2 with a unique $r \leq n$ and some symmetric $q'_* \in \mathrm{GL}_{2r}(R)$. One should think of $(b', 1_{2(n-r)+1})$ as a block-matrix

$$\begin{pmatrix} B_{11} & 0 & B_{12} \\ 0 & 1_{2(n-r)+1} & 0 \\ B_{21} & 0 & B_{22} \end{pmatrix} \qquad \text{with} \qquad b' = \begin{pmatrix} B_{11} & B_{22} \\ B_{21} & B_{22} \end{pmatrix}$$

Since the class of $\Delta(q'_*)$ in $R^\times/(R^\times)^2$ is the inverse of the class of $\Delta(J_{2n+1}|\ker(b-1))$, the transformation class of q'_* is unique by lemma 2.4. Up to this the $\mathrm{SO}(q'_*, R)$-conjugacy-class of b' is unique. The bijectivity of the Cayley-transform map and property 5.1(c) then imply that there is a $p' \in Alt_{2r}(R)$, which is unique up to transformations with elements of $\mathrm{SO}(q'_*, R)$, such that $b = \tilde{C}(p')$. Now we consider the unimodular bilinear form $h' = p' + q'$ on R^{2r}, which is unique up to transformations with elements of $\mathrm{GL}_{2r}(R)$, and some unimodular skew symmetric form p_- on $R^{2(n-r)}$. The form $p_- \oplus h'$ on R^{2n} is then unimodular and R-Θ-semisimple i.e. corresponds to a R-Θ-semisimple transformation class h. For every R-semisimple $a \in R^\times$ we get $\mathcal{N}(h, a) = b$. Since the transformation class of h' is unique by the above considerations and by lemma 2.3 we conclude that the fibers of $\mathcal{N}$ correspond to the different choices for the R-semisimple element $a \in R^\times$.

(b) In the case that $R = \bar{F}$ is an algebraically closed field the claim can be obtained by an explicit calculation with diagonal and antidiagonal elements, where we finally use example 1.14.

In the case that R is arbitrary we consider the commutative diagram with exact rows and columns, which follows from the snake lemma:

$$\begin{array}{ccccccccccc}
 & & 1 & \longrightarrow & \mathrm{Spin}_{2n+1}(R) & = & \mathrm{Spin}_{2n+1}(R) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 1 \longrightarrow & R^\times & \longrightarrow & \mathrm{GSpin}_{2n+1}(R) & \xrightarrow{pr_{ad}} & \mathrm{SO}_{2n+1}(R) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \text{Spinnorm}} & & \\
1 \longrightarrow & \{\pm 1\} \longrightarrow & R^\times & \xrightarrow{r \mapsto r^2} & R^\times & \longrightarrow & R^\times/(R^\times)^2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

It follows from this diagram and lemma 5.4 that a matrix η_0 in the class $\mathcal{N}(h)$ has a preimage $\eta \in \mathrm{GSpin}_{2n+1}(R)$ such that $\mu(\eta) = \det(h) \cdot a^2$ and that the set $\{x \in \mathrm{GSpin}_{2n+1}(\bar{F}) | pr_{ad}(x) = \eta_0,\ \mu(x) = \det(h) \cdot a^2\}$ just consists of $\pm\eta$. On the other hand by example 1.14 an element $\eta' \in \mathrm{GSpin}_{2n+1}(\bar{F})$ matching with (h, a) satisfies $\mu(\eta') = \det(h) \cdot a^2$. From the validity of the proposition over $\bar{F}$ now follows that either η or $-\eta$ matches with (h, a). This element has all desired properties. □

Corollary 5.6. *For each semisimple $\eta \in GSpin_{2n+1}(F)$ there exists an F-Θ-semisimple $(h \cdot J^{-1}, a)\Theta \in (GL_{2n}(F) \times F^\times) \rtimes \langle\Theta\rangle$ matching with η.*

Proof: By 5.5(a) for $R = F$ there exists $(h, a_1) \in \mathrm{GL}_{2n}(F) \times F^\times$ with $pr_{ad}(\eta) \in \mathcal{N}(h)$ and by (b) there exists $\eta_1 \in \mathrm{GSpin}_{2n+1}(F)$ matching with $(h \cdot J^{-1}, a_1)\Theta$ such that $pr_{ad}(\eta_1) = pr_{ad}(\eta)$. It follows $\eta = \eta_1 \cdot b$ for some $b \in F^\times \simeq Center(\mathrm{GSpin}_{2n+1}(F))$. Then $(h \cdot J^{-1}, a_1 \cdot b)\Theta$ matches with η. □

Lemma 5.7. *For $G = GL_{2n} \times \mathbb{G}_m$ let $\gamma_1, \gamma_2 \in G(\mathcal{O}_F), g_F \in G(\bar{F})$ be such that $\gamma_2\Theta = g_F \cdot \gamma_1\Theta \cdot g_F^{-1}$ with Θ as in example 1.9. Then there exists $g_R \in G(\mathcal{O}_F)$ with $\gamma_2\Theta = g_R \cdot \gamma_1\Theta \cdot g_R^{-1}$.*

Proof: Write $\gamma_i = (h_i \cdot J_{2n}^{-1}, a_i)$, $g_F = (h_F, \tilde{a})$. Then the assumption means: $h_2 = h_F \cdot h_1 \cdot {}^t h_F$ and $a_2 = \tilde{a} \cdot a_1 \cdot \det(h_F)^{-1} \cdot \tilde{a}^{-1}$ which implies $\det(h_F) \in \mathcal{O}_F^\times$. By lemma 2.11(a) there exists $h_R \in \mathrm{GL}_{2n}(\mathcal{O}_F)$ with $h_2 = h_R \cdot h_1 \cdot {}^t h_R$. This implies $\det(h_R)^2 = \det(h_F)^2$. If $\det(h_R) = -\det(h_F)$ then $h_F^{-1} \cdot h_R \in \mathrm{O}(h_1)(\bar{F})$ where $\mathrm{O}(h_1) = \{h \in \mathrm{GL}_{2n} \mid {}^t h {\cdot} h_1 {\cdot} h = h_1\} = \mathrm{O}(q_{+,1}) \times (\mathrm{Sp}(p_{-,1} \oplus p_{0,1}) \cap \mathrm{O}(q_{-,1} \oplus q_{0,1}))$ has determinant -1. This implies $M_{+,1} \neq 0$ so that we get an element $h_\epsilon \in \mathrm{O}(h_1)(\mathcal{O}_F)$ of determinant -1. Replacing h_R by $h_R \cdot h_\epsilon$ we can now assume $\det(h_R) = \det(h_F)$. With $g_R = (h_R, 1)$ we now have $\gamma_2\Theta = g_R \cdot \gamma_1\Theta \cdot g_R^{-1}$. □

Lemma 5.8. *Let $(h, a) = (sJ, a) \in GL_{2n}(\mathcal{O}_F) \times \mathcal{O}_F^\times$ be $\mathcal{O}_F$-Θ-semisimple and $b = (1_{2(n-r)+1}, b_*) \in SO_{2n+1}(\mathcal{O}_F)$ a representing element of $\mathcal{N}(h)$. With p_*, q_*, p_- as in 5.3 assume that we have matching topologically unipotent elements $u_- \in Sp_{2(n-r)}(F) \simeq Sp(p_-)$ and $v_- \in SO_{2(n-r)+1}(F) \simeq$* $(\ker(b-1)(F) \cap SO_{2n+1}(F))$

and an additional topologically unipotent element $u_* \in SO(q_*)(F) \cap Sp(p_*)(F) \simeq Cent(b_*, SO(q_*)(F))$.
Then the element $\gamma\Theta = (s,a)\Theta\cdot(u_-,u_*) = (u_-,u_*)\cdot(s,a)\Theta \in (GL_{2n}(F) \times F^\times) \rtimes \langle\Theta\rangle$ matches with some element $\eta \in GSpin_{2n+1}(F)$, which projects to $\beta := (v_-^2, u_*^2)\cdot b = b\cdot(v_-^2,u_*^2) \in SO_{2n+1}(F)$.

Proof: It is easy to prove the existence of $\eta \in \mathrm{GSpin}_{2n+1}(\bar{F})$ with the desired properties by calculations with diagonal elements as in the proof of lemma 4.3(c).

To get η as an element of $\mathrm{GSpin}_{2n+1}(F)$ we observe that the determinant of γJ_{2n}^{-1} equals the spinor norm of β as an element of $F^\times/(F^\times)^2$: This is already clear by 5.4 for the residually semisimple parts, but both topologically unipotent parts lead to the neutral element in $F^\times/(F^\times)^2$, since $2 \neq p$ by assumption. Now one argues as in the proof of 5.5(b) to get η as an F-rational element. $\square$

Theorem 5.9. *(BC_2) is true.*

Proof: We observe that every pair of BC-matching (topologically unipotent) elements $\bar{\gamma} \in \mathrm{SO}_5(F)$ and $\eta_1 \in \mathrm{Sp}_4(F)$ can be obtained from a pair of (topologically unipotent) elements $\gamma \in \mathrm{GSp}_4(F) \simeq \mathrm{GSpin}_5(F)$ and $\eta\Theta = \Theta\eta \in (\widetilde{\mathrm{GL}_4 \times \mathrm{GL}_1})(F)$ such that $\bar{\gamma} = pr_{ad}(\gamma)$ and $\eta = (\eta_1, a) \in (\mathrm{GL}_4 \times \mathrm{GL}_1)^\Theta(F) \simeq (\mathrm{Sp}_4 \times \mathrm{GL}_1)(F)$ and such that γ^2 matches with $\eta\Theta$ in the sense of 1.11. This follows immediately from the definition of BC-matching 1.12 and example 1.14.

If we apply lemma 3.8 in the case $G = \mathrm{GSpin}_5 \simeq \mathrm{GSp}_4,\ T = \mathbb{G}_m, H = \mathrm{SO}_5$ and lemma 3.6 we get

$$O^{st}_{\gamma^2}(1, \mathrm{GSp}_4) = O^{st}_{\bar{\gamma}}(1, \mathrm{SO}_5)$$

Since we have $O^{st}_\eta(1, \mathrm{Sp}_4 \times \mathrm{GL}_1) = O^{st}_{\eta_1}(1, \mathrm{Sp}_4)$ by lemma 3.8 the statement of (BC_2) is equivalent to the identity

$$O^{st}_\gamma(1, \mathrm{GSp}_4) = O^{st}_\eta(1, \mathrm{Sp}_4 \times \mathrm{GL}_1)$$

for matching topologically unipotent $\gamma \in \mathrm{GSp}_4(F)$ and $\eta\Theta = \Theta\eta \in (\widetilde{\mathrm{GL}_4 \times \mathrm{GL}_1})(F)$. In the case that η is strongly Θ-regular this has been proved in [Fl2, ch. II]. The general case follows by the germ expansion principle as in [Hal2], [Rog]. $\square$

Corollary 5.10 (Fundamental lemma for $\mathrm{Sp}_4 \leftrightarrow \widetilde{\mathrm{PGL}_5}$)**.** *If $\gamma\Theta \in \widetilde{PGL_5}(F)$ and $h \in Sp_4(F)$ are matching semisimple elements then we have*

$$O^{st}_{\gamma\Theta}(1, \widetilde{PGL_5}) = O^{st}_h(1, Sp_4).$$

Proof: This follows from theorem 5.9, (BC_1) (compare 3.3) and theorem 4.7. $\square$

Theorem 5.11. *Let $G = GL_{2n} \times \mathbb{G}_m$. If $\gamma\Theta \in \tilde{G}(F)$ and $\eta \in GSpin_{2n+1}(F)$ are matching semisimple elements and if conjecture (BC_m) is true for all $m \leq n$ then we have*

(i)
$$O^{st}_{\gamma\Theta}(1, \tilde{G}) = O^{st}_\eta(1, GSpin_{2n+1}).$$

Proof: Let $h = pr_{ad}(\eta) \in \mathrm{SO}_{2n+1}(F)$. In view of lemma 3.8 we have to prove

(ii) $$O^{st}_{\gamma\Theta}(1, \tilde{G}) = O^{st}_h(1, \mathrm{SO}_{2n+1}).$$

The proof is now similar to the proof of Theorem 4.7, so that we leave the details to the reader:

In step Step 1 one has to prove that the nonvanishing of one side of (**i**) implies the nonvanishing of the other side and that we can reduce to the situation where $\gamma \in G(\mathcal{O}_F)$, $h \in \mathrm{SO}_{2n+1}(\mathcal{O}_F)$ and the topological Jordan decompositions are of the form $\gamma\Theta = (u_-, u_*) \cdot s\Theta$ and $h = (v_-^2, u_*^2) \cdot b$ where the residually semisimple $s\Theta$ matches with the residually semisimple part η_s of η, such that $b = pr_{ad}(\eta_s)$, with BC-matching $u_- \in \mathrm{Sp}_{2(n-r)}(\mathcal{O}_F)$ and $v_- \in \mathrm{SO}_{2(n-r)+1}(\mathcal{O}_F)$, and with $u_* \in Cent\,(b_*, \mathrm{SO}(q_*, \mathcal{O}_F))$. Here we write $b = (1_{2(n-r)+1}, b_*) \in \mathrm{SO}_{2(n-r)+1}(\mathcal{O}_F) \times \mathrm{SO}(q_*, \mathcal{O}_F)_{ess}$, where q_* denotes the restriction of J_{2n+1} to the orthogonal complement of $\ker(b-1) \simeq \mathcal{O}_F^{2(n-r)+1}$ but can be identified with the form q_* on the module M_* attached to $h = s \cdot J_{2n}$ in 5.3.

Step 2: As in the proof of theorem 4.7 we get from lemma 2.10 the fact that we obtain all relevant conjugacy classes in the stable conjugacy class of h if we let v'_- vary through a set of representatives of the stable conjugacy class of v_- in $\mathrm{SO}_{2(n-r)+1}(F)$ and u'_* through a set of representatives of the stable conjugacy class of u_* in $Cent(b_*, \mathrm{SO}(q_*))$ and then consider all $h' = b \cdot ((v'_+)^2, (u'_*)^2)$. In view of the identity

$$Cent(b, \mathrm{SO}_{2n+1}) \quad \simeq \quad \mathrm{SO}_{2(n-r)+1} \times Cent(b_*, \mathrm{SO}(q_*)) \times \{\pm 1\}$$

we can compute the orbital integral using the Kazhdan-lemma 3.5, lemma 3.7 and lemma 3.6 for $N = 2$:

(iii) $$O^{st}_h(1, \mathrm{SO}_{2n+1}) = \sum_{v'_+ \sim v_+} O_{v'_+}(1, \mathrm{SO}_{2(n-r)+1}) \cdot \sum_{u'_* \sim u_*} O_{u'_*}(1, Cent(b_*, \mathrm{SO}(q_*))).$$

Step 3: We can repeat this argument in the Θ-twisted situation, since by lemma 5.7 the class of the residually semisimple part $(s,a)\Theta$ of $\gamma\Theta$ is the only $G(F)$-conjugacy class inside the stable class of $(s,a)\Theta$, which meets $G(\mathcal{O}_F)$ and since the Kazhdan-Lemma 3.5 holds for $\tilde{G}$ by the same lemma. We remark that $G^{(s,a)\Theta} \simeq \mathrm{Sp}_{2(n-r)} \times \mathrm{Cent}(b_*, SO(q_*)) \times \mathbb{G}_m$ by the definition of Θ and lemma 2.9(e), so $G^{(s,a)\Theta}$ is connected and we can use lemma 3.8 to get rid of the $\mathbb{G}_m$ factors in the following orbital integrals. If we denote by u'_- a set of representatives for the $\mathrm{Sp}_{2(n-r)}(F)$-conjugacy classes in the stable class of $u_- \in \mathrm{Sp}_{2(n-r)}(\mathcal{O}_F)$ we get

(iv) $$O^{st}_{\tilde{\gamma}\Theta}(1, \tilde{G}) = \sum_{u'_- \sim u_-} O_{u'_-}(1, \mathrm{Sp}_{2(n-r)}) \cdot \sum_{u'_* \sim u_*} O_{u'_*}(1, Cent(b_*, \mathrm{SO}(q_*))).$$

Step 4: Since v_- and u_- are BC-matching, the theorem follows from (BC_{n-r}), (**iii**) and (**iv**).

$\square$

6 Comparison between $\mathbf{SO}_{2n+2}$ and $\mathbf{Sp}_{2n}$

Let R be as in 2.2.

Lemma 6.1. *Let N be a free R-module.*

(a) If p is a unimodular symplectic form on N and if $\beta \in Sp(p, R)$ is R-semisimple then there exists a β-invariant orthogonal (with respect to p) direct sum decomposition $N = N_+ \oplus N_- \oplus N_*$ such that β acts as identity on N_+, as $-id$ on N_- and $\beta_* = \beta|N_* \in Sp(p_*)$ satisfies $\beta_* - \beta_*^{-1} \in GL(N_*)$, where p_* is the restriction of p to N_*.

(b) If q is a unimodular symmetric bilinear form on N and $b \in O(q, R)$ is R-semisimple then there exists a b-invariant orthogonal (with respect to q) direct sum decomposition $N = N_+ \oplus N_- \oplus N_*$ such that b acts as identity on N_+, as $-id$ on N_- and $b_* = b|N_* \in O(q_*)$ satisfies $b_* - b_*^{-1} \in GL(N_*)$, where q_* is the restriction of q to N_*.

Proof: The proof of lemma 2.8 can be adapted with obvious modifications: We have $b - b^{-1} = b^{-1} \cdot (b-1) \cdot (b+1)$, so that $b - b^{-1} \in \mathrm{GL}(N_*)$ is equivalent to $b-1, b+1 \in \mathrm{GL}(N_*)$. $\square$

Lemma 6.2. *Let $b \in GL_n(R)$ satisfy $b - b^{-1} \in GL_n(R)$. Then the following holds:*

(a) If $q \in GL_n(R)$ is symmetric and $b \in O(q, R)$ then the matrix $p = q \cdot (b - b^{-1})$ is unimodular skew-symmetric and we have $b \in Sp(p, R)$.

(b) If $p \in GL_n(R)$ is skew-symmetric and $b \in Sp(p, R)$ then the matrix $q = p \cdot (b - b^{-1})^{-1}$ is unimodular symmetric and we have $b \in SO(q, R)$.

(c) Under the conditions of (a) and (b) we have:

$$Cent(b, O(q)) = Cent(b, Sp(p)) = Cent(b, SO(q)).$$

(d) The above statements and formulas are invariant under the substitutions $b \mapsto g^{-1}bg, q \mapsto {}^tgqg, p \mapsto {}^tgpg$ for $g \in GL_n(R)$.

The elementary proof is left to the reader as an exercise. $\square$

(6.3) The explicit norm map $\mathcal{N}$. If $s \in \mathrm{O}_{2n+2}$ with $\det(s) = -1$ denotes a reflection, we can identify the semidirect product $\mathrm{SO}_{2n+2} \rtimes \langle\Theta\rangle$ where $\Theta = int(s)$ with the orthogonal group O_{2n+2}.

Let $\mathrm{O}_{2n+2}(R)^-_{Rss}/conj$ be the set of $\mathrm{SO}_{2n+2}(R)$-conjugacy classes of R-semisimple (=R-Θ-semisimple) elements of $h \in \mathrm{O}_{2n+2}(R)$ with $\det(h) = -1$. Recall that $Sp_{2n}(R)_{Rss}/conj$ is the set of conjugacy classes of R-semisimple elements in $Sp_{2n}(R)$. We define a norm map

$$\mathcal{N} : \mathrm{O}_{2n+2}(R)^-_{Rss}/conj \longrightarrow \mathrm{Sp}_{2n}(R)_{Rss}/conj$$

as follows: If $b \in \mathrm{O}_{2n+2}(R)$ represents a class of the left hand side, we decompose $N = R^{2n+2} = N_+ \oplus N_- \oplus N_*$ as in lemma 6.1(b). Let $b_+ = id_{N_+}, b_- = -id_{N_-}$ and $b_* = b|N_*$. Let q_* be the restriction of the form J_{2n+2} to N_*. We may think of q_* as a symmetric matrix after introducing a basis of N_*. Since $b_* \in \mathrm{Sp}(p_*)$ for $p_* = q_* \cdot (b_* - b_*^{-1})$ by lemma 6.2(a) we have $\det(b_*) = 1$. Therefore $-1 = \det(b) = \det(b_+) \cdot \det(b_-) \cdot \det(b_*) = 1 \cdot (-1)^{rankN_-} \cdot 1$, i.e. $rank(N_-)$ is odd $= 1 + 2r_-$. Since $rank(N_*)$ is even by lemma 6.2 we have $rank(N_+) = 1 + 2r_+$ for some $r_+ \in \mathbb{N}_0$. Now we equip the R-module $M = M_+ \oplus M_- \oplus N_*$ where $M_+ \simeq R^{2r_+}, M_- \simeq R^{2r_-}$ with the alternating form $p = J_{2r_+} \oplus J_{2r_-} \oplus p_*$ and the linear automorphism $\beta = id_{M_+} \times -id_{M_-} \times b_* \in \mathrm{Sp}(p)$. Identifying the symplectic space $(M, p) \simeq (R^{2n}, J_{2n})$ we can think of β as an element of $\mathrm{Sp}_{2n}(R)$. The conjugacy class of β in $\mathrm{Sp}_{2n}(R)$ does not depend on the choices we made (apply lemma 6.2(d)) and is the desired $\mathcal{N}(b)$. It is clear that $\mathcal{N}(b)$ is R-semisimple.

Proposition 6.4. *Let R be as in 2.2.*

(a) *The map $\mathcal{N} : O_{2n+2}(R)^-_{Rss}/conj \longrightarrow Sp_{2n}(R)_{Rss}/conj$ is well defined. Each $b \in O_{2n+2}(R)^-_{Rss}/conj$ matches with $\mathcal{N}(b)$ in the sense of Θ-endoscopy (compare examples 1.10, 1.15).*

(b) *The map $\mathcal{N}$ is surjective, if $R = \mathcal{O}_F$. Its fibers are of order $2 = \#(R^\times/(R^\times)^2)$ and describe the two different pairs (q_+, q_-) of classes of unimodular quadratic forms on (M_+, M_-) such that $\Delta(q_+) \cdot \Delta(q_-) \equiv \det(q_*)^{-1} \mod (R^\times)^2$.*

Proof: (a)That $\mathcal{N}$ is well defined is already clear. By the definition of matching we can work over $R = \bar{F}$, so that we may assume that $\gamma = b \cdot s^{-1} \in \mathrm{SO}_{2n+2}(R)$ has diagonal form $\gamma = diag(t_1, \ldots, t_{n+1}, t_{n+1}^{-1}, \ldots, t_1^{-1})$, where s is the reflection defined in 1.10. With the standard basis $(e_i)_{1 \le i \le 2n+2}$ of R^{2n+2} we can compute:

$$\begin{aligned} M_+ &= \langle e_i, e_{2n+3-i} \mid t_i = 1,\ 1 \le i \le n \rangle \oplus \langle t_{n+1} \cdot e_{n+1} + e_{n+2} \rangle \\ M_- &= \langle e_i, e_{2n+3-i} \mid t_i = -1, 1 \le i \le n \rangle \oplus \langle t_{n+1} \cdot e_{n+1} - e_{n+2} \rangle \\ M_* &= \langle e_i, e_{2n+3-i} \mid t_i \neq \pm 1, 1 \le i \le n \rangle \end{aligned}$$

The corresponding description of $N = N_+ \oplus N_- \oplus N_*$ can be arranged such that:

$$\begin{aligned} N_\pm &= \langle e_i', e_{2n+3-i}' \mid t_i = \pm 1,\ 1 \le i \le n \rangle \\ N_* &= \langle e_i', e_{2n+3-i}' \mid t_i \neq \pm 1, 1 \le i \le n \rangle \end{aligned}$$

where $e_i' = (-1)^i \cdot (t_i - t_i^{-1})^{-1} e_i$ if $t_i \neq \pm 1$ and $1 \le i \le n$ and $e_j' = e_j$ else. With respect to this new basis of M_* the symplectic form given by $p_* = q_* \cdot (b_* - b_*^{-1})$ has standard form J_{2g}, so that the symplectic form p on R^{2n} can be assumed to be of standard form J_{2n} with respect to the basis $e_1', \ldots, e_n', e_{n+3}', \ldots, e_{2n+2}'$. The symplectic transformation $\beta = id_{N_+} \times (-id_{N_-}) \times b_*$ in $\mathcal{N}(b)$ has the diagonal form $diag(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$ with respect to this basis. The claim now follows from example 1.15.

(b) Let $\beta \in \mathrm{Sp}_{2n}(R)_{Rss}$. We decompose $N = R^{2n} = N_+ \oplus N_- \oplus N_*$ as in lemma 6.1(a). Since this decomposition is J_{2n}-orthogonal the restrictions p_+, p_-, p_* of the symplectic form J_{2n} to N_+, N_- and N_* are unimodular, so these spaces have even rank: $N_+ \simeq R^{2r_+}$, $N_- \simeq R^{2r_-}$, $N_* \simeq R^{2g}$. If we view p_* as skew symmetric matrix and $\beta_* \in \mathrm{Sp}(p_*) \subset \mathrm{SL}_{2g}$ we can form the symmetric matrix (bilinear form) $q_* = p_* \cdot (\beta_* - \beta_*^{-1})^{-1}$ and get $\beta_* \in \mathrm{SO}(q_*)$. For $\epsilon_\pm \in R^\times/(R^\times)^2$ we consider the symmetric bilinear forms $q_+ = \epsilon_+ \cdot J_{1+2r_+}$ on $M_+ = R^{1+2r_+}$ and $q_- = \epsilon_- \cdot J_{1+2r_-}$ on $M_- = R^{1+2r_-}$. By lemma 2.4 there are two different choices of pairs (ϵ_+, ϵ_-) such that the quadratic space $(M, q) = (M_+, q_+) \oplus (M_-, q_-) \oplus (N_*, q_*)$ is isomorphic to the standard space (R^{2n+2}, J'_{2n+2}). For these two choices the element $b = id_{M_+} \times (-id_{M_-}) \times \beta_* \in \mathrm{O}(q, R)^-$ can be viewed as an element of $\mathrm{O}_{2n+2}(R)^-_{Rss}$, which maps to β under $\mathcal{N}$. It is clear from the constructions that the two classes just obtained are all $\mathrm{SO}_{2n+2}(R)$-conjugacy classes in $\mathrm{O}_{2n+2}(R)^-_{Rss}$ mapping to β under $\mathcal{N}$.

□

Lemma 6.5. *Let $\gamma_1 \in O_{2n+2}(\mathcal{O}_F)^-$ be R-Θ-semisimple.*

(a) If $\gamma_2 := g_F^{-1} \cdot \gamma_1 \cdot g_F \in O_{2n+2}(\mathcal{O}_F)^-$ for $g_F \in SO_{2n+2}(F)$ there exists $g_R \in SO_{2n+2}(\mathcal{O}_F)$ with $\gamma_2 = g_R^{-1} \cdot \gamma_1 \cdot g_R$.

(b) There is a unique $SO_{2n+2}(\mathcal{O}_F)$-conjugacy class $\{\gamma_1'\}$ in $SO_{2n+2}(\mathcal{O}_F)$ different from $\{\gamma_1\}$ such that for every $g_F \in SO_{2n+2}(\bar{F})$ with $\gamma_2 := g_F^{-1} \cdot \gamma_1 \cdot g_F \in O_{2n+2}(\mathcal{O}_F)^-$ there either exists $g_R \in SO_{2n+2}(\mathcal{O}_F)$ with $\gamma_2 = g_R^{-1} \cdot \gamma_1 \cdot g_R$ or $g_R' \in SO_{2n+2}(\mathcal{O}_F)$ with $\gamma_2 = (g_R')^{-1} \cdot \gamma_1' \cdot g_R'$

Proof: (a) Let $M = \mathcal{O}_F^{2n+2} = M_{+,i} \oplus M_{-,i} \oplus M_{*,i}$ for $i = 1, 2$ be the orthogonal decompositions with respect to γ_i as in lemma 6.1(b) and let $q_{\pm,i}, q_{*,i}$ denote the restrictions of the standard form J'_{2n+2} to these subspaces. Since $M_{\pm,i}$ are eigenspaces of γ_i the relation $\gamma_2 := g_F^{-1} \cdot \gamma_1 \cdot g_F$ implies $g_F(M_{\pm,2} \otimes_{\mathcal{O}_F} F) = M_{\pm,1} \otimes_{\mathcal{O}_F} F$. The relation $g_F \in \mathrm{SO}_{2n+2}(F)$ implies the corresponding result for the orthogonal complements and implies that we have isomorphisms of quadratic spaces:

$$\begin{array}{rcll} (M_{\pm,2} \otimes_{\mathcal{O}_F} F, q_{\pm,2}) & \xrightarrow{\sim} & (M_{\pm,1} \otimes_{\mathcal{O}_F} F, q_{\pm,1}) & \text{and} \\ g_{F,*}: \quad (M_{*,2} \otimes_{\mathcal{O}_F} F, q_{*,2}) & \xrightarrow{\sim} & (M_{*,1} \otimes_{\mathcal{O}_F} F, q_{*,1}) \,. & \end{array}$$

Since the quadratic spaces are defined over $\mathcal{O}_F$ and become isomorphic over F and since the forms are unimodular, the spaces are isomorphic over $\mathcal{O}_F$ by lemma 2.4, i.e. there exists $g_R' \in \mathrm{SO}_{2n+2}(\mathcal{O}_F)$ inducing isomorphisms

$$(M_{\pm,2}, q_{\pm,2}) \xrightarrow{\sim} (M_{\pm,1}, q_{\pm,1}) \quad \text{and} \quad g'_{R,*}: \quad (M_{*,2}, q_{*,2}) \xrightarrow{\sim} (M_{*,1}, q_{*,1})\,.$$

If $\gamma_{*,i}$ denotes the restriction of γ_i to $M_{*,i}$ we get $\gamma_{*,2} = g_{F,*}^{-1} \cdot \gamma_{*,1} \cdot g_{F,*}$ and $\gamma_{*,3} := g'_{R,*} \cdot \gamma_{*,2} \cdot (g'_{R,*})^{-1} \in \mathrm{SO}(q_{*,1}, \mathcal{O}_F)$. We have $g_{F,3} := g_{F,*} \cdot (g'_{R,*})^{-1} \in \mathrm{SO}(q_{*,1}, F)$. Now it follows from $\gamma_{*,3} = g_{F,3}^{-1} \cdot \gamma_{*,1} \cdot g_{F,3}$ and lemma 2.10 that there exists $g_* \in \mathrm{SO}(M_{*,1}, q_{*,1})$

satisfying $g_* \cdot \gamma_{*,3} \cdot g_*^{-1} = \gamma_{*,1}$. Then $g_R := (id_{M_{+,1}} \times id_{M_{-,1}} \times g_*) \cdot g'_R \in \mathrm{SO}_{2n+2}(\mathcal{O}_F)$ satisfies $g_R \cdot \gamma_2 \cdot g_R^{-1} = \gamma_1$.

(b) Let us assume that $g_F \in \mathrm{SO}_{2n+2}(\bar{F})$ satisfies $\gamma_2 := g_F^{-1} \cdot \gamma_1 \cdot g_F \in \mathrm{O}_{2n+2}(\mathcal{O}_F)^-$. We only know that the quadratic spaces become isomorphic over $\bar{F}$, but we have the additional discriminant conditions $\Delta(q_{+,1}) \cdot \Delta(q_{-,1}) \cdot \Delta(q_{*,1}) = \Delta(q_{+,2}) \cdot \Delta(q_{-,2}) \cdot \Delta(q_{*,2})$ and $\Delta(q_{*,1}) = \Delta(p_{*,1}) \cdot \det(\gamma_{*,1} - \gamma_{*,1}^{-1}) = 1 \cdot \det(g_{F,*}(\gamma_{*,2} - \gamma_{*,2}^{-1})g_{F,*}^{-1}) = \Delta(p_{*,2}) \cdot \det(\gamma_{*,2} - \gamma_{*,2}^{-1}) = \Delta(q_{*,2})$ in $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$, where we use the fact that the $p_{*,i} := q_{*,i} \cdot (\gamma_{*,i} - \gamma_{*,i}^{-1})$ are unimodular skew symmetric by lemma 6.2 and thus have square determinants. The isomorphy-type of the quadratic spaces $(M_{\pm,1}, q_{\pm,1})\,, (M_{*,1}, q_{*,1})$ being fixed this means that there are two choices for the equivalence class of $q_{+,2}$ but the isomorphy-type of the other quadratic spaces $(M_{-,2}, q_{-,2})$ and $(M_{*,2}, q_{*,2})$ are then uniquely determined. To construct γ'_1 we change the quadratic forms $q_{+,1}$ on $M_{+,1}$ and $q_{-,1}$ on $M_{-,1}$ to the other isomorphy-typ but make no change for $M_{*,1}$, consider an isomorphism of quadratic spaces $\iota : R^{2n+2} \xrightarrow{\sim} M_{+,1} \oplus M_{-,1} \oplus M_{*,1}$ with respect to these modified forms on $M_{\pm,1}, M_{*,1}$ and the standard form J'_{2n+2} on R^{2n+2}, and put finally $\gamma'_1 = \iota^{-1} \circ \gamma_1 \circ \iota$. The statement of (b) now follows as in part (a). $\square$

Lemma 6.6. *In the notations of 6.3 let $b \in O_{2n+2}(\mathcal{O}_F)^-$ be residually semisimple and $\beta = 1_{2r_+} \times (-1_{2r_-}) \times b_* \in Sp_{2n}(\mathcal{O}_F)$ a representing element of $\mathcal{N}(b)$ with $b_* - b_*^{-1} \in GL_{2g}(\mathcal{O}_F)$.*

Assume we have BC-matching topologically unipotent elements $u_+ \in SO_{2r_++1}(F)$ and $v_+ \in Sp_{2r_+}(F)$ resp. $u_- \in SO_{2r_-+1}(F)$ and $v_- \in Sp_{2r_-}(F)$ and an additional topologically unipotent element $u_* \in Cent(b_*, SO(q_*, F)) \simeq Cent(b_*, Sp(p_*, F))$. We form the topologically unipotent elements $u = u_+ \times u_- \times u_* \in Cent(b, SO_{2n+2}(F))$ and $v = v_+ \times v_- \times u_* \in Cent(\beta, Sp_{2n}(F))$.

Then the elements $g := bu = ub \in O_{2n+2}(F)^-$ and $\gamma := \beta v = v\beta \in Sp_{2n}(F)$ match.

Proof: As in the proof of lemma 4.6 we work in the case $F = \bar{F}$ and assume that g resp. γ lie in the diagonal tori. The same holds for the residually semisimple parts b resp. β and the topologically unipotent parts u and v. As the matching of b and β is already proved in 6.4 we only have to examine the topologically unipotent elements. We can arrange the diagonal matrices $u_\pm \in \mathrm{SO}(q_\pm, F)$ such that their middle entries 1 correspond to the eigenvectors $t_{n+1} \cdot e_{n+1} \pm e_{n+2} \in M_\pm$, which get lost by the construction of $N_\pm$. Then the claim follows immediately from the definition of BC-matching 1.12, example 1.15 and the constructions in the proof of proposition 6.4. $\square$

Remark 6.7. The surjectivity statement of Proposition 6.4(b) is not true if R is a field, for example a p-adic field F: Let $\Delta \in F^*$ denote a non square and

$$a_i = \frac{\lambda_i^2 + \Delta}{\lambda_i^2 - \Delta}, \qquad b_i = \frac{2 \cdot \lambda_i}{\lambda_i^2 - \Delta} \qquad \text{for some } \lambda_i \in F^*, \quad i = 1, 2.$$

If we consider the matrix

$$\beta \quad = \quad \begin{pmatrix} a_1 & & & b_1 \\ & a_2 & b_2 & \\ & b_2\Delta & a_2 & \\ b_1\Delta & & & a_1 \end{pmatrix} \quad \in \quad \mathrm{Sp}_4(F)$$

then we have $N_* = N$ and $\beta_* = \beta$ for $N = F^4$ and can compute

$$\begin{aligned} q_* \quad &:= \quad p_* \cdot (\beta_* - \beta_*^{-1})^{-1} \quad = \quad J_4 \cdot antidiag(2b_1, 2b_2, 2\Delta b_2, 2\Delta b_1)^{-1} \\ &= \quad diag\left(\frac{1}{2b_1}, \frac{-1}{2b_2}, \frac{1}{2\Delta b_2}, \frac{-1}{2\Delta b_1}\right) \quad = \quad \frac{-1}{2\Delta b_1} \cdot diag\left(-\Delta, \Delta \cdot \frac{b_1}{b_2}, -\frac{b_1}{b_2}, 1\right). \end{aligned}$$

Thus the quadratic form q_* on N is anisotropic if $b_1 \cdot b_2^{-1}$ is not a norm of the extension $F\sqrt{\Delta}/F$. In this case (N, q_*) cannot be obtained as direct summand of the six dimensional quadratic split space (F^6, J_6'). The considerations of 6.3 and 6.4(b) then show that the conjugacy class of β is not in the image of $\mathcal{N}$.

The following theorem is again the fundamental lemma for a stable endoscopic lift modulo the BC-conjecture. But the non surjectivity of $\mathcal{N}$ in the case of local fields forces us to include the vanishing statement for orbital integrals of elements, that do not match.

Theorem 6.8. *Assume that conjecture (BC_m) is true for all $m \leq n$.*

(a) If $g \in \widetilde{SO}_{2n+2}(F) = O_{2n+2}(F)$ with $\det(g) = -1$ and $\gamma \in Sp_{2n}(F)$ are matching semisimple elements then we have

(i) $$O_g^{st}(1, \widetilde{SO}_{2n+2}) \quad = \quad O_\gamma^{st}(1, Sp_{2n}).$$

(b) If the semisimple $\gamma \in Sp_{2n}(F)$ matches with no element of $\widetilde{SO}_{2n+2}(F)$, then we have $O_\gamma^{st}(1, Sp_{2n}) = 0$.

Proof: Since the proof of (a) is similar to the proofs of theorems 4.7 and 5.11 we will omit the details. We remark that (b) is an immediate corollary of the considerations in Step 1: If the right hand side of (**i**) does not vanish one can construct an element $g \in \widetilde{\mathrm{SO}}_{2n+2}(F)$ matching with γ using proposition 6.4 and lemma 6.6.

We just remark that by lemma 6.5(b) we have to deal with two representatives of residually semisimple elements s and s' in step 3 when we compute $O_g^{st}(1, \mathrm{O}_{2n+2})$. But observe that the centralizers SO_{2n+2}^s and $\mathrm{SO}_{2n+2}^{s'}$ can be identified, since the two equivalence classes of symmetric unimodular forms on a free $\mathcal{O}_F$-module of odd rank have representatives which are scalar multiples of each other. Therefore we can use the same collections of topologically unipotent elements for s as for s'. The appearance of s' thus introduces just an additional factor 2 in the computation. But since the centralizers SO_{2n+2}^s and $\mathrm{SO}_{2n+2}^{s'}$ have two connected components, there appears an additional factor $\frac{1}{2}$ when we apply the Kazhdan-lemma 3.5, which cancels the factor 2.

$\square$

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