On double coset decompositions for the algebraic group G_2

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1 Introduction

(1.1) The base rings. We denote by F a p-adic field with ring of integers $\mathcal{O}_F$, prime ideal $\mathfrak{p}$ and uniformizing element $\varpi = \varpi_F$. Let $val : F \to \mathbb{Z} \cup \{\infty\}$ be the normalized (i.e. $val(\varpi) = 1$) valuation. The residue field of characteristic p is denoted $\kappa = \kappa_F = \mathcal{O}_F/\mathfrak{p}$. By $\bar{F}$ we denote an algebraic closure of F. In the following we will assume that $p \neq 2$. Let G denote a connected reductive group scheme over $\mathcal{O}_F$.

(1.2) It has been observed by M. Schröder in his thesis [Schr] that one can compute orbital integrals using a decomposition $G(F) = \bigcup_{i\in I} H(F)g_iG(\mathcal{O}_F)$, where $H \subset G$ is some maximal reductive subgroup of G. In his case $G = \mathrm{GSP}_4$ and $H = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 | \det(g_1) = \det(g_2)\}$. This method has been used in [W1] and [Fl] to compute the orbital integrals for the unit element in the Hecke-algebra for GSP_4 and for GL_4 with an outer automorphism. Meanwhile Weissauer has obtained the generalization of the decomposition to all unramified classical groups [W2] [W3].

(1.3) To get the decomposition one can look for a representation (ρ, V) of G such that H becomes the stabilizer of one vector $e_0 \in V$. Then the map $g \mapsto \rho(g^{-1})x_0$ induces an isomorphism $H(F)\backslash G(F)/G(\mathcal{O}_F) \simeq G(\mathcal{O}_F)\backslash G(F)e_0$ and it remains to determine the $G(\mathcal{O}_F)$-orbits inside the $G(F)$-orbit of e_0.

(1.4) If $B \subset G$ denotes a Borel and if we assume that $G(\mathcal{O}_F)$ is a hyperspecial maximal compact, then it follows from the Iwasawa decomposition that each $G(\mathcal{O}_F)$-orbit inside $G(F)e_0$ meets $B(F)e_0$. If we consider weight vectors in V, which generate an $\mathcal{O}_F$-stable lattice L, we furthermore get elements v in each $G(\mathcal{O}_F)$-orbit, such that the coefficient of the highest weight vector in the decomposition of v has

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minimal valuation among all coefficients. Then the action of $\mathcal{O}_F$-valued points in the opposite Borel B^- enables us to kill the coefficients of all weight vectors, which differ from the highest weight vector by a negative root.

We carry out this program for the algebraic group G_2 of type G_2 and the two maximal reductive subgroups $H_1 \simeq \mathrm{SL}_3$, $H_2 \simeq \mathrm{SO}_4$. It happens that we arrive in a subset of $G(F)e_0$, which lies in just one orbit of $T(\bar{F})$, where $T = B \cap B^-$ is a maximal torus. To be more precise, if $T_S \subset T$ denotes the stabilizer of this T-action, then we arrive in one orbit of $(T/T_S)(F)$. The exact sequence $T(F) \to (T/T_S)(F) \to H^1(F, T_S)$ then shows, that we arrive in a finite set of $T(F)$-orbits (Propositions 3.3 and 5.9).

In the case of the group $H_2 \simeq \mathrm{SO}_4$ this finite set can be described as the the set of those pairs $(\alpha, \beta) \in F^*/(F^*)^2 \times F^*/(F^*)^2$, for which the Hilbert symbol $(\alpha, \beta)_H$ equals 1.

2 G_2 as simple algebraic group

(2.1) In this section we denote by R an arbitrary integral domain, by F its field of fractions.

Let $V = V_7 = R^7$ be the standard free R-module of rank 7 with standard basis $e_3, e_2, e_1, e_0, e_{-1}, e_{-2}, e_{-3}$. Let the dual basis of V^* be $x_3, x_2, x_1, x_0, x_{-1}, x_{-2}, x_{-3}$. (Sometimes we will take the x_i be the coordinates of an element $v \in V_7$: i.e. we abbreviate x_i for $x_i(v)$.)

If G_2 denotes the simple split algebraic group of type G_2 over R, then it is well known [Asch], that G_2 can be realized as the group of those automorphisms of V_7 which simultaneously respect the symmetric bilinear form

$$Q \quad = \quad -x_0^2 + x_1x_{-1} + x_2x_{-2} + x_3x_{-3} \quad \in \quad Sym^2(V^*)$$

and the alternating trilinear form

$$f \quad = \quad x_0 \wedge (x_1 \wedge x_{-1} + x_2 \wedge x_{-2} + x_3 \wedge x_{-3}) + x_1 \wedge x_2 \wedge x_3 + x_{-1} \wedge x_{-2} \wedge x_{-3} \in \Lambda^3(V^*)$$

(2.2) We let the group SL_3 act on V via

$$\sigma(A) \begin{pmatrix} x_3 \\ x_2 \\ x_1 \\ x_0 \\ x_{-1} \\ x_{-2} \\ x_{-3} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Theta(A) \end{pmatrix} \begin{pmatrix} x_3 \\ x_2 \\ x_1 \\ x_0 \\ x_{-1} \\ x_{-2} \\ x_{-3} \end{pmatrix}$$

where $A \in \mathrm{SL}_3(R)$ and $\Theta(A) = W_3 \cdot {}^tA^{-1} \cdot W_3$ with

$$W_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(2.3) SL_2 subgroups. From now on we use the ordering $e_{-2}, e_3, e_1, e_0, e_{-1}, e_{-3}, e_2$ of our basis and introduce three actions $\alpha_i^\vee$ of the group $H = \mathrm{SL}_2$ on V:

$$\textbf{(i)} \quad \alpha_1^\vee\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\begin{pmatrix} x_{-2} \\ x_3 \\ x_1 \\ x_0 \\ x_{-1} \\ x_{-3} \\ x_2 \end{pmatrix} = \begin{pmatrix} a & -b & & & & & \\ -c & d & & & & & \\ & & a^2 & 2ab & b^2 & & \\ & & ac & ad+bc & bd & & \\ & & c^2 & 2cd & d^2 & & \\ & & & & & a & b \\ & & & & & c & d \end{pmatrix}\begin{pmatrix} x_{-2} \\ x_3 \\ x_1 \\ x_0 \\ x_{-1} \\ x_{-3} \\ x_2 \end{pmatrix}$$

$$\textbf{(ii)} \quad \alpha_2^\vee(B) = \sigma(W_\tau)\alpha_1^\vee(B')\sigma(W_\tau)^{-1} \qquad \alpha_3^\vee(B) = \sigma(W_\tau)^{-1}\alpha_1^\vee(B)\sigma(W_\tau)$$

$$\text{with} \quad W_\tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot B \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is well known [Asch] that:

$$\textbf{(iii)} \qquad G_2 = \langle \sigma(\mathrm{SL}_3), \alpha_1^\vee(\mathrm{SL}_2) \rangle.$$

In fact a straightforward computation shows that the forms f and Q are invariant under the action of $\alpha_1^\vee(\mathrm{SL}_2)$ and under the σ-action of SL_3. Thus G_2 contains $\alpha_i^\vee(\mathrm{SL}_2)$ for all $i = 1, 2, 3$.

We furthermore introduce the following embeddings $\beta_i^\vee$ of SL_2 in G_2:

$$\beta_1^\vee\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} \qquad \beta_2^\vee\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma\begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}$$

$$\beta_3^\vee\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma\begin{pmatrix} 1 & & \\ & d & c \\ & b & a \end{pmatrix}.$$

For $\beta_2^\vee$ we rewrite this in terms of 7×7-matrices with respect to the unusual ordering of the basis:

$$\textbf{(iv)} \qquad \beta_2^\vee\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ & x & y & & & & \\ & z & w & & & & \\ & & & 1 & & & \\ & & & & x & -y & \\ & & & & -z & w & \\ & & & & & & 1 \end{pmatrix}$$

(2.4) The root system. The torus $T = \{t = (t_2^{-1}, t_3, t_1, 1, t_1^{-1}, t_3^{-1}, t_2) \mid t_1t_2t_3 = 1\}$, which is the image of the diagonal torus in SL_3, is a maximal split torus in G_2.

As Borel subgroup P_{12} of G_2 we can take all elements in G_2 which act via upper triangular matrices on V with respect to the ordered basis $(e_{-2}, e_3, e_1, e_0, e_{-1}, e_{-3}, e_2)$.

We remark that the maps

$$t \mapsto \gamma^\vee \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \qquad \text{for } \gamma \in \{\alpha_i, \beta_i | i = 1, 2, 3\}$$

are just the positive coroots of G_2 with respect to (P_{12}, T) and that the images of the upper triangular unipotent matrices in SL_2 under the above defined $\gamma^\vee$ are the unipotent subgroups of G_2 belonging to the positive roots γ. The simple roots are $\alpha = \alpha_1, \beta = \beta_2$. The other roots satisfy $\alpha_3 = \beta + \alpha, \alpha_2 = \beta + 2\alpha, \beta_3 = \beta + 3\alpha, \beta_1 = 2\beta + 3\alpha$. We have $\alpha(t) = 1/t_1$ and $\beta(t) = t_1/t_3$.

(2.5) Parabolic and Levi subgroups. There exist two maximal parabolic subgroups including P_{12}:

- $P_1 = \beta_2^\vee(\mathrm{SL}_2) \cdot P_{12}$ is the stabilizer of the line $R \cdot e_2$,
- $P_2 = \alpha_1^\vee(\mathrm{SL}_2) \cdot P_{12}$ is the stabilizer of the plane $\langle e_2, e_{-3}\rangle$.

Similarly we introduce the Borel subgroup P_{12}^- of elements acting by lower triangular matrices on the ordered basis of V and the parabolic subgroups $P_1^- = \beta_2^\vee(\mathrm{SL}_2) \cdot P_{12}^-$ respectively $P_2^- = \alpha_1^\vee(\mathrm{SL}_2) \cdot P_{12}^-$.
Thus P_1^- leaves invariant the line $R \cdot e_{-2}$ and therefore also the line spanned by the alternating bilinear form $f(.,.,e_{-2})$ obtained by contracting f with e_{-2}. This form can be written as $-x_0 \wedge x_2 - x_{-1} \wedge x_{-3}$. Its kernel is P_1^--invariant and spanned by e_1, e_3, e_{-2}.

(2.6) Furthermore we consider the following **maximal reductive subgroups**, which are not Levi subgroups of a parabolic:

- $H_1 = \sigma(\mathrm{SL}_3)$
- $H_2 = \beta_1^\vee(\mathrm{SL}_2) \cdot \alpha_1^\vee(\mathrm{SL}_2) \simeq \mathrm{SL}_2 \times \mathrm{SL}_2/\{\pm 1_2\}$, where $\{\pm 1_2\}$ is diagonally embedded. We remark that $\langle x_3, x_2, x_{-2}, x_{-3}\rangle$ is isomorphic to the tensor product of two twodimensional representations of the two SL_2 factors, i.e. the actions $\beta_1^\vee(\mathrm{SL}_2)$ and $\alpha_1^\vee(\mathrm{SL}_2)$ commute.

Lemma 2.7. *The subgroup H_1 is the stabilizer of $e_0 \in V_7$.*

Proof: This is essentially [RS, lemma 2]. □

Lemma 2.8. *H_2 is the stabilizer of $s_0 = x_1x_{-1} - x_0^2 \in Sym^2(V_7^*)$.*

Proof: The inclusion $H_2 \subset Stab_{G_2}(s_0)$ follows immediately from the definition of $\alpha_1^\vee$ and $\beta_1^\vee$. Thus let $g \in Stab_{G_2}(s_0)(F)$. The space $W_4 = \langle e_3, e_2, e_{-2}, e_{-3}\rangle$, being the kernel of the quadratic form s_0, is invariant under g, and so is the orthogonal complement with respect to Q, namely the space $W_3 = \langle e_1, e_0, e_{-1}\rangle$. Thus we get $g \in \mathrm{O}(W_3, s_0) \times \mathrm{O}(W_4, Q - s_0)$. Since the action of H_2 on W_4 exhausts the group $\mathrm{SO}(W_4, Q - s_0)$ we can modify g by an element of $H_2(F)$ such that the new g acts either as identity on W_4 or as the reflection which interchanges e_2 and e_{-2} and fixes e_3, e_{-3}. In the latter case we would get as summand of $g.f$ a term of the form $(gx_1) \wedge x_{-2} \wedge x_3 + (gx_{-1}) \wedge x_2 \wedge x_{-3}$, but this cannot occur as summand in f. So g acts as identity on W_4. Now it follows immediately from $g.f = f$ that g fixes x_1, x_{-1} and x_0, i.e. $g = id_7 \in H_2(F)$. □

(2.9) Representations of G_2. For $m, n \in \mathbb{N}_0$ we denote by $\chi_{m,n} : T \to \mathbb{G}_m$ the character $m \cdot \alpha_2 + n \cdot \beta_1$. We denote by $V_{m,n}$ the irreducible representation with highest weight $\chi_{m,n}$. We list the multiplicities of weights in some low dimensional representations:

(v)

ρ	$\chi_{0,0}$	$\chi_{1,0}$	$\chi_{0,1}$	$\chi_{2,0}$	$\chi_{1,1}$	$\chi_{3,0}$	$\chi_{0,2}$	$\chi_{2,1}$	$\chi_{4,0}$	$\dim(\rho)$
$Sym^2(V_7)$	4	2	1	1						28
$\Lambda^3(V_7)$	5	3	1	1						35
$Sym^2(V_{2,0})$	24	18	12	11	5	3	2	1	1	378

By comparing these multiplicities of weights with [Hum, 22.4, table 2] for the irreducibles we follow:

(vi)

$$\begin{aligned} Sym^2(V_7) &= V_{0,0} \oplus V_{2,0}, \\ \Lambda^3(V_7) &= V_{0,0} \oplus V_7 \oplus V_{2,0}, \\ Sym^2(V_{2,0}) &= V_{4,0} \oplus V_{0,2} \oplus V_{1,1} \oplus 2 \cdot V_{2,0} \oplus V_{0,0}. \end{aligned}$$

3 Schröder decomposition for H_1

(3.1) To carry out the program of the introduction for $G = G_2, H = H_1$ we take the vector $e_0 \in V_7$, the 7-dimensional representation of G_2, and we have to determine the $G_2(\mathcal{O}_F)$-orbits in $V_7 = F^7$. There exist two invariants of an element $x \in V_7$: $B(x)$ is a $G_2(F)$ invariant, and since $G_2(\mathcal{O}_F)$ stabilizes the lattice $L = \mathcal{O}_F^7$ we have the numerical invariant $dep(x) := \max\{n \in \mathbb{Z} | \varpi^{-n} \cdot x \in L\}$ (especially $dep(0) = \infty$). Since $val(B(x)) \geq 0$ for $x \in L$ we have $2 \cdot dep(x) \leq val(B(x))$.

Conversely if we have $a \in F$ and $d \in \mathbb{Z} \cup \{\infty\}$ such that $2d \leq val(a)$, we can form the element $x = x(a,d) = (x_i)_{3 \geq i \geq -3} := (0, 0, \varpi^d, 0, a \cdot \varpi^{-d}, 0, 0)$ which satisfies $B(x) = a$ and $dep(x) = d$. In the case $d = \infty, a = 0$ we simply put $x = 0$.

Lemma 3.2. *With the above notations every $x \in F^7$ lies in the $G_2(\mathcal{O}_F)$-orbit of $x(a,d)$ where $a = B(x), d = dep(x)$.*

Proof: Let $x = (x_3, x_2, x_1, x_0, x_{-1}, x_{-2}, x_{-3}) \in F^7$. The case $x = 0$ being trivial we can assume that $d = dep(x) = min_{3 \geq i \geq -3} val(x_i) \in \mathbb{Z}$. We will apply elements of $\alpha_i^\vee(\mathrm{SL}_2(\mathcal{O}_F))$ or $\sigma(\mathrm{SL}_3(\mathcal{O}_F)) \subset G_2(\mathcal{O}_F)$ to x until we have $x = x(a,d)$:

If $val(x_i) > d$ for $i \neq 0$ we can apply $\alpha_1^\vee(A)$ for some unipotent upper triangular matrix $A \in \mathrm{SL}_2(\mathcal{O}_F)$ to get $val(x_1) = d$ (observe $2 \in \mathcal{O}_F^*$). If $\min_{3 \geq i \geq 1} val(x_i) > d = \min_{-1 \geq i \geq -3} val(x_i)$ we can apply $\alpha_1^\vee(W_2) \in G_2(\mathcal{O}_F)$ to achieve $\min_{3 \geq i \geq 1} val(x_i) = d$. Then we can use a suitable $\sigma(A)$ for $A \in \mathrm{SL}_3(\mathcal{O}_F)$ to get $d = val(x_1)$.

Now we apply $\alpha_1^\vee(A)$, where $A \in \mathrm{SL}_2(\mathcal{O}_F)$ has entries $a = d = 1, b = 0, c = -\frac{x_0}{x_1}$, to get an element with $x_0 = 0$ but unchanged x_1. Applying once more some $\sigma(A)$ for $A \in \mathrm{SL}_3(\mathcal{O}_F)$ we can then achieve $x_3 = x_2 = 0, x_1 = \varpi^d$ and still $x_0 = 0$. Applying some $\beta_1^\vee(A)$ for $A \in SL_2(\mathcal{O}_F)$ we can furthermore achieve that additionally the condition $x_{-3} = 0$ is satisfied.

Now we can finish in applying $\alpha_3^\vee(A)$, where $A \in \mathrm{SL}_2(\mathcal{O}_F)$ has entries $a = d = 1, c = 0, b = -\frac{x_{-2}}{x_1}$, to get the further condition $x_{-2} = 0$. Thus we have $x_i = 0$ for $i \neq \pm 1$ and $x_1 = \varpi^d$. Since $a = B(x) = x_1 x_{-1}$ we must have $x_{-1} = a \cdot \varpi^{-d}$, i.e. by changing x in its $G_2(\mathcal{O}_F)$-orbit we have achieved $x = x(a,d)$. □

Proposition 3.3. *With $H_1 \simeq SL_3 \subset G_2$ we have a disjoint decomposition:*

$$(\mathbf{vii}) \quad G_2(F) \quad = \quad \dot{\bigcup_{n \in \mathbb{N}_0}} H_1(F) \cdot \alpha_1^\vee(g_n) \cdot G_2(\mathcal{O}_F) \qquad \textit{where } \; g_n = \begin{pmatrix} 1 & \varpi^{-n} \\ 0 & 1 \end{pmatrix}.$$

Proof: First notice that $\alpha_1^\vee(g_n^{-1})(e_0) = (0, 0, -2\varpi^{-n}, 1, 0, 0, 0)$ lies in the $G_2(\mathcal{O}_F)$-orbit of $x(-1, -n)$. For $g \in G_2(F)$ we can write the element $g^{-1}e_0$ by lemma 3.2 in the form $k' \cdot x(-1, -n)$ for suitable $n \in \mathbb{N}_0$ and $k' \in G_2(\mathcal{O}_F)$ since $B(e_0) = -1$. This implies $g^{-1}e_0 = k \cdot \alpha_1^\vee(g_n^{-1})(e_0)$ i.e. $h := g \cdot k \cdot \alpha_1^\vee(g_n^{-1}) \in Stab_{G_2}(e_0)$ for some $k \in G_2(\mathcal{O}_F)$. So we get $g = h \cdot \alpha_1^\vee(g_n) \cdot k^{-1}$, where $h \in H_1(F)$ by lemma 2.7. □

Corollary 3.4. *We have*

$$G_2(F) \quad = \quad H_1(F) \cdot \alpha_1^\vee \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot T(F) \cdot G_2(\mathcal{O}_F)$$

Proof: In the notations of proposition 3.3 we have $\alpha_1^\vee(g_n) = t_n \cdot \alpha_1^\vee(g_0) \cdot t_n^{-1}$ where $t_n \in T(F)$ satisfies $\alpha(t_n) = \varpi^{-n}$, e.g. $t_n = \alpha^\vee(\varpi^{-n}) \cdot \beta^\vee(\varpi^{-n})$. (here $\alpha^\vee$ is the long and $\beta^\vee$ the short coroot). □

4 The open parabolic orbit

The case $G = G_2, H = H_2$ is more difficult than the case $H = H_1$, since the group H_2 is smaller and we have to work in the bigger representation $V_{2,0}$. To prepare the double coset decomposition for H_2 we analyze the open P_1^--orbit inside the G_2-orbit of s_0:

(4.1) We recall that H_2 is the stabilizer of $s_0 = x_1x_{-1} - x_0^2 \in Sym^2(V_7)$.

The bilinear form B induces an isomorphism of G_2 modules $V^* \simeq V$. In the following we furthermore identify $Sym^2(V_7^*) \simeq Sym^2(V_7)$ with the space Sym_7 of symmetric 7×7-matrices. If we denote the elementary matrices by $E_{kl} = (\delta_{ik}\delta_{jl})_{i,j}$, the elements $F_{kl} = E_{kl} + E_{lk},\ k \leq l$ form a basis of Sym_7 (thus $F_{kk} = 2E_{kk}$), which is identified with the basis $x_k \cdot x_l$ of $Sym^2(V^*)$. The entries of a matrix are arranged such that the indices $k, l \in \{-3, -2, \ldots, 3\}$ have the order $-2 < 3 < 1 < 0 < -1 < -3 < 2$. The induced right action on $Sym^2(V^*)$ corresponds to the right action $S \mapsto {}^t gSg =: S.g$ of $g \in G_2 \subset \mathrm{GL}_7$ on Sym_7. The element $s_0 = x_1x_{-1} - x_0^2 \in Sym^2(V_7)$ corresponds to $S_0 := F_{1,-1} - F_{0,0} \in Sym_7$.

Since the trivial subrepresentation $V_{0,0} \subset Sym_7$ is spanned by $S' = F_{1,-1} + F_{3,-3} + F_{-2,2} - F_{0,0}$, we can realize H_2 as the stabilizer of

$$S_0' := 2 \cdot (F_{1,-1} - F_{0,0}) - S' = -F_{-2,2} - F_{3,-3} + F_{1,-1} - F_{0,0}.$$

(4.2) The action of $\beta_2^\vee(\mathbf{SL}_2) \times \alpha_2^\vee(\mathbf{SL}_2)$ on S_0'. For $\phi, \psi \in F^*$ we consider the elements

$$S_1' \quad = \quad S_0' \,.\, \beta_2^\vee \begin{pmatrix} 1 & \phi \\ -\frac{1}{2\phi} & \frac{1}{2} \end{pmatrix} \quad = \quad -F_{-2,2} - \frac{1}{\phi} F_{3,-1} - \phi \cdot F_{1,-3} - F_{0,0} \qquad \text{and}$$

$$S_2' \quad := \quad S_1' \,.\, \alpha_2^\vee \begin{pmatrix} \frac{1}{2} & \frac{\phi\psi}{2} \\ -\frac{1}{\phi\psi} & 1 \end{pmatrix} \quad = \quad S(\psi, \phi^2),$$

where we use the following notation for $\rho, \psi \in \bar{F}^*$:

$$\textbf{(viii)} \qquad S(\psi,\rho) \quad := \quad \begin{pmatrix} -\frac{1}{\rho\psi^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\rho\psi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\psi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\rho\psi^2 \end{pmatrix} \quad =$$

$$F_{0,0} + \frac{1}{2} \cdot \left(-\psi F_{-1,-1} + \psi\rho F_{-3,-3} - \psi^2\rho F_{2,2} - \frac{1}{\psi} F_{1,1} + \frac{1}{\psi\rho} F_{3,3} - \frac{1}{\psi^2\rho} F_{-2,-2} \right)$$

Lemma 4.3. *Let $x = (x_{i,j})_{-3\leq i,j\leq 3}$ denote an element of the orbit $\mathcal{O} := G_2(\bar{F})S_0'$. Then we have:*

(a)

$$(\mathbf{ix}) \qquad \det\begin{pmatrix} x_{-1,-1} & x_{-1,-3} & x_{-1,2} \\ x_{-3,-1} & x_{-3,-3} & x_{-3,2} \\ x_{2,-1} & x_{2,-3} & x_{2,2} \end{pmatrix} = (x_{2,2})^2.$$

(b) x lies in the $P_1^-(\bar{F})$ orbit of $S(1,1)$ if and only if $x_{2,2} \neq 0$.

(c) If $x_{2,2} \neq 0$, $x_{i,2} = x_{2,i} = 0$ for all $i \neq 2, -2$ and if additionally $x_{-1,-3} = x_{-3,-1} = 0$, then $(x_{i,j})$ is of the form $S(\psi, \rho)$ for suitable $\psi, \rho \in \bar{F}^*$.

(d) If we only assume $x_{2,2} \neq 0$ and $x_{i,2} = x_{2,i} = 0$ for all $i \neq 2, -2$ then x is of the form

$$x = \begin{pmatrix} \frac{1}{\det(A)} & & & & \\ & A' & & & \\ & & 2 & & \\ & & & A & \\ & & & & \det(A) \end{pmatrix},$$

where A denotes a matrix in $GL_2(\bar{F})$ and

$$A' = \frac{1}{\det(A)} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof: The formula (a) is clear for $(x_{i,j})$ of the form $S(\psi, \rho)$. The set of all $S(\psi, \rho)$ for $\psi, \rho \in \bar{F}$ is just the $T(\bar{F})$-orbit of $S(1,1)$. The formula remains true under the action of the unipotent subgroups of P_1^-, since P_1^- leaves invariant the line $\bar{F} \cdot e_2$ and the space generated by e_{-1}, e_{-3}, e_2, so that the unipotent elements do not change either side of (**ix**). But the stabilizer of $S(1,1)$ inside the 9-dimensional group P_1^- is a 1-dimensional group of type SO_2. The orbit $P_1^- \cdot S(1,1)$ is of dimension 8, which is the dimension ($= 14 - 6$) of the total orbit $\mathcal{O} \simeq G_2/H_2$. It is thus open inside $\mathcal{O}$ and its closure has to be $\mathcal{O}$, since $\mathcal{O}$ is irreducible. Since (**ix**) is true on the open dense orbit $P_1^- \cdot S(1,1)$, it is thus true on all of $\mathcal{O}$, i.e. (a) is proved.

The "only if" part of (b) being clear, let $(x_{i,j})$ be an element of $\mathcal{O}$ satisfying $x_{2,2} \neq 0$. By transforming it with unipotent elements inside P_1^- we can achieve that the assumptions of (c) are satisfied. Thus it remains to prove (c):

For an element x of $\mathcal{O}$ satisfying the assumptions of (c) we conclude from (a) and $x_{2,2} \neq 0$ that $x_{-1,-1} \neq 0 \neq x_{-3,-3}$. It follows that the stabilizer of x inside P_1^- is of dimension ≤ 1, so that the P_1^--orbit of x is of dimension at least 8 and thus has to be open inside $\mathcal{O}$. From the irreducibility of $\mathcal{O}$ we conclude that x lies in the P_1^--orbit of $S(1,1)$. Then it is easy to see that x is of the form $S(\psi, \rho)$.

(d) follows from (c) by considering the action of $\beta_2^\vee(\mathrm{SL}_2)$ on elements of the form $S(\psi, \rho)$. □

Lemma 4.4. *Let $\Gamma = Gal(\bar{F}/F)$ denote the Galois group of some perfect field F.*

(a) The element $S(\psi,\rho)$ lies in the rational orbit $G_2(F)S_0'$ if and only if the 2-cocycle $c = c_{-\psi\rho,\rho} : \Gamma\times\Gamma \to \mu_2 = \{\pm 1\}$

$$c(\sigma,\tau) \quad = \quad \begin{cases} -1 & \text{if } \sigma(\sqrt{-\rho\psi}) = -\sqrt{-\rho\psi} \text{ and } \tau(\sqrt{\rho}) = -\sqrt{\rho} \\ 1 & \text{else.} \end{cases}$$

represents the trivial class in $H^2(\Gamma,\mu_2)$.

(b) If F is a p-adic field, then $S(\psi,\rho)$ lies in the rational orbit $G_2(F)S_0'$ if and only if the Hilbert symbol satisfies $(\psi,\rho)_H = 1$.

Proof: (a) An F-rational element in the orbit $G_2(\bar{F})S_0'$ can be written in the form $s \in (G_2/H_2)(F)$. It lies in the rational orbit $G_2(F)S_0'$ if and only if the class $\delta(s) \in H^1(\Gamma, H_2)$ vanishes. Here $\delta(s)$ denotes the class of the 1-cocycle $\sigma \mapsto (s')^{-1}\sigma(s')$, if s is represented by $s' \in G_2(\bar{F})$. From the long exact cohomology sequence attached to the short exact sequence $1 \to \mu_2 \to \mathrm{SL}_2\times\mathrm{SL}_2 \to H_2 \to 1$ one concludes that this is the case if and only if the coboundary $\delta_2(\delta(s))$ vanishes in $H^2(\Gamma,\mu_2)$. From the relations

$$S(\psi,\rho) \quad = \quad S_0' \cdot \beta_2^\vee \begin{pmatrix} 1 & \sqrt{\rho} \\ -\frac{1}{2\sqrt{\rho}} & \frac{1}{2} \end{pmatrix} \cdot \alpha_2^\vee \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{\rho\psi}}{2} \\ -\frac{1}{\sqrt{\rho\psi}} & 1 \end{pmatrix} \qquad \text{and}$$

$$\begin{pmatrix} 1 & \sqrt{\rho} \\ -\frac{1}{2\sqrt{\rho}} & \frac{1}{2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & -\sqrt{\rho} \\ \frac{1}{2\sqrt{\rho}} & \frac{1}{2} \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & -\sqrt{\rho} \\ \frac{1}{\sqrt{\rho}} & 0 \end{pmatrix}$$

one concludes that the 1-cocycle $\delta(s)$ satisfies

$$\sigma \quad \mapsto \begin{cases} \beta_2^\vee \begin{pmatrix} 0 & -\sqrt{\rho} \\ \frac{1}{\sqrt{\rho}} & 0 \end{pmatrix} \cdot \alpha_2^\vee \begin{pmatrix} 0 & -\sqrt{\rho\psi} \\ \frac{1}{\sqrt{\rho\psi}} & 0 \end{pmatrix} & \text{if } \sigma(\sqrt{\rho}) = -\sqrt{\rho} \\ 1 & \text{if } \sigma(\sqrt{\rho}) = \sqrt{\rho}. \end{cases}$$

Now an easy calculation shows

$$\beta_2^\vee \begin{pmatrix} 0 & -\sqrt{\rho} \\ \frac{1}{\sqrt{\rho}} & 0 \end{pmatrix} \cdot \alpha_2^\vee \begin{pmatrix} 0 & \sqrt{\rho\psi} \\ -\frac{1}{\sqrt{\rho\psi}} & 0 \end{pmatrix} \quad = \quad \alpha_1^\vee \begin{pmatrix} 0 & \sqrt{-\psi} \\ -\frac{1}{\sqrt{-\psi}} & 0 \end{pmatrix} \cdot \beta_1^\vee \begin{pmatrix} 0 & -\zeta \\ \frac{1}{\zeta} & 0 \end{pmatrix},$$

where $\zeta = \rho\sqrt{-\psi}^3$, since both sides equal the antidiagonal matrix with entries $\rho\psi^2, -\rho\psi, -\psi, -1, -\frac{1}{\psi}, -\frac{1}{\rho\psi}, \frac{1}{\rho\psi^2}$. Thus we can lift the 1-cocycle $\delta(s) : \Gamma \to H_2(\bar{F})$ to the 1-cochain

$$\begin{aligned} \gamma : \Gamma \quad &\to \quad \mathrm{SL}_2\times\mathrm{SL}_2 \\ \sigma \quad &\mapsto \quad \begin{cases} A & \text{if } \sigma(\sqrt{\rho}) = -\sqrt{\rho} \\ 1 & \text{if } \sigma(\sqrt{\rho}) = \sqrt{\rho}, \end{cases} \qquad \text{where} \\ A \quad &= \quad \left(\begin{pmatrix} 0 & \sqrt{-\psi} \\ -\frac{1}{\sqrt{-\psi}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\rho\sqrt{-\psi}^3 \\ \rho^{-1}(\sqrt{-\psi})^{-3} & 0 \end{pmatrix} \right) \end{aligned}$$

and the calculation of the coboundary $c = \delta_2(\delta(s))$ of γ gives:

$$\begin{aligned} c(\sigma,\tau) &= \gamma(\sigma)^{-1}\cdot\gamma(\sigma\tau)\cdot{}^{\sigma}\gamma(\tau)^{-1} \\ &= \begin{cases} 1 & \text{if } \tau(\sqrt{\rho}) = \sqrt{\rho} \\ A\cdot{}^{\sigma}A^{-1} & \text{if } \tau(\sqrt{\rho}) = -\sqrt{\rho} \text{ and } \sigma(\sqrt{\rho}) = \sqrt{\rho} \\ A^{-1}\cdot{}^{\sigma}A^{-1} & \text{if } \tau(\sqrt{\rho}) = -\sqrt{\rho} \text{ and } \sigma(\sqrt{\rho}) = -\sqrt{\rho}. \end{cases} \end{aligned}$$

The relations $A^{-1} = -A$ and ${}^{\sigma}\!A = \pm A$ if $\sigma(\sqrt{-\psi}) = \pm\sqrt{-\psi}$ now imply $c = c_{-\psi\rho,\rho}$.

(b) From (a) and the well known relation between the Hilbert symbol and the cohomology group $H^2(F,\mu_2)$ it follows that $S(\psi,\rho) \in G_2(F)S_0'$ if and only if $(-\psi\rho,\rho)_H = 1$. The claim now follows from the bilinearity of the Hilbert symbol and the relation $(-\rho,\rho)_H = 1$. □

5 Schröder decomposition for H_2

(5.1) By the Iwasawa-decomposition $G_2(F) = P_{12}(F)\cdot G_2(\mathcal{O}_F) = T(F)\cdot U_{12}(F)\cdot G_2(\mathcal{O}_F)$, where U_{12} denotes the unipotent radical of the Borel P_{12}, every $G_2(\mathcal{O}_F)$-orbit in $G_2(F)S_0$ meets $U_{12}(F)S_0$. We introduce the elements

$$\begin{aligned} &S_1 = F_{-1,2},\quad S_2 = F_{-1,-3} + 2F_{0,2},\quad S_3 = F_{1,2} + 2F_{0,-3},\quad S_4 = F_{1,-3},\\ &S_5 = F_{2,2},\quad S_6 = F_{-3,2},\quad S_7 = F_{-3,-3}.\end{aligned}$$

In the next table we describe the right action of the unipotent subgroups corresponding to the positive (with respect to P_{12}) roots $\alpha_2,\alpha_3,\beta_2,\beta_3$ on the S_i. Their action on S_5, S_6, S_7 is trivial. Also the action of the unipotent subgroup corresponding to β_1 is trivial on S_i for $0 \le i \le 8$.

	$\beta_2^{\vee}\begin{pmatrix}1 & -e\\ 0 & 1\end{pmatrix}$	$\alpha_3^{\vee}\begin{pmatrix}1 & e\\ 0 & 1\end{pmatrix}$	$\alpha_2^{\vee}\begin{pmatrix}1 & e\\ 0 & 1\end{pmatrix}$	$\beta_3^{\vee}\begin{pmatrix}1 & e\\ 0 & 1\end{pmatrix}$
$S_0 = F_{1,-1} - F_{0,0} - S'$	$S_0 + eS_4$	$S_0 - eS_3 - e^2S_7$	$S_0 - eS_2 - e^2S_5$	$S_0 + eS_1$
$S_1 = F_{2,-1}$	$S_1 + eS_6$	$S_1 - eS_5$	S_1	S_1
$S_2 = F_{-1,-3} + 2F_{0,2}$	$S_2 + eS_7$	$S_2 + eS_6$	$S_2 + 2eS_5$	S_2
$S_3 = F_{1,2} + 2F_{0,-3}$	S_3	$S_3 + 2eS_7$	$S_3 + eS_6$	$S_3 + eS_5$
$S_4 = F_{1,-3}$	S_4	S_4	$S_4 - eS_7$	$S_4 + eS_6$

Now an easy calculation shows:

$$\begin{aligned} & S_0.\beta_3^{\vee}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}.\alpha_2^{\vee}\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}.\alpha_3^{\vee}\begin{pmatrix}1 & -c\\ 0 & 1\end{pmatrix}.\beta_2^{\vee}\begin{pmatrix}1 & -d\\ 0 & 1\end{pmatrix} \\ \text{(x)}\quad = \;& S_0 + aS_1 + bS_2 + cS_3 + dS_4 + (ac-b^2)S_5 + (ad-bc)S_6 + (bd-c^2)S_7 \end{aligned}$$

The orbit $U_{12}(F)s_0$ is thus described as a subset of the 7-dimensional affine space $A_7 = S_0 + \langle S_i \rangle_{1 \le i \le 7}$. By applying $\alpha_2^\vee, \alpha_3^\vee, \beta_2^\vee, \beta_3^\vee$ of some integral unimodular matrices we can achieve

$$min(val(ac-b^2), val(ad-bc), val(bd-c^2)) \;\le\; min(val(a), val(b), val(c), val(d))$$

(5.2) We still have an action of $\alpha_1^\vee(\mathrm{SL}_2(\mathcal{O}_F))$ on $R_4 = \langle S_1, S_2, S_3, S_4 \rangle$ and on $R_3 := \langle S_5, S_6, S_7 \rangle$, which respects the orbit structure, i.e. acts on $U_{12}(F)S_0$. As right representation of $\alpha_1^\vee(\mathrm{SL}_2)$ we can identify the space R_4 with the homogeneous polynomials of degree 3 in X, Y and the space R_3 with the symmetric 2×2-matrices. It is easy to see that we have a correspondence:

S_1	S_2	S_3	S_4	S_5	S_6	S_7
Y^3	$3Y^2X$	$3YX^2$	X^3	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

Here an element $g \in \mathrm{SL}_2$ with entries a, b, c, d acts via $X \mapsto aX + bY, Y \mapsto cX + dY$ on the homogeneous polynomials and via $S \mapsto {}^tg \cdot S \cdot g =: S.g$ on the symmetric matrices S.

Using this action of $\alpha_1^\vee(\mathrm{SL}_2(\mathcal{O}_F))$ we can now achieve that $ad - bc = 0$ and that $val(ac-b^2) \le val(bd-c^2)$. We introduce the notations $\Delta = ac - b^2$ and $\delta = -(bd-c^2)/\Delta$. This implies $c = a\delta,\ d = b\delta$, i.e. we have to consider elements of the form

$$\textbf{(xi)} \qquad \begin{aligned} S \;&=\; S_0 + aS_1 + bS_2 + a\delta S_3 + b\delta S_4 + \Delta S_5 - \delta\Delta S_7, \qquad \text{where} \\ & a, b, \Delta \in F,\ \delta \in \mathcal{O}_F, \quad \Delta = a^2\delta - b^2, \quad val(\Delta) \le val(a), val(b). \end{aligned}$$

(5.3) We rewrite (**xi**) in matrix form

$$S \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & b\delta & a\delta \\ 0 & 0 & 0 & -1 & 0 & 2a\delta & 2b \\ 0 & 0 & \frac{1}{2} & 0 & 0 & b & a \\ 0 & -\frac{1}{2} & b\delta & 2a\delta & b & -2\delta\Delta & 0 \\ -\frac{1}{2} & 0 & a\delta & 2b & a & 0 & 2\Delta \end{pmatrix}$$

Now we apply elements in the opposite unipotent radical $U_{12}^-(\mathcal{O}_F)$ to get a representative of each $G_2(\mathcal{O}_F)$-orbit in some standard form (compare [W2],[W3]): It is straightforward to get zeros in the last row (and the last column) apart from the lower right entry 2Δ. All entries of the final result can be obtained by a laborious brute force computation, but the result is already clear from lemma 4.3(d):

$$S_{13} := S.\alpha_3^\vee\begin{pmatrix}1 & 0\\ \frac{a}{2\Delta} & 1\end{pmatrix}.\alpha_2^\vee\begin{pmatrix}1 & 0\\ \frac{-b}{2\Delta} & 1\end{pmatrix}.\beta_3^\vee\begin{pmatrix}1 & 0\\ -\frac{a\delta}{2\Delta} & 1\end{pmatrix}.\beta_1^\vee\begin{pmatrix}1 & 0\\ -\frac{ab}{2\Delta^2} & 1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{8\Delta} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{a^2}{8\Delta^2} & -\frac{b}{4\Delta} & 0 & 0 & 0 & 0\\ 0 & -\frac{b}{4\Delta} & -\frac{\delta}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{a^2}{2\Delta} & b & 0\\ 0 & 0 & 0 & 0 & b & -2\delta\Delta & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 2\Delta\end{pmatrix}$$

(5.4) To transform S_{13} with elements of $\beta_2^\vee(\mathrm{SL}_2(\mathcal{O}_F))$ into the final diagonal form we have to distinguish three cases.

Case 1: $val(\delta\Delta) \leq val(b),\ val(\delta\Delta) \leq val(\frac{a^2}{\Delta})$. In this case we get:

$$S_{14} := S_{13}.\beta_2^\vee\begin{pmatrix}1 & 0\\ -\frac{b}{2\Delta\delta} & 1\end{pmatrix} = S(\delta^{-1}, -4\delta^2\cdot\Delta),$$

using 2.3(**iv**) and the notation introduced in (**viii**).

Case 2: $val(\frac{a^2}{\Delta}) < val(\delta\Delta),\ val(\frac{a^2}{\Delta}) \leq val(b)$. We get:

$$S_{14} : = S_{13}.\beta_2^\vee\begin{pmatrix}1 & -\frac{2b\Delta}{a^2}\\ 0 & 1\end{pmatrix} = S(a^2/\Delta, -4\Delta^3/a^4).$$

Case 3: $val(b) < val(\delta\Delta),\ val(b) < val(\frac{a^2}{\Delta})$

In this case we have $val(b^2) < val(\delta\Delta) + val(\frac{a^2}{\Delta}) = val(\delta a^2)$ which implies that $-\Delta = b^2 - a^2\delta$ is a square. We can furthermore choose $\mu \in \mathcal{O}_F^*$ such that $\mu^2\cdot b = b + 2\delta\Delta + \frac{a^2}{8\Delta}$ and define $\nu = -1 - \frac{a^2}{4\Delta b} \in \mathcal{O}_F$. We get

$$\begin{aligned}S_{14} : &= S_{13}.\beta_2^\vee\begin{pmatrix}1 & 1/2\\ 0 & 1\end{pmatrix}.\beta_2^\vee\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}.\beta_2^\vee\begin{pmatrix}1 & 0\\ \nu & 1\end{pmatrix}\\ &= S(2\Delta/b, -b^2/\Delta).\end{aligned}$$

In all three cases S_{14} is of the form $S(\alpha,\beta)$. Since $val(\Delta) = val(\alpha^2\beta) \leq val(\alpha\beta)$ and $val(\Delta) \leq val(\alpha)$ by (**xi**), we have $val(\alpha) \leq 0$ and $val(\alpha\beta) \leq 0$. Since $S(\alpha,\beta).\beta_2^\vee(W_2) = S(-\alpha\beta,\beta^{-1})$ we can achieve furthermore $val(\beta) \leq 0$.

By inspection of all three cases we observe that the Hilbert symbol satisfies $(\alpha,\beta)_H = 1$, as predicted in lemma 4.4(b). We still have an action of $T(\mathcal{O}_F)$ at our disposal: $S(\alpha,\beta).(t_i) = S(\alpha\cdot t_1^{-2}, \beta\cdot t_1^2 t_3^{-2})$. Thus α and β can be changed by squares of units.

We finally remark that we can modify β by non-square units in the case $val(\alpha) = 0$ using the action of $\alpha_1^\vee(\mathrm{SL}_2(\mathcal{O}_F))$ and α by non-square units in the case $val(\beta) = 0$ using the action of $\beta_2^\vee(\mathrm{SL}_2(\mathcal{O}_F))$.

(5.5) Notation: Let $\varepsilon \in \mathcal{O}_F^* \backslash (\mathcal{O}_F^*)^2$ be a fixed non-square unit. We define

$$\textbf{(xii)} \quad \mathcal{R} \quad = \quad \big\{(\alpha,\beta) \mid \alpha = \varepsilon_1 \varpi^{-m},\ \beta = \varepsilon_2 \varpi^{-n}, \quad m,n \geq 0, \quad \varepsilon_1, \varepsilon_2 \in \{1, \varepsilon\}\big\}$$

Proposition 5.6. *Under the assumptions of 1.1 a set of representatives for the $G_2(\mathcal{O}_F)$-orbits inside $G_2(F)S_0$ is*

$$\{S(\alpha,\beta) \,|\, (\alpha,\beta) \in \mathcal{R},\ (\alpha,\beta)_H = 1,\ \varepsilon_1 = 1 \textit{ if } n = 0,\ \varepsilon_2 = 1 \textit{ if } m = 0\}$$

using the notations introduced in (**viii**),(**xii**).

Proof: By our above considerations every $G_2(\mathcal{O}_F)$-orbit has a representative of the stated form. It follows already from lemma 4.4 that every $S(\alpha,\beta)$ as in the proposition lies in $G_2(F)S_0$, but we still have to prove that different $S(\alpha,\beta)$ lie in different $G_2(\mathcal{O}_F)$-orbits, i.e. that each $S(\alpha,\beta)$ as in the proposition is determined by its $G_2(\mathcal{O}_F)$-orbit.

Write $\alpha = \varepsilon_1 \cdot \varpi^{-m},\ \beta = \varepsilon_2 \cdot \varpi^{-n}$ as in (**xii**). For $x \in Sym_7(F)$ the quantity $dep_{2,0}(x) := \max_{n\in\mathbb{Z}} \varpi^{-n} \cdot x \in Sym_7(\mathcal{O}_F)$ is invariant under the action of $G_2(\mathcal{O}_F) \subset \mathrm{GL}_7(\mathcal{O}_F)$. We have $dep_{2,0}(S(\alpha,\beta)) = val(\alpha^2\beta) = 2m+n$. If $m > 0$ we have furthermore $\varpi^{2m+n}S(\alpha,\beta) \cong -\frac{1}{2}\varepsilon_1^2\varepsilon_2 \cdot F_{2,2} \mod \varpi \cdot Sym_7(\mathcal{O}_F)$. Under the action of $g \in G_2(\mathcal{O}_F) \subset \mathrm{GL}_7(\mathcal{O}_F)$ the coefficient of $F_{2,2}$ is modified by multiplication with $g_{2,2}^2 \mod \varpi$. This implies that the class of ε_2 in $\mathcal{O}_F^*/(\mathcal{O}_F^*)^2$ is determined by the $G_2(\mathcal{O}_F)$-orbit of $S(\alpha,\beta)$.

Now we consider $Q = S(\alpha,\beta) \cdot S(\alpha,\beta) \in Sym^2(Sym_7(F)) \simeq Sym^2(Sym^2(V_7^*)) =: W$. Here we consider $Sym^2(V)$ always as a quotient of the tensor product $V \otimes V$ and may thus write $F_{i,j} \cdot F_{k,l} \in Sym^2(Sym_7(F))$ resp. $(x_ix_j)\cdot(x_kx_l) \in Sym^2(Sym^2(V_7^*))$. We remark that this expression is commutative by definition, i.e. invariant under substitutions $x_i \leftrightarrow x_j, x_k \leftrightarrow x_l, (x_i,x_j) \leftrightarrow (x_k,x_l)$, but not associative. In fact we have a $G_2(F)$-equivariant idempotent projection

$$P_1 : W \to W,\ (x_ix_j)\cdot(x_kx_l) \mapsto \frac{1}{3}\left((x_ix_j)\cdot(x_kx_l) + (x_ix_k)\cdot(x_jx_l) + (x_ix_l)\cdot(x_jx_k)\right),$$

such that we can identify the image $P_1(W)$ with $Sym^4(V_7^*)$. Now $\ker(P_1)$ is the image of W under the idempotent $P_2 = id - P_1$. From 2.9(**vi**) we deduce

$$Sym^2(Sym_7) \;=\; Sym^2(V_{2,0}) \oplus V_{2,0} \oplus V_{0,0} \;=\; V_{4,0} \oplus V_{0,2} \oplus V_{1,1} \oplus 3\cdot V_{2,0} \oplus 2\cdot V_{0,0}$$

and by a weight analysis we conclude

$$\begin{aligned} Sym^4(V_7^*) &= V_{4,0} \oplus V_{2,0} \oplus V_{0,0} \quad \text{and} \\ \ker(P_1) &= V_{0,2} \oplus V_{1,1} \oplus 2 \cdot V_{2,0} \oplus V_{0,0}. \end{aligned}$$

We get

$$\begin{aligned} P_2(Q) = -\frac{1}{3} \big(& \alpha^3\beta^2 (F_{2,2} \cdot F_{-3,-3} - F_{2,-3} \cdot F_{2,-3}) - \alpha^3\beta(F_{2,2} \cdot F_{-1,-1} - F_{2,-1} \cdot F_{2,-1}) \\ & +\alpha^2\beta(F_{-1,-1} \cdot F_{-3,-3} - F_{-1,-3} \cdot F_{-1,-3} + 2F_{2,2} \cdot F_{0,0} - 2F_{2,0} \cdot F_{2,0}) \\ & -\alpha\beta(F_{1,1} \cdot F_{2,2} - F_{1,2} \cdot F_{1,2} + 2F_{-3,-3} \cdot F_{0,0} - 2F_{-3,0} \cdot F_{-3,0}) \\ & +\alpha(F_{2,2} \cdot F_{3,3} - F_{2,3} \cdot F_{2,3} + 2F_{-1,-1} \cdot F_{0,0} - 2F_{-1,0} \cdot F_{-1,0}) \\ & +\beta(F_{1,1} \cdot F_{-3,-3} - F_{1,-3} \cdot F_{1,-3}) - (F_{1,1} \cdot F_{-1,-1} - F_{1,-1} \cdot F_{1,-1}) \\ & -(F_{2,2} \cdot F_{-2,-2} - F_{2,-2} \cdot F_{2,-2}) - (F_{3,3} \cdot F_{-3,-3} - F_{3,-3} \cdot F_{3,-3}) \\ & + \text{ analogous terms with negative exponents of } \alpha, \beta \, \big) \end{aligned}$$

Since $val(\alpha), val(\beta) \leq 0$ we get that $dep_{0,2}(P_2(Q)) := \max_{n\in\mathbb{Z}} \varpi^{-n} \cdot P_2(Q) \in Sym^2(Sym^2(\mathcal{O}_F^7)) \cap \ker P_1$, which is an invariant of the $G_2(\mathcal{O}_F)$-orbit of Q, equals $3m + 2n + val(3)$. Furthermore if $n = -val(\beta) > 0$ we get that the coefficient of $F_2 = F_{2,2} \cdot F_{-3,-3} - F_{2,-3} \cdot F_{2,-3}$ is multiplied with $(xw - yz)^2 \mod \varpi$ under the action of $g \in G_2(\mathcal{O}_F)$, where $x_2.g = x \cdot x_2 + y \cdot x_{-3} + ...$ and $x_{-3}.g = z \cdot x_2 + w \cdot x_{-3}$. Thus the class of ε_1 in $\mathcal{O}_F^*/(\mathcal{O}_F^*)^2$ is an invariant of the $G_2(\mathcal{O}_F)$-orbit of $S(\alpha, \beta)$ in the case $n > 0$.

Summarizing $2m + n$ and $3m + 2n$ are determined by the $G_2(\mathcal{O}_F)$-orbit of $S(\alpha, \beta)$, as are ε_1 if $m > 0$ and ε_2 if $n > 0$. The claim follows. □

(5.7) To get a concrete set of representatives for the double cosets we make some explicit computations: We observe that for every pair $(\alpha, \beta) \in \mathcal{R}$ with $(\alpha, \beta)_H = 1$ at least one of the following four conditions is satisfied:

<u>Condition 1:</u> $\beta = (2\gamma)^2$. We put in the notations of (**xi**): $a = 0,\ b = \alpha\gamma,\ \delta = \alpha^{-1} \in \mathcal{O}_F$. Thus $\Delta = -b^2 = -\alpha^2\gamma^2 = -\frac{\alpha^2\beta}{4},\ val(\gamma) \leq 0,\ val(\Delta) \leq val(b), val(a)$, so that we can start the procedure 5.3. We have $-2\delta\Delta = \alpha\beta/2$, so that $val(\delta\Delta) = val(\alpha) + val(\beta) = val(\alpha) + 2val(\gamma) \leq val(\alpha) + val(\gamma) = val(b)$ and we are in case 1 above arriving at the desired $S(\delta^{-1}, -4\delta^2 \cdot \Delta) = S(\alpha, \beta)$.

<u>Condition 2:</u> $\alpha = \gamma^2$. We put in the notations of (**xi**): $a = \frac{1}{4}\alpha\gamma(\beta - 1),\ b = \frac{1}{4}\alpha(\beta + 1),\ \delta = \alpha^{-1} \in \mathcal{O}_F$. Thus $\Delta = \delta a^2 - b^2 = \frac{1}{16}\alpha^2((\beta-1)^2 - (\beta+1)^2) = -\frac{\alpha^2\beta}{4},\ val(\alpha) \leq val(\gamma) \leq 0,\ val(\Delta) \leq val(b), val(a)$ so that we can start the procedure 5.3. We have $-2\delta\Delta = \alpha\beta/2$, so that $val(\delta\Delta) = val(\alpha) + val(\beta) \leq val(\gamma) + val(\beta + 1) = val(b)$ and we are in case 1 above arriving again at the desired $S(\delta^{-1}, -4\delta^2 \cdot \Delta) = S(\alpha, \beta)$.

<u>Condition 3:</u> β is $-\alpha$ up to square, say $\beta = -\frac{(2\gamma)^2}{\alpha}$. We put in the notations of (**xi**): $a = \gamma\alpha,\ b = 0,\ \delta = \alpha^{-1} \in \mathcal{O}_F$. Thus $\Delta = \delta(\gamma\alpha)^2 = \gamma^2\alpha = -\frac{\alpha^2\beta}{4},\ val(\gamma) =$

$\frac{1}{2}(val(\alpha)+val(\beta)) \le 0,\ val(\Delta) \le val(a), val(b)$ so that we can start the procedure 5.3. We have $-2\delta\Delta = \alpha\beta/2,\ -\frac{a^2}{2\Delta} = -\frac{\alpha}{2}$, so that S_{13} is already the desired $S(\alpha,\beta)$.

<u>Condition 4:</u> $\alpha = \varepsilon\gamma_1^2,\ \beta = \varepsilon\gamma_2^2$ and -1 is not a square in F. Then -1 is a norm of the unramified extension $F(\sqrt{\varepsilon})/F$, i.e. $-1 = a_0^2 - \varepsilon b_0^2$ with $a_0, b_0 \in \mathcal{O}_F$. We put in the notations of (**xi**): $a = \frac{1}{2}\gamma_1^3\gamma_2\varepsilon^2 a_0,\ b = \frac{1}{2}\gamma_1^2\gamma_2\varepsilon^2 b_0,\ \delta = \alpha^{-1} \in \mathcal{O}_F$. Thus $\Delta = \delta a^2 - b^2 = \frac{1}{4}\gamma_1^4\gamma_2^2\varepsilon^3(a_0^2 - \varepsilon b_0^2) = -\frac{1}{4}\gamma_1^4\gamma_2^2\varepsilon^3 = -\frac{\alpha^2\beta}{4},\ val(\Delta) \le val(b), val(a)$ so that we can start the procedure 5.3. We have $-2\delta\Delta = \alpha\beta/2$, so that $val(\delta\Delta) = val(\alpha) + val(\beta) = 2val(\gamma_1) + 2val(\gamma_2) \le 2val(\gamma_1) + val(\gamma_2) = val(b)$ and we are again in case 1 above arriving once more at the desired $S(\delta^{-1}, -4\delta^2 \cdot \Delta) = S(\alpha,\beta)$.

(5.8) For $a, b \in F, \delta \in \mathcal{O}_F$ we introduce the notation

$$\mu(a,b,\delta) \quad = \quad \beta_3^\vee\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} \cdot \alpha_2^\vee\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix} \cdot \alpha_3^\vee\begin{pmatrix}1 & -a\delta\\ 0 & 1\end{pmatrix} \cdot \beta_2^\vee\begin{pmatrix}1 & -b\delta\\ 0 & 1\end{pmatrix}$$

and the following sets corresponding to the four conditions in the proof of 5.6:

$$\begin{aligned}
I_1 &= \left\{(0, \varpi^{-n-m}\varepsilon_1, \varpi^m\varepsilon_1^{-1}) \,\middle|\, \begin{matrix} n, m \ge 0,\ \varepsilon_1 \in \{1, \varepsilon\}, \\ \varepsilon_1 = \varepsilon \text{ if } m \text{ even},\ \varepsilon_1 = 1 \text{ if } n = 0\end{matrix}\right\} \\
I_2 &= \left\{\left(\frac{\varpi^{-3n}}{4}\cdot(\varepsilon_1\varpi^{-m} - 1), \frac{\varpi^{-2n}}{4}\cdot(\varepsilon_1\varpi^{-m} + 1), \varpi^{2n}\right) \,\middle|\, \begin{matrix} n, m \ge 0,\ \varepsilon_1 \in \{1, \varepsilon\} \\ \varepsilon_1 = 1 \text{ if } n = 0\end{matrix}\right\}
\end{aligned}$$

$$\begin{aligned}
I_3 &= \left\{(\varpi^{-n-m}\varepsilon_1, 0, \varpi^m\varepsilon_1^{-1}) \,\middle|\, \begin{matrix} n, m \ge 0,\ \varepsilon_1 \in \{1, \varepsilon\},\ m \text{ odd if } -1 \in (F^*)^2 \\ \varepsilon_1 = \varepsilon \text{ if } m \text{ even},\ \varepsilon_1 = 1 \text{ if } n = 0\end{matrix}\right\} \\
I_4 &= \{(\varpi^{-3n-m}\cdot a_0, \varpi^{-2n-m}\cdot b_0, \varepsilon^{-1}\varpi^{2n}) \mid n, m > 0,\ -1 \notin (F^*)^2\} \\
I &= I_1 \cup I_2 \cup I_3 \cup I_4
\end{aligned}$$

Here $a_0, b_0 \in \mathcal{O}_F$ are fixed satisfying the condition $-1 = a_0^2 - \varepsilon \cdot b_0^2$.

Corollary 5.9. *We have a disjoint decomposition*

$$G_2(F) \quad = \quad \dot{\bigcup_{(\alpha,\beta,\delta)\in I}} H_2(F)\cdot\mu(\alpha,\beta,\delta)\cdot G_2(\mathcal{O}_F)$$

□

Corollary 5.10. *There exist finitely many g_i such that we have a decomposition*

$$G_2(F) \quad = \quad \dot{\bigcup_{i\in I}} H_2(F)\cdot g_i\cdot T(F)\cdot G_2(\mathcal{O}_F)$$

In fact $T(F)$ acts on the set of all $S(\alpha,\beta)$ such that the orbits have representatives with $0 \le m, n \le 1$. □

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