The Valuative Section Conjecture

Oberseminar Arithmetische Homotopietheorie
Wintersemester 2014/15

Let X be a normal, geometrically irreducible variety over a field k. Let $\bar{k}$ be a separable closure of k, and denote by $\overline{X} = X \times_k \bar{k}$ the base change of X to k. If we denote by $\mathrm{Gal}_k = \mathrm{Gal}(\bar{k}|k)$ the absolute Galois group of k and by $\pi_1(X, \bar{x})$ resp. $\pi_1(\overline{X}, \bar{x})$ the étale resp. geometric fundamental group of X with basepoint $\bar{x}$, we get an exact sequence

$$\pi_1(X/k): \qquad 1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}_k \to 1.$$

By functoriality of the fundamental group, any rational point $a \in X(k)$ gives rise to a conjugacy class of sections $[s_a]$ of $\pi_1(X/k)$. We therefore get a map

$$X(k) \to \mathcal{S}_{\pi_1(X/k)}, \qquad a \mapsto [s_a]$$

from $X(k)$ to the set $S_{\pi_1(X/k)}$ of conjugacy classes of sections of $\pi_1(X/k)$. So geometric information (the existence of rational points) is stored inside the fundamental group.

For certain (so-called *anabelian*) varieties, the arithmetic information of the fundamental group is believed to encode even more geometric data. In a letter to Faltings in 1983 ([Gro97]), Grothendieck made the following conjecture:

Conjecture (*Section Conjecture*). *If X is a smooth, projective and geometrically connected curve of genus ≥ 2 over a number field k, then the above map $X(k) \to S_{\pi_1(X/k)}$ is a bijection.*

Attempts to understand this conjecture soon led to considering the case where k is a p-adic field instead of a number field. This is known as the *p-adic section conjecture*.

The injectivity of the above map was already mentioned in the letter of Grothendieck; it can be derived from the Mordell-Weil theorem in the case of number fields and from the theorem of Mattuck-Tate in the p-adic case. A real analogue was proven by Mochizuki, and a birational version of the p-adic section conjecture was proven by Koenigsmann. But until today, neither the original nor the p-adic version of the section conjecture have been proven. Even the list of curves known to fulfill the section conjecture is limited as it only contains examples with no sections at all.

In their paper [PS14], Pop and Stix change the viewpoint from rational points to valuations. From now on, let us fix the following notation: k is a p-adic field with valuation ring $\mathfrak{o}$, X (as before) a smooth, projective and geometrically connected curve of genus ≥ 2 over k. Let K denote the function field of X and $\tilde{K}$ be the function field of the maximal pro-étale cover $\tilde{X}$ of X. An $\mathfrak{o}$-valuation w of K is a valuation whose valuation ring contains $\mathfrak{o}$. To every such $\mathfrak{o}$-valuation w with prolongation $\tilde{w}$ to $\tilde{K}$ we can associate the decomposition group $D_{\tilde{w}|w}$ inside $\mathrm{Gal}(\tilde{K}|K) \cong \pi_1(X, \bar{x})$.

Theorem. *In the above context, let $s : \mathrm{Gal}_k \to \pi_1(X, \bar{x})$ be a section of $\pi_1(X/k)$. Then there exists an $\mathfrak{o}$-valuation w of K with prolongation $\tilde{w}$ to $\tilde{K}$ such that $s(\mathrm{Gal}_k) \subset D_{\tilde{w}|w}$.*

This is an approximation to the section conjecture: If all the $\mathfrak{o}$-valuations above were in fact k-valuations, then the p-adic section conjecture would follow immediately.

The main ingredients of the proof are the so-called Brauer group method, which leads to a valuation coming close to having the claimed properties, and then profound knowledge of the relations of the inertia groups used in a fixed point method. The aim of this seminar is to understand this proof and the techniques behind it.

Time and Place: Thursdays, 9-11 a.m.; INF 288/MathI HS 2

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Talks

Please contact me directly if you want to have Talk 2 or 3. The other talks will be distributed at the end of the first session on Thursday, 16/10/14.

1 Introduction

2 Models and stable reduction

Give a short overview of models of curves as in [Rom13, §2]: normal/regular/stable/semi-stable models, existence of a regular model, intersection theory, blow-ups, blow-downs, contractions, minimal regular model. Most of this should be known anyway, so proofs can be sketched or omitted. Then state the stable reduction theorem of Deligne-Mumford. If time allows, explain the idea of the proof (either following Deligne-Mumford as in [Rom13] or following Artin-Winters ([AW71])).

3 The zoo of valuations

The main part of this talk is concerned with the structure of the $\mathfrak{o}$-valuations of K ([PS14, Appendix A]). Define the space of all $\mathfrak{o}$-valuations $\mathrm{Val}_{\mathfrak{o}}(K)$ with its patch topology and the Riemann-Zariski space of K/k with its limit constructible topology. Prove that the center map induces a homeomorphism between these two spaces (§A.1, see also [Vaq00, §7]). The next topic is the description of the types of valuations (§A.2-A.4). The rigid analytic description is not necessary in the following talks and can be (but does not have to be) omitted. Present and prove the table in §A.5 and describe the valuations of the universal cover (§A.6). Afterwards, define decomposition and inertia groups of points and valuations and describe their behaviour under refinements ([PS14, Appendix B]).

4 Separation properties for inertia groups

[PS14], §2: The main purpose of this talk is Proposition 2.1, which separates inertia subgroups of valuation of type 1v and positive genus. It is derived from the localization sequence of a model by various limit procedures (on the base, the coefficients and the models) and Pontryagin duality. Emphasize the fact that the matrix $\bar{\rho}_n$ in (2.3) is the intersection matrix.

5 Log geometry

Give an introduction into logarithmic geometry, following [Ill02] and/or [Sti02, §3]: (fs) monoids, log structures, (Kummer) log etale covers ([Ill02, §1],[Sti02, §3.1]). Define the logarithmic fundamental group ([Ill02, 4.6], [Sti02, 3.3.7]) and determine the log fundamental group of a strict Henselian local noetherian fs-log scheme ([PS14, 3.2 (1)], [Sti02, 3.1.1], [Ill02, 4.7]). Explain the connection between log and tame fundamental group and Fujiwara-Kato purity ([Ill02, 4.7 (c) & 7.6], [Sti02, Example (3) after 3.3.7]). Define log blow-ups ([Ill02, 6.1], [Sti02, §3.3.2]) and explain the behaviour of log fundamental groups under log blow-ups ([Ill02, 6.10],[Sti02, 3.3.11]). Also mention the injectivity result of [Sti02, 6.2.11].

Concentrate on explaining similarities and differences between the log and non-log versions instead of giving full proofs.

6 The structure and properties of logarithmic inertia groups

[PS14], §3: Based on the results of the previous talk, present the structure of the logarithmic inertia group of a valuation. The central result of this talk is a comparison result between logarithmic inertia groups (3.6), which will be important in the proof of the main theorem. Also define visible valuations and give equivalent descriptions. Describe the relation between the kernel of the specialization and log-specialization maps.

7 Lichtenbaum-Tate duality and relative Brauer group

Define the Lichtenbaum-Tate pairing $\mathrm{Br}(X) \times \mathrm{Pic}(X) \to \mathrm{Br}(k)$ ([Lic69, §3]) and the notions of period and index. The duality isomorphism $\mathrm{Br}(X) \cong \mathrm{Pic}(X)^*$ ([Lic69, Theorem 4]), whose proof can only be sketched, implies the theorem of Roquette and Lichtenbaum ([Lic69, Theorem 3]): the order of the relative Brauer group $\mathrm{Br}(X/k) = \ker(\mathrm{Br}(k) \to \mathrm{Br}(X))$ equals the index of X. Show that period and index are powers of p if $\pi_1(X/k)$ splits ([Sti10, 12-15] using [Lic69, Theorem 6 & 7]). Use this result to determine the pro-ℓ Brauer group of the decomposition pro-cover of a section ([PS14, §4.4]).

8 The Brauer group method

In this talk, the properties of the Brauer group are used to detect to each section a candidate valuation: State and prove local-global and local-semilocal principles for the Brauer group ([PS14, §4.1-4.3]). (Some facts needed for §4.2 can be found in [Gro68, §6].) These principles, together with the last theorem of the previous talk, are then used to show ([PS14, §4.5]) that the image of an ℓ-Sylow subgroup of Gal_k under a section of $\pi_1(X/k)$ is contained in a decomposition subgroup of some valuation. This theorem will be the starting point of the proof of the main theorem.

9 The valuative section conjecture

[PS14], §5: This talk comprises the statement and proof of the main theorem. After translating the theorem into a statement about the existence of a fixed point under a group action, the result from the previous talk is applied to find a candidate for such a fixed point. Afterwards, use the properties for inertia groups that were shown in the talks 4 and 6 as well as a combinatorial argument to derive that this candidate leads to one with the claimed properties. The result of Mochizuki ([PS14, Lemma 5.3]) can be used as a black box.

10 Arithmetic and uniqueness properties

In the first part, collect some arithmetic properties of valuations occuring in the valuative section conjecture ([PS14, §6]). Ideally, one would like to exclude all valuations of type other than $2h$ to deduce the p-adic section conjecture, but this has not been achieved yet. Afterwards, state and prove the uniqueness properties of the valuation as in [PS14], §7. The results of Tamagawa on resolution of non-singularities and of Nakamura on the anabelian wight filtration should be quoted without proof.

Literatur

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