On SK_1 of Iwasawa algebras

joint work with Peter Schneider

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The setup

R commutative ring,

$\mathcal{L}$ a R-Lie algebra, finitely generated free as R-module

$[\ ,\ ] : \mathcal{L} \wedge \mathcal{L} \to \mathcal{L}$

$$\bigwedge \mathcal{L} := < x \wedge y \mid [x, y]_L = 0 >_R \ \subseteq \ \ker[\ ,\ ]$$

Question: When does $\bigwedge \mathcal{L} = \ker[\ ,\ ]$ hold?

A counterexample

Assume that $2 \in R^\times$.
$V := R^4$ with standard basis $e_1, \ldots, e_4$ and
$W := \bigwedge^2 V / R(e_1 \wedge e_2 + e_3 \wedge e_4)$ (rank 5)

$$\partial : \bigwedge^2 V \xrightarrow{\mathrm{pr}} W \ .$$

Note that $\ker \partial$ does not contain any nonzero vector of the form $a \wedge b$.
$\mathcal{L}' := V \oplus W$ with bracket

$$[\ ,\ ] : \bigwedge^2 \mathcal{L}' \xrightarrow{\mathrm{pr}} \bigwedge^2 V \xrightarrow{\partial} W \xrightarrow{\subseteq} \mathcal{L}'$$

makes $\mathcal{L}'$ into a 2-step nilpotent Lie algebra over R with center $Z(\mathcal{L}') = [\mathcal{L}', \mathcal{L}'] = W$ and $e_1 \wedge e_2 + e_3 \wedge e_4 \in \ker[\ ,\ ] \setminus \bigwedge \mathcal{L}'$.

Chevalley orders

F field of characteristic zero

$\mathfrak{g}$ a F-split reductive Lie algebra over F with center $\mathfrak{z}$, Cartan subalgebra $\mathfrak{h}$ and root system Φ, $[X_\alpha, X_{-\alpha}] = -H_\alpha$

$Q^\vee := \sum_{\alpha\,\epsilon\,\Phi} \mathbb{Z}H_\alpha \subseteq \mathfrak{h}$ coroot lattice
$P^\vee := \{h \,\epsilon\, \sum_{\alpha\,\epsilon\,\Phi} \mathbb{Q}H_\alpha : \beta(h) \,\epsilon\, \mathbb{Z} \text{ for any } \beta \,\epsilon\, \Phi\} \subseteq \mathfrak{h}$ coweight lattice of the root system Φ

$\mathfrak{h}_\mathbb{Z} \subseteq \mathfrak{h}$ $\mathbb{Z}$-lattice such that $Q^\vee \subseteq \mathfrak{h}_\mathbb{Z} \subseteq P^\vee \oplus \mathfrak{z}$,

$$\mathfrak{g}_\mathbb{Z} := \mathfrak{h}_\mathbb{Z} + \sum_{\alpha\,\epsilon\,\Phi} \mathbb{Z}X_\alpha \subseteq \mathfrak{g} .$$

$\mathfrak{g}_\mathbb{Z}$ is a $\mathbb{Z}$-Lie subalgebra (Chevalley order) of $\mathfrak{g}$.

$\mathfrak{g}_R := R \otimes_\mathbb{Z} \mathfrak{g}_\mathbb{Z}$ is a R-Lie algebra.

Theorem

If 2 *and* 3 *are invertible in* R *then* $\ker[\ ,\ ] = \bigwedge \mathfrak{g}_R$.

Kostant had proved the case $R = \mathbb{C}$ by different methods. This is an integral version of his result.

Uniform pro-p-groups

G (topologically) finitely generated pro-p group with

1. $[G,G] \subseteq G^p$,
2. $G_i/G_{i+1} \xrightarrow[\cong]{.p} G_{i+1}/G_{i+2}$ for all $i \geq 1$,

where $G_1 := G$ and $G_i := [G, G_i]G^p$ is the lower p-central series, is called *uniform* pro-p group.

Facts:

1. $G \xrightarrow[\cong]{.p^{i-1}} G_i$ is homeomorphic (but not homomorphic in general).
2. (Lazard) A pro-finite group G is a p-adic Lie group $\iff$ G has an open characteristic subgroup which is uniform.

The associated Lie algebra

The operations

$$x + y := \lim_{n\to\infty} (x^{p^n} y^{p^n})^{\frac{1}{p^n}}$$

$$(x, y) := \lim_{n\to\infty} [x^{p^n}, y^{p^n}]^{\frac{1}{p^{2n}}}$$

make G into a $\mathbb{Z}_p$ Lie algebra, denoted $\mathcal{L} := \mathcal{L}(G)$,

and we have an equivalence of categories

$$\{G \text{ uniform}\} \longleftrightarrow \{\mathcal{L} \text{ with } (\mathcal{L}, \mathcal{L}) \subseteq p\mathcal{L}, \text{i.e., powerful}\}$$

Iwasawa algebras

G pro-finite group

$$\Lambda(G) := \varprojlim_{U \lhd G \text{ open}} \mathbb{Z}_p[G/U]$$

Iwasawa algebra

$$\Lambda_\infty(G) := \varprojlim_{U \lhd G \text{ open}} \mathbb{Q}_p[G/U]$$

$$SK_1(\mathbb{Z}_p[G/U]) := \ker\left(K_1(\mathbb{Z}_p[G/U]) \longrightarrow K_1(\mathbb{Q}_p[G/U])\right)$$

is known to be *finite!*

$$SK_1(\Lambda(G)) := \ker\left(K_1(\Lambda(G)) \longrightarrow K_1(\Lambda_\infty(G))\right) \cong \varprojlim SK_1(\mathbb{Z}_p[G/U])$$

by a result of Fukaya and Kato.

A homological description

`Oliver:` for H finite we have

$$\oplus_{A\subseteq H} H_2(A,\mathbb{Z}) \longrightarrow H_2(H,\mathbb{Z}) \longrightarrow SK_1(\mathbb{Z}_p[H]) \longrightarrow 0$$

where A runs trough all abelian subgroups of H.
Dualizing with $-^\vee := \mathrm{Hom}_{cts}(-,\mathbb{Q}_p/\mathbb{Z}_p)$ and taking limits gives:

$$SK_1(\Lambda(G))^\vee = \ker\left(H^2(G,\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \varinjlim_N \prod_{A\subseteq G/N} H^2(A,\mathbb{Q}_p/\mathbb{Z}_p)\right).$$

A cohomological criterion

If G has no torsion, then $SK_1 = 0 \iff$

$$0 \longrightarrow H^1(G, \mathbb{Q}_p/\mathbb{Z}_p)/p \xrightarrow{\delta} H^2(G, \mathbb{F}_p) \xrightarrow{\text{res}} \prod_{A \subseteq G} H^2(A, \mathbb{F}_p)$$

is exact. As a consequence of Whiteheads Lemma and a result of Lazard we obtain

Corollary

If G is a compact p-adic Lie group such that $L(G) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{L}(\mathcal{G})$ is semi-simple, then $SK_1(\Lambda(G))$ is finite.

The uniform case

`Lazard:` $$H^*(G,\mathbb{F}_p) = \bigwedge H^1(G,\mathbb{F}_p)$$

$V := G/G^p$.

Then $SK_1 = 0 \iff$

$$0 \longrightarrow \bigwedge V \xrightarrow{\subseteq} \bigwedge^2 V \xrightarrow{\delta^\vee} G^{ab}[p] \longrightarrow 0\ ,$$

is exact

The uniform case

`Lazard:` $$H^*(G,\mathbb{F}_p) = \bigwedge H^1(G,\mathbb{F}_p)$$

$V := G/G^p$.

Then $SK_1 = 0 \iff$

$$0 \longrightarrow \bigwedge V \xrightarrow{\subseteq} \bigwedge^2 V \xrightarrow{\delta^\vee} G^{ab}[p] \longrightarrow 0\ ,$$

is exact $\iff$

$$\bigwedge V = \ker \partial$$

$$\begin{aligned} \text{where } \partial \quad &: V \wedge V \longrightarrow (G^p/[G^p,G])[p] \\ gG^p \wedge hG^p &\longmapsto [g,h] \bmod [G^p,G] \end{aligned}$$

A Lie criterion

$$SK_1 = 0 \Longleftrightarrow \bigwedge \mathcal{L} = \ker[\ ,\ ]$$

Vanishing of SK_1

$R = \mathbb{Z}_p$ for $p \neq 2, 3$
$\mathfrak{g}$ a $\mathbb{Q}_p$-split reductive Lie algebra
$\mathfrak{g}_{\mathbb{Z}} \subseteq \mathfrak{g}$ a Chevalley order.
Then, for any $n \geq 1$, $p^n \mathfrak{g}_{\mathbb{Z}_p}$ corresponds to unique uniform p-adic Lie group $G(p^n)$ with $\mathbb{Z}_p$-Lie algebra

$$\mathcal{L}(G(p^n)) = p^n \mathfrak{g}_{\mathbb{Z}_p} \ .$$

Theorem

In the above setting we have

$$SK_1(\Lambda(G(p^n))) = 0 \ .$$

Examples

$\mathcal{G}$ a split reductive group scheme over $\mathbb{Z}$

$$G(p^n) := \ker\left(\mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{Z}/p^n)\right)$$

satisfies conditions of the theorem, e.g. for $m \geq 1$

$$\ker\left(SL_d(\mathbb{Z}_p) \to SL_d(\mathbb{Z}_p/p^m)\right).$$

Iwasawa Main Conjecture

Uniqueness-statements in Main Conjectures of Iwasawa theory:

$$\begin{tikzcd}
SK_1(\Lambda(G)) \arrow[d, hook] & \mathcal{L}, \mathcal{L}' \arrow[rr, mapsto] & & {[X_E]} \\
K_1(\Lambda(G)) \arrow[r] \arrow[d, "DET"'] & K_1(\Lambda(G)_S) \arrow[rr, two heads, "\partial"] \arrow[d, "DET"'] & & K_0(S-\mathrm{tor}) \\
Maps(\mathrm{Irr}(G), \overline{\mathbb{Z}_p}^\times) \arrow[r] & Maps(\mathrm{Irr}(G), \overline{\mathbb{Q}_p} \cup \{\infty\}) & &
\end{tikzcd}$$

That is, if $SK_1(\Lambda(G)) = 1$ and if $\mathcal{L}$ is induced form $\Lambda(G) \cap \Lambda(G)_S^\times$ (no poles), then $\mathcal{L}$ is unique with

1. $\partial\mathcal{L} = [X_E]$,
2. $DET(\mathcal{L})$ satisfies some interpolation property.