On the leading terms of
Zeta isomorphisms and p-adic L-functions
in non-commutative Iwasawa theory

Dedicated to John Coates

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ABSTRACT. We discuss the formalism of Iwasawa theory descent in the setting of the localized K_1-groups of Fukaya and Kato. We then prove interpolation formulas for the 'leading terms' of the global Zeta isomorphisms that are associated to certain Tate motives and of the p-adic L-functions that are associated to certain critical motives.

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1. Introduction

In the last few years there have been several significant developments in non-commutative Iwasawa theory.

Firstly, in [10], Coates, Fukaya, Kato, Sujatha and the second named author formulated a main conjecture for elliptic curves without complex multiplication. More precisely, if F_∞ is any Galois extension of a number field F which contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc} of F and is such that $\mathrm{Gal}(F_\infty/F)$ is a compact p-adic Lie group with no non-trivial p-torsion, then Coates et al formulated a $\mathrm{Gal}(F_\infty/F)$-equivariant main conjecture for any elliptic curve which is defined over F, has good reduction at all places above p and whose Selmer group (over F_∞) satisfies a certain natural torsion condition.

Then, in [14], Fukaya and Kato formulated a natural main conjecture for any compact p-adic Lie extension of F and any critical motive M which is defined over F and has good ordinary reduction at all places above p.

The key feature of [10] is the use of the localization sequence of algebraic K-theory with respect to a canonical Ore set. However, the more general approach of [14] is rather more involved and uses a notion of 'localized K_1-groups' together with Nekovar's theory of Selmer complexes and the (conjectural) existence of certain canonical p-adic L-functions. See [37] for a survey.

The p-adic L-functions of Fukaya and Kato satisfy an interpolation formula which involves both the 'non-commutative Tamagawa number conjecture' (this is a natural refinement of the 'equivariant Tamagawa number conjecture' formulated by Flach and the first named author in [6] and hence also implies the 'main conjecture of non-abelian Iwasawa theory' discussed by Huber and Kings in [17]) as well as a local analogue of the non-commutative Tamagawa number conjecture. Indeed, by these means, at each continuous finite dimensional p-adic representation ρ of $\mathrm{Gal}(F_\infty/F)$, the 'value at ρ' of the p-adic L-function is explicitly related to the value at the central critical point of the complex L-function associated to the 'ρ^*-twist' $M(\rho^*)$ of M. However, if the Selmer module of $M(\rho^*)$ has strictly positive rank (and recent work of Mazur and Rubin [19] shows that this should often be the case), then both sides of the Fukaya-Kato interpolation formula are equal to zero.

The main aim of the present article is therefore to extend the formalism of Fukaya and Kato in order to obtain an interesting interpolation formula for all representations ρ as above. To this end we shall introduce a notion of 'the leading term at ρ' for elements of suitable localized K_1-groups. This notion is defined in terms of the Bockstein homomorphisms that have already played significant roles (either implicitly or explicitly) in work of Perrin-Riou [25, 27], of Schneider [32, 31, 30, 29] and of Greither and the first named author [8, 3] and have been systematically incorporated into Nekovar's theory of Selmer complexes [22]. We then give two explicit applications of this approach in the setting of extensions F_∞/F with $F_{\mathrm{cyc}} \subseteq F_\infty$. We show first that the 'p-adic Stark conjecture at $s = 1$', as formulated by Serre [33] and interpreted by Tate in [35], can be reinterpreted as providing interpolation formulas for the leading terms of the global Zeta isomorphisms associated to certain Tate motives in terms of the leading terms at $s = 1$ (in the classical sense) of the p-adic Artin L-functions that are constructed by combining Brauer induction with the fundamental results of Deligne and Ribet and of Cassou-Nogues. We then also prove an interpolation formula for the leading terms of the Fukaya-Kato p-adic L-functions which involves the leading term at the central critical point of the associated complex L-function, the Neron-Tate pairing and Nekovar's p-adic height pairing.

In a subsequent article we shall apply the approach developed here to describe the leading terms of the 'algebraic p-adic L-functions' that are introduced by the first named author in [4], and we shall use the resulting description to prove that the main conjecture of Coates et al for an extension F_∞/F and an elliptic curve E implies the equivariant Tamagawa number conjecture for the motive $h^1(E)(1)$ at each finite degree subextension of F_∞/F. We note that this result provides a partial converse to the theorem of Fukaya and Kato which shows

that, under a natural torsion hypothesis on Selmer groups, the main conjecture of Fukaya and Kato specialises to recover the main conjecture of Coates et al. The main contents of this article are as follows. In §2 we recall some basic facts regarding (non-commutative) determinant functors and the localized K_1-groups of Fukaya and Kato. In §3 we discuss the formalism of Iwasawa theory descent in the setting of localized K_1-groups and we introduce a notion of the leading terms at p-adic representations for the elements of such groups. We explain how this formalism applies in the setting of the canonical Ore sets introduced by Coates et al, we show that it can be interpreted as taking values after 'partial derivation in the cyclotomic direction', and we use it to extend several well known results concerning Generalized Euler Poincare characteristics. In §4 we recall the 'global Zeta isomorphisms' that are conjectured to exist by Fukaya and Kato, and in §5 we prove an interpolation formula for the leading terms of the global Zeta isomorphisms that are associated to certain Tate motives. Finally, in §6, we prove an interpolation formula for the leading terms of the p-adic L-functions that are associated to certain critical motives. We shall use the same notation as in [37].

It is clear that the recent developments in non-commutative Iwasawa theory are due in large part to the energy, encouragement and inspiration of John Coates. It is therefore a particular pleasure for us to dedicate this paper to him on the occasion of his sixtieth birthday.
This collaboration was initiated during the conference held in Boston in June 2005 in recognition of the sixtieth birthday of Ralph Greenberg. The authors are very grateful to the organizers of this conference for the opportunity to attend such a stimulating meeting.

2. Preliminaries

2.1. Determinant functors. For any ring R we write $B(R)$ for the category of bounded complexes of (left) R-modules, $C(R)$ for the category of bounded complexes of finitely generated (left) R-modules, $P(R)$ for the category of finitely generated projective (left) R-modules, $C^p(R)$ for the category of bounded (cohomological) complexes of finitely generated projective (left) R-modules. By $D^p(R)$ we denote the category of perfect complexes as full triangulated subcategory of the derived category $D^b(R)$ of the homotopy category of $B(R)$. We write $(C^p(R), \text{quasi})$ and $(D^p(R), \text{is})$ for the subcategory of quasi-isomorphisms of $C^p(R)$ and isomorphisms of $D^p(R)$, respectively.
For each complex $C = (C^\cdot, d_C)$ and each integer r we define the r-fold shift $C[r]$ of C by setting $C[r]^i = C^{i+r}$ and $d_{C[r]}^i = (-1)^r d_C^{i+r}$ for each integer i.
We first recall that for any (associative unital) ring R there exists a Picard category $\mathcal{C}_R$ and a determinant functor

$$\mathbf{d}_R : (C^p(R), \text{quasi}) \to \mathcal{C}_R$$

with the following properties (for objects C, C' and C'' of $\mathrm{C}^p(R)$)

d) [1] If $0 \to C' \to C \to C'' \to 0$ is a short exact sequence of complexes, then there is a canonical isomorphism

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C'')$$

which we take as an identification.

e) If C is acyclic, then the quasi-isomorphism $0 \to C$ induces a canonical isomorphism

$$\mathbf{1}_R \to \mathbf{d}_R(C).$$

f) For any integer r one has $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$.

g) the functor $\mathbf{d}_R$ factorizes over the image of $\mathrm{C}^p(R)$ in the category of perfect complexes $\mathrm{D}^p(R)$, and extends (uniquely up to unique isomorphisms) to $(\mathrm{D}^p(R), \mathrm{is})$.

h) For each $C \in \mathrm{D}(R)$ we write $H(C)$ for the complex which has $H(C)^i = H^i(C)$ in each degree i and in which all differentials are 0. If $H(C)$ belongs to $\mathrm{D}^p(R)$ (in which case we shall say that $C^{\cdot}$ is *cohomologically perfect*), then there are canonical isomorphisms

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(H(C)) \cong \prod_{i \in \mathbb{Z}} \mathbf{d}_R(H^i(C))^{(-1)^i}.$$

i) If R' is any further ring and Y an (R', R)-bimodule which is both finitely generated and projective as an R'-module, then the functor $Y \otimes_R - : \mathrm{P}(R) \to P(R')$ extends to a commutative diagram

$$\begin{array}{ccc} (\mathrm{D}^p(R), \mathrm{is}) & \xrightarrow{\mathbf{d}_R} & \mathcal{C}_R \\ Y \otimes_R^{\mathbb{L}} - \downarrow & & \downarrow Y \otimes_R - \\ (\mathrm{D}^p(R'), \mathrm{is}) & \xrightarrow{\mathbf{d}_{R'}} & \mathcal{C}_{R'}. \end{array}$$

In particular, if $R \to R'$ is a ring homomorphism and $C \in \mathrm{D}^p(R)$, we often write $\mathbf{d}_R(C)_{R'}$ in place of $R' \otimes_R \mathbf{d}_R(C)$.

REMARK 2.1. Unless R is a regular ring, property d) does not in general extend to arbitrary distinguished triangles. The second displayed isomorphism in h) is induced by properties d) and f). However, whilst a precise description of the first isomorphism in h) is important for the purposes of explicit computations, it is actually rather difficult to find in the literature. Here we use the description given by Knudsen in [18, §3].

REMARK 2.2. We have to distinguish between at least two inverses of a map $\phi : \mathbf{d}_R(C) \to \mathbf{d}_R(D)$ with $C, D \in \mathrm{C}^p(R)$. The inverse with respect to composition will be denoted by $\bar{\phi} : \mathbf{d}_R(D) \to \mathbf{d}_R(C)$ while $\phi^{-1} := \overline{\mathrm{id}}_{\mathbf{d}_R(D)^{-1}} \cdot \phi \cdot \mathrm{id}_{\mathbf{d}_R(C)^{-1}} : \mathbf{d}_R(C)^{-1} \to \mathbf{d}_R(D)^{-1}$ is the unique isomorphism such that $\phi \cdot \phi^{-1} = \mathrm{id}_{\mathbf{1}_R}$ under the identification $\mathbf{d}_R(X) \cdot \mathbf{d}_R(X)^{-1} = \mathbf{1}_R$ for $X = C, D$. If $D = C$, then $\phi : \mathbf{d}_R(C) \to \mathbf{d}_R(C)$ corresponds uniquely to an element of $K_1(R) \cong \mathrm{Aut}_{\mathcal{C}_R}(\mathbf{1}_R)$

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[1]The listing starts with d) to be compatible with the notation of [37] where a)-c) describe properties of the category $\mathcal{C}_R$.

by the rule $\phi \cdot \mathrm{id}_{\mathbf{d}_R(C)^{-1}} : \mathbf{1}_R \to \mathbf{1}_R$. Under this identification $\overline{\phi}$ and ϕ^{-1} agree in $K_1(R)$ and are inverse to ϕ. Furthermore, the following relation between $\circ$ and $\cdot$ is easily verified: if $\phi : A \to B$ and $\psi : B \to C$ are morphisms in $\mathcal{C}_R$, then one has $\psi \circ \phi = \psi \cdot \phi \cdot \mathrm{id}_{B^{-1}}$.

We shall use the following

CONVENTION: If $\phi : \mathbf{1} \to A$ is a morphism and B an object in $\mathcal{C}_R$, then we write $B \xrightarrow{\;\cdot\; \phi\;} B \cdot A$ for the morphism $\mathrm{id}_B \cdot \phi$. In particular, any morphism $B \xrightarrow{\;\phi\;} A$ can be written as $B \xrightarrow{\;\cdot\; (\mathrm{id}_{B^{-1}} \cdot\; \phi)\;} A$.

REMARK 2.3. The determinant of the complex $C = [P_0 \xrightarrow{\phi} P_1]$ (in degree 0 and 1) with $P_0 = P_1 = P$ is by definition $\mathbf{d}_R(C) \stackrel{def}{=\!=} \mathbf{1}_R$ and is defined even if ϕ is not an isomorphism (in contrast to $\mathbf{d}_R(\phi)$). But if ϕ is an isomorphism, i.e. if C is acyclic, then by e) there is also a canonical map $\mathbf{1}_R \xrightarrow{acyc} \mathbf{d}_R(C)$, which is in fact nothing else then

$$\mathbf{1}_R =\!=\mathbf{d}_R(P_1)\mathbf{d}_R(P_1)^{-1} \xrightarrow{\mathbf{d}(\phi)^{-1} \cdot \mathrm{id}_{\mathbf{d}(P_1)^{-1}}} \mathbf{d}_R(P_0)\mathbf{d}_R(P_1)^{-1} =\!= \mathbf{d}_R(C)$$

(and which depends in contrast to the first identification on ϕ). Hence, by Remark 2.2 the composite $\mathbf{1}_R \xrightarrow{acyc} \mathbf{d}_R(C) \stackrel{def}{=\!=} \mathbf{1}_R$ corresponds to $\mathbf{d}_R(\phi)^{-1} \in K_1(R)$. In order to distinguish between the above identifications between $\mathbf{1}_R$ and $\mathbf{d}_R(C)$ we also say that C is *trivialized by the identity* when we refer to $\mathbf{d}_R(C) \stackrel{def}{=\!=} \mathbf{1}_R$ (or its inverse with respect to composition). Obviously, if $\phi = \mathrm{id}_P$, then the above identifications coincide.

REMARK 2.4. Let $\mathcal{O} = \mathcal{O}_L$ be the ring of integers of a finite extension L of $\mathbb{Q}_p$ and let A be a finite $\mathcal{O}$-module. For any morphism of the form $a : \mathbf{1}_\mathcal{O} \to \mathbf{d}_\mathcal{O}(A)$, in particular for that induced by some exact sequence of the form $0 \to \mathcal{O}^n \to \mathcal{O}^n \to A \to 0$, we obtain a canonical element $c = c(a) \in L^\times = \mathrm{Aut}_{\mathcal{C}_L}(\mathbf{1}_L)$ by means of the composite

$$\mathbf{1}_L \xrightarrow{a_L} L \otimes_\mathcal{O} \mathbf{d}_\mathcal{O}(A) =\!= \mathbf{d}_L(L \otimes_\mathcal{O} A) \xrightarrow{acyc} \mathbf{1}_L$$

where the map 'acyc' is induced by property e). As an immediate consequence of the elementary divisor theorem one checks that $\mathrm{ord}_L(c) = \mathrm{length}_\mathcal{O}(A)$.

2.2. THE LOCALIZED K_1-GROUP. In [14] a *localized K_1-group* was defined for any full subcategory Σ of $C^p(R)$ which satisfies the following four conditions:

   (i) $0 \in \Sigma$,
   (ii) if C, C' are in $C^p(R)$ and C is quasi-isomorphic to C', then $C \in \Sigma \Leftrightarrow C' \in \Sigma$,
   (iii) if $C \in \Sigma$, then also $C[n] \in \Sigma$ for all $n \in \mathbb{Z}$,

(iv) if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in $C^p(R)$ with $C', C'' \in \Sigma$ then also $C \in \Sigma$.

Since we want to apply the same construction to a subcategory which is not necessarily closed under extensions, we weaken the last condition to

(iv′) if C' and C'' belong to Σ, then $C' \oplus C''$ belongs to Σ.

DEFINITION 2.5. (Fukaya-Kato) Assume that Σ satisfies (i), (ii), (iii) and (iv′). The *localized K_1-group* $K_1(R, \Sigma)$ is defined to be the (multiplicatively written) abelian group which has as generators symbols of the form $[C, a]$ for each $C \in \Sigma$ and morphism $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$ and relations

(0) $[0, \mathrm{id}_{\mathbf{1}_R}] = 1$,
(1) $[C', \mathbf{d}_R(f) \circ a] = [C, a]$ if $f : C \to C'$ is an quasi-isomorphism with C (and thus C') in Σ,
(2) if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in Σ, then
$$[C, a] = [C', a'] \cdot [C'', a'']$$
where a is the composite of $a' \cdot a''$ with the isomorphism induced by property d),
(3) $[C[1], a^{-1}] = [C, a]^{-1}$.

REMARK 2.6. Relation (3) is a simple consequence of the relations (0)-(2). Note also that this definition of $K_1(R, \Sigma)$ makes no use of the conditions (iii) and (iv′) that the category Σ is assumed to satisfy. In particular, if Σ satisfies (iv) (rather than only (iv′)), then the above definition coincides with that given in [14, §1]. We shall often refer to a morphism in $\mathcal{C}_R$ of the form $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ or $a : \mathbf{d}_R(C) \to \mathbf{1}_R$ as a *trivialization* (of C).

We now assume given a left denominator set S of R and we let $R_S := S^{-1}R$ denote the corresponding localization and Σ_S the full subcategory of $C^p(R)$ consisting of all complexes C such that $R_S \otimes_R C$ is acyclic. For any $C \in \Sigma_S$ and any morphism $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$ we write $\theta_{C,a}$ for the element of $K_1(R_S)$ which corresponds under the canonical isomorphism $K_1(R_S) \cong \mathrm{Aut}_{\mathcal{C}_{R_S}}(\mathbf{1}_{R_S})$ to the composite

$$\mathbf{1}_{R_S} \to \mathbf{d}_{R_S}(R_S \otimes_R C) \to \mathbf{1}_{R_S}$$

where the first arrow is induced by a and the second by the fact that $R_S \otimes_R C$ is acyclic. Then it can be shown that the assignment $[C, a] \mapsto \theta_{C,a}$ induces an isomorphism of groups

$$\mathrm{ch}_{R, \Sigma_S} : K_1(R, \Sigma_S) \cong K_1(R_S)$$

(cf. [14, prop. 1.3.7]). Hence, if Σ is any subcategory of Σ_S we also obtain a composite homomorphism

$$\mathrm{ch}_{R, \Sigma} : K_1(R, \Sigma) \to K_1(R, \Sigma_S) \cong K_1(R_S).$$

In particular, we shall often use this construction in the following case: $A^\cdot \in \Sigma_S$ and Σ is equal to smallest full subcategory $\Sigma_{A^\cdot}$ of $C^p(R)$ that contains $A^\cdot$ and also satisfies the conditions (i), (ii), (iii) and (iv) that are described above. (With this definition, it is easily seen that $\Sigma_{A^\cdot} \subset \Sigma_{\mathrm{ss}}$).

3. LEADING TERMS

In this section we define a notion of the leading term at a continuous finite dimensional p-adic representation of elements of suitable localized K_1-groups. To do this we introduce an appropriate 'semisimplicity' hypothesis and use a natural construction of Bockstein homomorphisms. We also discuss several alternative characterizations of this notion. We explain how this formalism applies in the context of the canonical localizations introduced in [10] and we use it to extend several well known results concerning Generalized Euler Poincare characteristics.

3.1. BOCKSTEIN HOMOMORPHISMS. Let G be a compact p-adic Lie group which contains a closed normal subgroup H such that the quotient group $\Gamma := G/H$ is topologically isomorphic to $\mathbb{Z}_p$. Then we fix a topological generator γ of Γ and denote by

$$\theta \in \mathrm{H}^1(G, \mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{Z}_p)$$

the unique homomorphism $G \twoheadrightarrow \Gamma \to \mathbb{Z}_p$ which sends γ to 1. We write $\Lambda(G)$ for the Iwasawa algebra of G. Since $\mathrm{H}^1(G, \mathbb{Z}_p) \cong \mathrm{Ext}^1_{\Lambda(G)}(\mathbb{Z}_p, \mathbb{Z}_p)$ by [23, prop. 5.2.14], the element θ corresponds to a canonical extension of $\Lambda(G)$-modules

(1) $$0 \to \mathbb{Z}_p \to E_\theta \to \mathbb{Z}_p \to 0.$$

Indeed, one has $E_\theta = \mathbb{Z}_p^2$ upon which G acts by $\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}$.

For any $A^{\cdot}$ in $B(\Lambda(G))$ we endow the complex $A^{\cdot} \otimes_{\mathbb{Z}_p} E_\theta$ with the natural diagonal G-action. Then (1) induces an exact sequence in $B(\Lambda(G))$ of the form

$$0 \to A^{\cdot} \to A^{\cdot} \otimes_{\mathbb{Z}_p} E_\theta \to A^{\cdot} \to 0.$$

In $D^b(\Lambda(G))$ we thus obtain a morphism, the cup product by θ (depending on the choice of γ),

$$A^{\cdot} \xrightarrow{\theta \cup -} A^{\cdot}[1]$$

which we also denote just by θ.

Now let $\rho : G \to GL_n(\mathcal{O})$ be a (continuous) representation of G on $T_\rho = \mathcal{O}^n$, where $\mathcal{O} = \mathcal{O}_L$ denotes the ring of integers of a finite extension L of $\mathbb{Q}_p$. We will be mainly interested in the morphism induced by θ after forming the derived tensor product with $\mathcal{O}^n$ considered as right $\Lambda(G)$-module via the transpose ρ^t of ρ

$$\mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^{\cdot} \xrightarrow{\theta_*} \mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^{\cdot}[1].$$

In particular, we have homomorphisms

$$\mathbb{T}\mathrm{or}_i^{\Lambda(G)}(T_\rho, A^{\cdot}) \xrightarrow{H^{-i}(\theta_*)} \mathbb{T}\mathrm{or}_{i-1}^{\Lambda(G)}(T_\rho, A^{\cdot})$$

of the hypertor groups $\mathbb{T}\mathrm{or}_i^{\Lambda(G)}(T_\rho, A^{\cdot}) := H^{-i}(\mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^{\cdot})$. We refer to the map

$$\mathfrak{B}_i := H^{-i}(\theta_*)$$

as the *Bockstein morphism (in degree i).*

3.2. THE CASE $G = \Gamma$. In this section we consider the case $G = \Gamma$ and take the trivial Γ-module $\mathbb{Z}_p$ for ρ. We set $T := \gamma - 1$.

For any complex $A^\cdot \in B(\Lambda(\Gamma))$ it is clear that the canonical short exact sequence

$$0 \to \Lambda(\Gamma) \xrightarrow{\times T} \Lambda(\Gamma) \to \mathbb{Z}_p \to 0$$

induces an exact triangle in $D^p(\Lambda(\Gamma))$ of the form

$$(2) \qquad A^\cdot \xrightarrow{\times T} A^\cdot \to \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} A^\cdot \to A^\cdot[1].$$

However, in order to be as concrete as possible, we choose to describe this result on the level of complexes. To this end we fix the following definition of the mapping cone of a map $f : A^\cdot \to B^\cdot$ of complexes:

$$\mathrm{cone}(f) := B^\cdot \oplus A^\cdot[1],$$

with differential in degree i equal to

$$d^i_{\mathrm{cone}(f)} := \begin{pmatrix} d^i_{B^\cdot} & f^i \\ 0 & -d^{i+1}_{A^\cdot} \end{pmatrix} : B^i \oplus A^{i+1} \to B^{i+1} \oplus A^{i+2}.$$

If $A^\cdot$ belongs to $C^p(\Lambda(\Gamma))$, then we set

$$\mathrm{cone}(A^\cdot) := \mathrm{cone}(A^\cdot \xrightarrow{T} A^\cdot)$$

and

$$A^\cdot_0 := \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G)} A^\cdot.$$

Then we have the following morphism of complexes $\pi : \mathrm{cone}(A^\cdot) \to A^\cdot_0$

$$\begin{array}{ccccccc}
\longrightarrow & A^{i-1} \oplus A^i & \xrightarrow{d^{i-1}_{\mathrm{cone}}} & A^i \oplus A^{i+1} & \xrightarrow{d^i_{\mathrm{cone}}} & A^{i+1} \oplus A^{i+2} & \xrightarrow{d^{i+1}_{\mathrm{cone}}} \\
& \pi^{i-1} \downarrow & & \pi^i \downarrow & & \pi^{i+1} \downarrow & \\
\longrightarrow & A_0^{i-1} & \xrightarrow{d^{i-1}_{A_0^\cdot}} & A_0^i & \xrightarrow{d^i_{A_0^\cdot}} & A_0^{i+1} & \xrightarrow{d^{i+1}_{A_0^\cdot}}
\end{array}$$

where, in each degree i, π^i sends $(a, b) \in A^i \oplus A^{i+1}$ to the image of a in $A^i/TA^i \cong A_0^i$. It is easy to check that π is a quasi-isomorphism.

Now from (2) we obtain short exact sequences

$$(3) \qquad 0 \to \mathrm{H}^i(A^\cdot)_\Gamma \to \mathbb{H}_{-i}(\Gamma, A^\cdot) \to \mathrm{H}^{i+1}(A^\cdot)^\Gamma \to 0$$

where $\mathbb{H}_i(\Gamma, A^\cdot) := \mathbb{T}\mathrm{or}_i^{\Lambda(\Gamma)}(\mathbb{Z}_p, A^\cdot)$ denotes hyper group homology of $A^\cdot$ with respect to Γ. Furthermore, for any $\Lambda(\Gamma)$-module M we write $M_\Gamma = M/TM$ and $M^\Gamma = {}_TM$ (kernel of multiplication by T) for the maximal quotient module, resp. submodule, of M upon which Γ acts trivially.

LEMMA 3.1. *In each degree i the homomorphism $\mathfrak{B}_i : \mathbb{H}_i(\Gamma, A^\cdot) \to \mathbb{H}_{i-1}(\Gamma, A^\cdot)$ can be computed as the composite*

$$\mathbb{H}_i(\Gamma, A^\cdot) \to \mathrm{H}^{-i+1}(A^\cdot)^\Gamma \xrightarrow{\kappa^{-i+1}(A^\cdot)} \mathrm{H}^{-i+1}(A^\cdot)_\Gamma \to \mathbb{H}_{i-1}(\Gamma, A^\cdot)$$

where the map

$$\kappa^i : \mathrm{H}^i(A^\cdot)^\Gamma \hookrightarrow \mathrm{H}^i(A^\cdot) \twoheadrightarrow \mathrm{H}^i(A^\cdot)_\Gamma$$

is induced by the identity.

Proof. As shown in [28, lem. 1.2], on the level of complexes θ is given by the map

$$\theta : \mathrm{cone}(A^{\cdot}) \to \mathrm{cone}(A^{\cdot})[1]$$

which sends $(a, b) \in A^i \oplus A^{i+1}$ to $(b, 0) \in A^{i+1} \oplus A^{i+2}$. Now let $\bar{a}$ be in $\ker d_{A_0^{\cdot}}^{-i}$ representing a class in $\mathbb{H}_i(\Gamma, A^{\cdot})$. Then there exists $(a, b) \in \ker(d_{\mathrm{cone}}^{-i})$ with $\pi^{-i}((a, b)) = \bar{a}$. Since $(a, b) \in \ker(d_{\mathrm{cone}}^{-i})$ one has $b \in \ker(d_A^{i+1})$ and $Tb = -d_{A^{\cdot}}^i(a)$. This implies that $d_{A^{\cdot}}^i(a)$ is divisible by T (in A^{i+1}) and that $b = -T^{-1}d_{A^{\cdot}}^i(a) \in A^{i+1}$. Thus θ maps (a, b) to $(-T^{-1}d_{A^{\cdot}}^i(a), 0)$ and the class in $\mathbb{H}_{i-1}(\Gamma, A^{\cdot})$ is represented by $-\overline{T^{-1}d_{A^{\cdot}}^i(a)} \in \ker(d_{A_0^{\cdot}}^{-i+1})$. By using the canonical exact sequence

$$0 \to A^{\cdot} \to \mathrm{cone}(A^{\cdot}) \to A^{\cdot}[1] \to 0$$

one immediately verifies that $\mathfrak{B}_i$ coincides with the map described in the lemma. □

From this description it follows immediately that the pair

(4) $(\mathbb{H}_i(\Gamma, A^{\cdot}), \mathfrak{B}_i)$

forms a (homological) complex, which by reindexing we consider as cohomological complex if necessary. We refer to the morphism $\mathfrak{B}_i$ defined here as the Bockstein morphism in degree i of the pair $(A^{\cdot}, T)$.

DEFINITION 3.2. *(Semisimplicity)* We shall say that a complex $A^{\cdot} \in C^p(\Lambda(\Gamma))$ is *semisimple* if the cohomology of the associated complex (4) is $\mathbb{Z}_p$-torsion (and hence finite) in all degrees. We let Σ_{ss} denote the full subcategory of $C^p(\Lambda(\Gamma))$ consisting of those complexes that are semisimple. For any $A^{\cdot} \in \Sigma_{\mathrm{ss}}$ we set

$$r_\Gamma(A^{\cdot}) := \sum_{i \in \mathbb{Z}}(-1)^{i+1} \dim_{\mathbb{Q}_p}(H^i(A^{\cdot})^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \in \mathbb{Z}.$$

REMARK 3.3. (i) If $A^{\cdot} \in \Sigma_{\mathrm{ss}}$, then the cohomology of $A^{\cdot}$ is a torsion $\Lambda(\Gamma)$-module in all degrees.
(ii) In each degree i Lemma 3.1 gives a canonical exact sequence

$$0 \to \mathrm{cok}(\kappa^{-i}) \to \ker(\mathfrak{B}_i)/\mathrm{im}(\mathfrak{B}_{i+1}) \to \ker(\kappa^{-i+1}) \to 0.$$

This implies that a complex $A^{\cdot} \in C^p(\Lambda(\Gamma))$ belongs to Σ_{ss} if and only if the map $\kappa^i(A^{\cdot}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is bijective in each degree i, and hence also that in any such case one has

$$r_\Gamma(A^{\cdot}) = \sum_{i \in \mathbb{Z}}(-1)^{i+1} \dim_{\mathbb{Q}_p}(H^i(A^{\cdot})_\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

DEFINITION 3.4. *(The canonical trivialization)* We write $(\mathbb{H}_{\cdot}(\Gamma, A^{\cdot}), 0)$ for the complex which has $(\mathbb{H}_{\cdot}(\Gamma, A^{\cdot}), 0)^i = \mathbb{H}_i(\Gamma, A^{\cdot})$ in each degree i and in which

all differentials are the zero map. If $A^\cdot \in \Sigma_{\mathrm{ss}}$, then we obtain a canonical trivialization of $\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} A^\cdot$ by taking the composite

$$(5) \quad t(A^\cdot) : \mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} A^\cdot)_{\mathbb{Q}_p} \cong \mathbf{d}_{\mathbb{Z}_p}((\mathbb{H}^\cdot(\Gamma, A^\cdot), 0))_{\mathbb{Q}_p}$$
$$= \mathbf{d}_{\mathbb{Z}_p}((\mathbb{H}^\cdot(\Gamma, A^\cdot), \mathfrak{B}^\cdot))_{\mathbb{Q}_p} \cong \mathbf{1}_{\mathbb{Q}_p}$$

where the first, resp. last, isomorphism uses property h), resp. e), of the functor $\mathbf{d}_{\mathbb{Z}_p}$.

REMARK 3.5. If $\mathbb{Q}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} A^\cdot$ is acyclic, then $t(A^\cdot)$ coincides with the trivialization obtained by applying property e) directly to $\mathbb{Q}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} A^\cdot$.

The category Σ_{ss} satisfies the conditions (i), (ii), (iii) and (iv') that are described in §2 (but does not satisfy condition (iv)). In addition, as the following lemma shows, the above constructions behave well on exact triangles of semisimple complexes.

LEMMA 3.6. *Let $A^\cdot, B^\cdot$ and $C^\cdot$ be objects of Σ_{ss} which together lie in an exact triangle in $D^p(\Lambda(\Gamma))$ of the form*

$$A^\cdot \to B^\cdot \to C^\cdot \to A^\cdot[1].$$

Then one has

$$r_\Gamma(B^\cdot) = r_\Gamma(A^\cdot) + r_\Gamma(C^\cdot)$$

and, with respect to the canonical isomorphism

$$\mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G)} B^\cdot)_{\mathbb{Q}_p} = \mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G)} A^\cdot)_{\mathbb{Q}_p} \mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G)} C^\cdot)_{\mathbb{Q}_p}$$

that is induced by the given exact triangle, one has

$$t(B^\cdot) = t(A^\cdot)t(C^\cdot).$$

Proof. Easy and left to the reader. □

We write ρ_{triv} for the trivial representation of Γ.

DEFINITION 3.7. *(The leading term)* For each $A^\cdot \in \Sigma_{\mathrm{ss}}$ and each morphism $a : \mathbf{1}_{\Lambda(\Gamma)} \to \mathbf{d}_{\Lambda(\Gamma)}(A^\cdot)$ in $\mathcal{C}_{\Lambda(\Gamma)}$ we define the *leading term* $(A^\cdot, a)^*(\rho_{\mathrm{triv}})$ of the pair $(A^\cdot, a)$ at ρ_{triv} to be equal to $(-1)^{r_\Gamma(A^\cdot)}$ times the element of $\mathbb{Q}_p \setminus \{0\}$ which corresponds via the isomorphisms $\mathbb{Q}_p^\times \cong K_1(\mathbb{Q}_p) \cong \mathrm{Aut}_{\mathcal{C}_{\mathbb{Q}_p}}(\mathbf{1}_{\mathbb{Q}_p})$ to the composite morphism

$$\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{\mathbb{Q}_p \otimes_{\Lambda(\Gamma)} a} \mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes_{\Lambda(\Gamma)} A^\cdot)_{\mathbb{Q}_p} \xrightarrow{t(A^\cdot)} \mathbf{1}_{\mathbb{Q}_p}.$$

After taking into account Lemma 3.6, this construction induces a well defined homomorphism of groups

$$(-)^*(\rho_{\mathrm{triv}}) : K_1(\Lambda(\Gamma), \Sigma_{\mathrm{ss}}) \to \mathbb{Q}_p^\times$$
$$[A^\cdot, a] \mapsto [A^\cdot, a]^*(\rho_{\mathrm{triv}}) := (A^\cdot, a)^*(\rho_{\mathrm{triv}}).$$

The reason for the occurrence of ρ_{triv} in the above definition will become clear in the next subsection. In the remainder of the current section we justify the name 'leading term' by explaining the connection between $(A^\cdot, a)^*(\rho_{\mathrm{triv}})$ and the leading term of an appropriate characteristic power series.

To this end we note that Σ_{ss} is a subcategory of the full subcategory of $C^p(\Lambda(\Gamma))$ consisting of those complexes C for which $Q(\Gamma) \otimes_{\Lambda(\Gamma)} C$ is acyclic (by Remark 3.3(i)), and hence that there exists a homomorphism

$$\mathrm{ch}_\Gamma := \mathrm{ch}_{\Lambda(\Gamma), \Sigma_{\mathrm{ss}}} : K_1(\Lambda(\Gamma), \Sigma_{\mathrm{ss}}) \to K_1(Q(\Gamma)) \cong Q(\Gamma)^\times.$$

If L is any field extension of $\mathbb{Q}_p$, and $\mathcal{O}$ the valuation ring in L, then the identification $\Lambda_\mathcal{O}(\Gamma) \cong \mathcal{O}[[T]]$ (which, of course, depends on the choice of $T = \gamma - 1$) allows any element $F \in Q_\mathcal{O}(\Gamma)^\times$ to be written uniquely as

$$(6) \qquad\qquad F(T) = T^r G(T)$$

with $r = r(F) \in \mathbb{Z}$ and $G(T) \in Q_\mathcal{O}(\Gamma)$ such that $G(0) \in L^\times$. The leading coefficient of F with respect to its Taylor series expansion in the Laurent series ring $L\{\{T\}\}$ is thus equal to $F^*(0) := G(0)$. Let $\mathfrak{p}$ be the kernel of the augmentation map $\Lambda(\Gamma) \to \mathbb{Z}_p$ and denote by R the localisation $\Lambda(\Gamma)_\mathfrak{p}$. This is a discrete valuation ring with uniformizer T and the residue class field $R/(T)$ is isomorphic to $\mathbb{Q}_p$. The notation of semi-simplicity extends naturally to R via the exact functor $R \otimes_{\Lambda(\Gamma)} -$: indeed, if $A^\cdot \in D^p(R)$, then the analogue of (2) induces a Bockstein map for $\mathrm{Tor}^R(R/(T), A^\cdot)$ and we say that $A^\cdot$ is *semisimple* if the associated complex $(\mathrm{Tor}^R(R/(T), A^\cdot), \mathfrak{B}_\cdot)$ is acyclic.

PROPOSITION 3.8. *Let* $a : \mathbf{1}_{\Lambda(\Gamma)} \to \mathbf{d}_{\Lambda(\Gamma)}(A^\cdot)$ *be a morphism in* $\mathcal{C}_{\Lambda(\Gamma)}$ *with* $A^\cdot$ *in* Σ_{ss}.

(i) (Order of vanishing) *For* $\mathcal{L} := [A^\cdot, a]$ *one has* $r(\mathrm{ch}_\Gamma(\mathcal{L})) = r_\Gamma(A^\cdot)$.

(ii) (Leading terms) *One has a commutative diagram of abelian groups*

$$
\begin{array}{ccc}
K_1(\Lambda(\Gamma), \Sigma_{\mathrm{ss}}) & \xrightarrow{\mathrm{ch}_\Gamma} & K_1(Q(\Gamma)) \\
{\scriptstyle (-)^*(\rho_{\mathrm{triv}})} \downarrow & & \downarrow {\scriptstyle (-)^*(0)} \\
\mathbb{Q}_p^\times & == & \mathbb{Q}_p^\times.
\end{array}
$$

Proof. It is easy to see that both maps $(-)^*(\rho_{\mathrm{triv}})$ and ch_Γ factor via flat base change $R \otimes_{\Lambda(\Gamma)} -$ through $K_1(R, \Xi)$, where Ξ denotes the full subcategory of $C^p(R)$ consisting of those complexes which are semisimple. Thus it suffices to show the commutativity of the above diagram with $K_1(\Lambda(\Gamma), \Sigma_{\mathrm{ss}})$ replaced by $K_1(R, \Xi)$. Moreover, by Lemma 3.9 below this is reduced to the case where $A^\cdot$ is a complex of the form $R \xrightarrow{a} R$ where R occurs in degrees -1 and 0 and a denotes multiplication by either T or 1. Further, since the complex $R \xrightarrow{\times 1} R$ is acyclic we shall therefore assume that a denotes multiplication by T.

Now $\mathrm{Mor}_{\mathcal{C}_R}(\mathbf{1}_R, \mathbf{d}(A^\cdot))$ is a $K_1(R)$-torsor and so all possible trivialization arise in the following way where $\epsilon \in R^\times$: The module homomorphisms $R \to A_i$ which send 1 to $1 \in R = A_{-1}$ and to $\epsilon \in R = A_0$, respectively, induce maps $can_1 : \mathbf{d}_R(R) \to \mathbf{d}_R(A_{-1})$ and $can_\epsilon : \mathbf{d}_R(R) \to \mathbf{d}_R(A_0)$ which give rise to

the trivialization $a_\epsilon := (can_1)^{-1} \cdot can_\epsilon : \mathbf{1}_R \to \mathbf{d}_R(A^\cdot)$. Setting $\mathcal{L}_\epsilon := [A^\cdot, a_\epsilon]$ one checks easily that $\mathrm{ch}_\Gamma(\mathcal{L}_\epsilon) = T^{-1}\epsilon$ and thus $\mathrm{ch}_\Gamma(\mathcal{L}_\epsilon)^*(0) = \epsilon(0)$. On the other hand, the bockstein map $\mathfrak{B}_1$ is just $\mathbb{Z}_p \xrightarrow{-1} \mathbb{Z}_p$ as one checks using the description of $\mathfrak{B}$ in the proof of Lemma 3.1. Thus $\mathcal{L}_\epsilon^*(\rho_{\mathrm{triv}})$ is equal to $(-1)^{r_\Gamma(A^\cdot)}$ times the determinant of

$$\mathbb{Q}_p \xrightarrow{\epsilon(0)} \mathbb{Q}_p \xrightarrow{(\mathfrak{B}_1)^{-1}_{\mathbb{Q}_p}=-1} \mathbb{Q}_p \xrightarrow{1} \mathbb{Q}_p.$$

Hence, observing that $r_\Gamma(A^\cdot) = -1 = r(\mathrm{ch}_\Gamma(\mathcal{L}_\epsilon))$, we have $\mathcal{L}_\epsilon^*(\rho_{\mathrm{triv}}) = \epsilon(0)$ which proves the Proposition. $\qquad\square$

LEMMA 3.9. *Let R be a discrete valuation ring with uniformizer T and assume that $A^\cdot \in C^p(R)$ is semisimple. Then $A^\cdot$ is isomorphic in $C^p(R)$ to the direct sum of finitely many complexes of the form $R \to R$ where the differential is equal to multiplication by either 1 or T.*

Proof. Assume that m is the maximal degree such that $A^m \neq 0$ and fix an isomorphism $D : R^d \cong A^m$. Let $(e_1, \ldots, e_d)$ be the standard basis of R^d. The semisimplicity of $A^\cdot$ is equivalent to the property that $T \cdot \mathrm{H}^i(A^\cdot) = 0$ for all $i \in \mathbb{Z}$ as can be seen easily by using an analogue of Lemma 3.1; indeed, both conditions are equivalent to the property that the analogues of the maps κ^i in that Lemma are isomorphisms for all i. Thus, for $1 \leq i \leq d$, Te_i is in the image of $D^{-1} \circ d^{m-1}$. We set $h_i = 1$ if e_i is already in this image and $h_i = T$, otherwise. By H we denote the diagonal matrix with the elements $h_1, \ldots, h_d$. Since the image of the map $R^d \xrightarrow{H} R^d$ coincides by definition with that of $D^{-1} \circ d^{m-1}$ we obtain a retraction $E : R^d \to A^{m-1}$ (i.e. with left inverse "$H^{-1} \circ D^{-1} \circ d^{m-1}$") making the following diagram commutative

$$
\begin{array}{ccccccccc}
\longrightarrow & 0 & \longrightarrow & R^d & \xrightarrow{H} & R^d & \longrightarrow & 0 & \longrightarrow \\
& \downarrow & & {\scriptstyle E}\downarrow & & {\scriptstyle D}\downarrow & & \downarrow & \\
\longrightarrow & A^{m-2} & \xrightarrow{d^{m-2}} & A^{m-1} & \xrightarrow{d^{m-1}} & A^m & \xrightarrow{d^m} & 0 & \longrightarrow \ .
\end{array}
$$

If $B^\cdot$ denotes the upper row of this diagram and $C^\cdot := A^\cdot/B^\cdot$ the associated quotient complex (not the mapping cone!), one checks readily that there exists a *split* exact sequence $0 \to B^\cdot \to A^\cdot \to C^\cdot \to 0$. Since $C^\cdot$ is again semisimple and has a strictly shorter length than $A^\cdot$ the proof is accomplished by induction. $\quad\square$

REMARK 3.10. It will be clear to the reader that analogous statements hold for all results of this subsection if we replace $\mathbb{Z}_p$ by $\mathcal{O}$, $\mathbb{Q}_p$ by L, $\Lambda(\Gamma)$ by $\Lambda_\mathcal{O}(\Gamma) := \mathcal{O}[[\Gamma]]$, and $\mathbb{Q}(\Gamma)$ by $Q_\mathcal{O}(\Gamma)$, the quotient field of $\Lambda_\mathcal{O}(\Gamma)$.

3.3. THE GENERAL CASE. For any continuous representation $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ we regard the complex

$$A^\cdot(\rho^*) := A^\cdot \otimes_{\mathbb{Z}_p} \mathcal{O}^n$$

as a complex of $\Lambda(G)$-modules by means of the following G-action: $g(a \otimes o) := ga \otimes \rho^*(g)o$ for $g \in G$, $a \in A^i$ and $o \in \mathcal{O}^n$.

DEFINITION 3.11. *(Semisimplicity at ρ)* We shall say that a complex $A^{\cdot} \in D^p(\Lambda(G))$ is *semisimple at ρ* if the cohomology of the associated complex $(\mathbb{H}_{\cdot}(G, A^{\cdot}(\rho^*)), \mathfrak{B}_{\cdot})$ is $\mathbb{Z}_p$-torsion in each degree. We let $\Sigma_{\text{ss}-\rho}$ denote the full subcategory of $C^p(\Lambda(G))$ consisting of all complexes that are semisimple at ρ, and we note that $\Sigma_{\text{ss}-\rho}$ satisfies the conditions (i), (ii), (iii) and (iv$'$) that are described in §2. For each $A^{\cdot} \in \Sigma_{\text{ss}-\rho}$ we set

$$r_G(A^{\cdot})(\rho) := \sum_{i \in \mathbb{Z}} (-1)^{i+1} \dim_L \left(\mathbb{H}_i(H, A^{\cdot}(\rho^*))^{\Gamma} \otimes_{\mathcal{O}} L \right) \in \mathbb{Z},$$

where $\mathbb{H}_i(H, -) := \mathbb{T}\text{or}_i^{\Lambda(H)}(\mathcal{O}, -)$ denotes hyper H-group homology.

DEFINITION 3.12. *(Finiteness at ρ)* Similarly we shall say that a complex $A^{\cdot} \in D^p(\Lambda(G))$ is *finite at ρ* if the cohomology groups $\mathbb{H}_i(G, A^{\cdot}(\rho^*))$ are $\mathbb{Z}_p$-torsion in each degree. We let $\Sigma_{\text{fin}-\rho}$ denote the full subcategory of $C^p(\Lambda(G))$ consisting of all complexes that are finite at ρ, and we note that $\Sigma_{\text{fin}-\rho}$ satisfies the conditions (i), (ii), (iii) and (iv) that are described in §2. In particular we have $\Sigma_{\text{fin}-\rho} \subseteq \Sigma_{\text{ss}-\rho}$.

In the next result we consider the tensor product $\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n$ as a natural $(\Lambda_{\mathcal{O}}(\Gamma), \Lambda(G))$-bimodule where $\Lambda_{\mathcal{O}}(\Gamma)$ acts by multiplication on the left and $\Lambda(G)$ acts on the right via the rule $(\tau \otimes o)g := \tau \bar{g} \otimes \rho(g)^t o$ for each $g \in G$ (with image $\bar{g}$ in Γ), $o \in \mathcal{O}^n$ and $\tau \in \Lambda_{\mathcal{O}}(\Gamma)$. For a complex $A^{\cdot} \in \Sigma_{\text{ss}-\rho}$, we write $t(A^{\cdot}(\rho^*))$ for the trivialization $t(A^{\cdot}_{\rho^*}) : \mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(\Gamma)}^{\mathbb{L}} A^{\cdot}_{\rho^*})_L \to \mathbf{1}_L$ defined in (5). By the following Lemma this amounts to the same as taking the composite

$$(7) \quad t(A^{\cdot}(\rho^*)) : \mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(G)}^{\mathbb{L}} A^{\cdot}(\rho^*))_L \cong \mathbf{d}_{\mathcal{O}}((\mathbb{H}_{\cdot}(G, A^{\cdot}(\rho^*)), 0))_L$$
$$= \mathbf{d}_{\mathcal{O}}((\mathbb{H}_{\cdot}(G, A^{\cdot}(\rho^*)), \mathfrak{B}_{\cdot}))_L \cong \mathbf{1}_L$$

where the first, resp. last, isomorphism uses property h), resp. e), of the functor $\mathbf{d}_{\mathcal{O}}$.

LEMMA 3.13. *Fix $A^{\cdot} \in C^p(\Lambda(G))$.*

(i) *One has $A^{\cdot} \in \Sigma_{\text{ss}-\rho}$ if and only if*

$$A^{\cdot}_{\rho} := (\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n) \otimes_{\Lambda(G)}^{\mathbb{L}} A^{\cdot} \cong \Lambda_{\mathcal{O}}(\Gamma) \otimes_{\Lambda_{\mathcal{O}}(G)}^{\mathbb{L}} A^{\cdot}(\rho^*)$$

belongs to Σ_{ss} (when considered as an object of $C^p(\Lambda_{\mathcal{O}}(\Gamma))$). In addition, in any such case the Bockstein map of $\mathbb{H}(G, A^{\cdot}(\rho^*))$ is equal to the Bockstein map of the pair $(A^{\cdot}_{\rho}, T)$ and one has natural isomorphisms in $C^p(\mathcal{O})$

$$\mathcal{O} \otimes_{\Lambda_{\mathcal{O}}(\Gamma)}^{\mathbb{L}} A^{\cdot}_{\rho} \cong \mathcal{O}^n \otimes_{\Lambda(G)}^{\mathbb{L}} A^{\cdot}$$

and in $B(\mathcal{O})$.

$$\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\Lambda(G)}^{\mathbb{L}} A^{\cdot} \cong \mathcal{O} \otimes_{\Lambda(H)}^{\mathbb{L}} A^{\cdot}.$$

(ii) *If $A^{\cdot} \in \Sigma_{\text{ss}-\rho}$, then $r_G(A^{\cdot})(\rho) = r_{\Gamma}(A^{\cdot}_{\rho})$.*

(iii) *If $A^{\cdot}, B^{\cdot}$ and $C^{\cdot}$ are objects of $\Sigma_{\mathrm{ss}-\rho}$ which together lie in an exact triangle in $D^p(\Lambda(G))$ of the form*

$$A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to A^{\cdot}[1],$$

then one has

$$r_G(B^{\cdot})(\rho) = r_G(A^{\cdot})(\rho) + r_G(C^{\cdot})(\rho)$$

and, with respect to the canonical isomorphism

$$\mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} B^{\cdot}(\rho^*))_L = \mathbf{d}_{\mathcal{O}}(\mathcal{O} \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} A^{\cdot}(\rho^*))_L \mathbf{d}_{O}(\mathcal{O} \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C^{\cdot}(\rho^*))_L$$

that is induced by the given exact triangle, one has

$$t(B^{\cdot}(\rho^*)) = t(A^{\cdot}(\rho^*)) \cdot t(C^{\cdot}(\rho^*)).$$

Proof. Claim (i) follows from the fact that the specified actions induce natural isomorphisms in $D^p(\mathcal{O})$ of the form

$$
\begin{aligned}
\mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^{\cdot} \quad &\cong \quad \mathcal{O} \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} A^{\cdot}(\rho^*) \\
&\cong \quad \mathcal{O} \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} \left(\Lambda_{\mathcal{O}}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} A^{\cdot}(\rho^*) \right) \\
&\cong \quad \mathcal{O} \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} \left((\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n) \otimes^{\mathbb{L}}_{\Lambda(G)} A^{\cdot} \right)
\end{aligned}
$$

and from the functorial construction of the Bockstein map. Claim (ii) follows directly by using claim (i) to compare the definitions of the terms $r_G(A^{\cdot})(\rho)$ and $r_{\Gamma}(A^{\cdot}_{\rho})$. To prove claim (iii) we observe that, by claim (i), the given triangle induces an exact triangle of semisimple complexes in $D^p(\Lambda(\Gamma))$ of the form

$$A^{\cdot}_{\rho} \to B^{\cdot}_{\rho} \to C^{\cdot}_{\rho} \to A^{\cdot}_{\rho}[1].$$

The equalities of claim (iii) thus follow from claim (ii) and the results of Lemma 3.6 as applied to the above triangle. $\qquad\square$

DEFINITION 3.14. *(The leading term at ρ)* For each complex $A^{\cdot} \in \Sigma_{\mathrm{ss}-\rho}$ and each morphism $a : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(A^{\cdot})$ we define the *leading term* $(A^{\cdot}, a)^*(\rho)$ of the pair $(A^{\cdot}, a)$ at ρ to be equal to $(-1)^{r_G(A^{\cdot})(\rho)}$ times the element of $L \setminus \{0\}$ which corresponds via the isomorphisms $L^{\times} \cong K_1(L) \cong \mathrm{Aut}_{\mathcal{C}_L}(\mathbf{1}_L)$ to the composite morphism

$$\mathbf{1}_L \xrightarrow{L^n \otimes_{\Lambda(G)} a} \mathbf{d}_L(L^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^{\cdot}) \xrightarrow{t(A^{\cdot}(\rho^*))} \mathbf{1}_L.$$

Since $\Sigma_{A^{\cdot}} \subset \Sigma_{\mathrm{ss}-\rho}$ Lemma 3.13 shows that the above construction induces a well-defined homomorphism of groups

$$(-)^*(\rho) : K_1(\Lambda(G), \Sigma_{A^{\cdot}}) \to L^{\times}$$
$$[A^{\cdot}, a] \mapsto [A^{\cdot}, a]^*(\rho) := (A^{\cdot}, a)^*(\rho).$$

If $A^{\cdot}$ is clear from context, then we often write $a^*(\rho)$ in place of $[A^{\cdot}, a]^*(\rho)$. It is easily checked that (in the case $G = \Gamma$ and $\rho = \rho_{\mathrm{triv}}$) these definitions are compatible with those given in §3.2. Further, in §3.4.3 we shall reinterpret the expression $[A^{\cdot}, a]^*(\rho)$ defined above as the leading term at $s = 0$ of a natural p-adic meromorphic function.

REMARK 3.15. If $A^{\cdot} \in \Sigma_{\text{fin}-\rho}$, then we set $[A^{\cdot}, a](\rho) := [A^{\cdot}, a]^{*}(\rho)$ and call this the *value* of $[A^{\cdot}, a]$ at ρ. Taking into account Remark 3.5 it is clear that this definition coincides with that given in [14, 4.1.5].

3.4. CANONICAL LOCALIZATIONS. We apply the constructions of §3.3 in the setting of the canonical localizations of $\Lambda(G)$ that were introduced in [10].

3.4.1. *The canonical Ore sets.* We recall from [10, §2-§3] that there are canonical left and right denominator sets S and S^{*} of $\Lambda(G)$ where

$$S := \{\lambda \in \Lambda(G) : \Lambda(G)/\Lambda(G) \cdot \lambda \text{ is a finitely generated } \Lambda(H)\text{-module}\}$$

and

$$S^{*} := \bigcup_{i \geq 0} p^{i} S.$$

We write S^{*}-tor for the category of finitely generated $\Lambda(G)$-modules M which satisfy $\Lambda(G)_{S^{*}} \otimes_{\Lambda(G)} M = 0$. We further recall from loc. cit. that a finitely generated $\Lambda(G)$-module M belongs to S^{*}-tor, if and only if $M/M(p)$ is finitely generated when considered as a $\Lambda(H)$-module (by restriction) where $M(p)$ is the submodule of M consisting of those elements which are annihilated by some power of p.

3.4.2. *Leading terms.* We use the notation of Definition 3.14. If $\rho : G \to \mathrm{GL}_{n}(\mathcal{O})$ is any continuous representation and $A^{\cdot}$ any object of $\Sigma_{S^{*}}$, then $\Sigma_{A^{\cdot}} \subset \Sigma_{S^{*}}$ and so there exists a canonical homomorphism

$$\mathrm{ch}_{G,A^{\cdot}} := \mathrm{ch}_{\Lambda(G),\Sigma_{A^{\cdot}}} : K_{1}(\Lambda(G), \Sigma_{A^{\cdot}}) \to K_{1}(\Lambda(G)_{S^{*}}).$$

In addition, the ring homomorphism $\Lambda(G)_{S^{*}} \to M_{n}(Q(\Gamma))$ which sends each element $g \in G$ to $\rho(g)\bar{g}$ where $\bar{g}$ is the image in Γ, induces a homomorphism

$$\rho_{*} : K_{1}(\Lambda(G)_{S^{*}}) \to K_{1}(M_{n}(Q_{\mathcal{O}}(\Gamma))) \cong K_{1}(Q_{\mathcal{O}}(\Gamma)) \cong Q_{\mathcal{O}}(\Gamma)^{\times}.$$

PROPOSITION 3.16. *Let $A^{\cdot}$ be a complex which belongs to $\Sigma_{S^{*}} \cap \Sigma_{\text{ss}-\rho}$.*

  (i) *(Order of vanishing) One has $r_{G}(A^{\cdot})(\rho) = r_{\Gamma}(A^{\cdot}_{\rho}) = r(\rho_{*} \circ \mathrm{ch}_{G,A^{\cdot}}(A^{\cdot}))$.*
  (ii) *(Leading terms) The following diagram of abelian groups commutes*

$$
\begin{array}{ccc}
K_{1}(\Lambda(G), \Sigma_{A^{\cdot}}) & \xrightarrow{\mathrm{ch}_{G,A^{\cdot}}} & K_{1}(\Lambda(G)_{S^{*}}) \\
{\scriptstyle (-)^{*}(\rho)} \downarrow & & \downarrow {\scriptstyle (\rho_{*}(-))^{*}(0)} \\
L^{\times} & =\!=\!=\!= & L^{\times},
\end{array}
$$

  where $(-)^{*}(0)$ denotes the 'leading term' homomorphism $K_{1}(Q_{\mathcal{O}}(\Gamma)) \to L^{\times}$ according to Proposition 3.8.

Proof. By Lemma 3.13(i) we have $\mathbb{H}_{i}(H, A^{\cdot}(\rho^{*})) = \mathrm{H}^{-i}(\mathcal{O} \otimes_{\Lambda(H)}^{\mathbb{L}} A^{\cdot}(\rho^{*})) = \mathrm{H}^{-i}(A^{\cdot}_{\rho})$. Taking also into account Proposition 3.8, (i) follows from Definition 3.2 and 3.11. Claim (ii) is proved by the same arguments as used in [14, lem.

4.3.10]. Indeed, one need only observe that the above diagram arises as the
following composite commutative diagram

$$
\begin{array}{ccc}
K_1(\Lambda(G), \Sigma_{A^{\cdot}}) & \xrightarrow{\mathrm{ch}_{\Lambda(G), \Sigma_{A^{\cdot}}}} & K_1(\Lambda(G)_{S^*}) \\
{\scriptstyle (\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n) \otimes_{\Lambda(G)} -} \downarrow & & \downarrow {\scriptstyle \rho_*} \\
K_1(\Lambda_{\mathcal{O}}(\Gamma), \Sigma_{\mathrm{ss}}) & \xrightarrow{\mathrm{ch}_{\Lambda_{\mathcal{O}}(\Gamma), \Sigma_{\mathrm{ss}}}} & K_1(Q_{\mathcal{O}}(\Gamma)) \\
{\scriptstyle (-)^*(\rho_{\mathrm{triv}})} \downarrow & & \downarrow {\scriptstyle (-)^*(0)} \\
L^{\times} & =\!=\!=\!= & L^{\times}
\end{array}
$$

where the lower square is as in Proposition 3.8. $\square$

Thus for any $F \in K_1(\Lambda(G)_{S^*})$ we write also $F^*(\rho)$ for the *leading term* $\rho_*(F)(0)$
of F at ρ. We note that, by Proposition 3.16 , this notation is consistent with
that of Definition 3.14 in the case that F belongs to the image of $\mathrm{ch}_{G,A^{\cdot}}$.
Similarly, we shall use the notation $F(\rho) := F^*(\rho)$ if $r(\rho_*(F)) = 0$.

3.4.3. *Partial derivatives.* We now observe that the constructions of the previ-
ous section allow an interpretation of the expression $(A^{\cdot}, a)^*(\rho)$ defined in §3.3
as the leading term at $s = 0$ of a natural p-adic meromorphic function.
At the outset we fix a representation of G of the form $\chi : G \twoheadrightarrow \Gamma \to \mathbb{Z}_p^{\times}$ of
infinite order and we set

$$
c_{\chi, \gamma} := \log_p(\chi(\gamma)) \in \mathbb{Q}_p^{\times}.
$$

We also fix $A^{\cdot} \in \Sigma_{S^*}$ and a morphism $a : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(A^{\cdot})$, put $\mathcal{L} := [A^{\cdot}, a]$,
and for any continuous representation $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ we set

$$
f_\rho(T) := \rho_*(\mathrm{ch}_{G,A^{\cdot}}(\mathcal{L})) \in K_1(Q_{\mathcal{O}}(\Gamma)) \cong Q_{\mathcal{O}}(\Gamma)^{\times}.
$$

Then, since the zeros and poles of elements in $Q_{\mathcal{O}}(\Gamma)$ are discrete, the function

$$
s \mapsto f_{\mathcal{L}}(\rho \chi^s) := f_\rho(\chi(\gamma)^s - 1)
$$

is a p-adic meromorphic function on $\mathbb{Z}_p$.

LEMMA 3.17. *Let $A^{\cdot}$ and a be as above and set $r := r_G(A^{\cdot})(\rho)$. Then,*

(i) *in any sufficiently small neighbourhood of 0 in $\mathbb{Z}_p$ one has*

$$
\mathcal{L}^*(\rho \chi^s) = \mathcal{L}(\rho \chi^s) = f_{\mathcal{L}}(\rho \chi^s)
$$

for all $s \neq 0$,

(ii) *$c_{\chi, \gamma}^r \mathcal{L}^*(\rho)$ is the leading coefficient at $s = 0$ of $f_{\mathcal{L}}(\rho \chi^s)$, and*

(iii) *if $r \geq 0$, one has*

$$
c_{\chi, \gamma}^r \mathcal{L}^*(\rho) = \frac{1}{r!} \frac{\mathrm{d}^r}{\mathrm{d} s^r} f_{\mathcal{L}}(\rho \chi^s)\big|_{s=0}.
$$

Proof. In any sufficiently small neighbourhood of 0 in $\mathbb{Z}_p$ one has $f_{\rho \chi^s}(0) \in L^{\times}$
for all $s \neq 0$. Since $f_{\rho \chi^s}(T) = f_\rho(\chi(\gamma)^s(T + 1) - 1)$ we may therefore deduce
from Proposition 3.16 that $\mathcal{L}^*(\rho \chi^s) = f_{\rho \chi^s}(0) = f_\rho(\chi(\gamma)^s - 1) = f_{\mathcal{L}}(\rho \chi^s)$ for
any such value of s.

In addition, if $r \geq 0$ and we factorize $f_\rho(T)$ as $T^r G_\rho(T)$ with $G_\rho(T) \in Q_\mathcal{O}(\Gamma)$, then $G_\rho(0) = f_\rho^*(0)$ and

$$
\begin{aligned}
\frac{1}{r!} \frac{\mathrm{d}^r}{\mathrm{d}s^r} f_\mathcal{L}(\rho\chi^s)\big|_{s=0} &= \lim_{0 \neq s \to 0} \frac{f_\rho(\chi(\gamma)^s - 1)}{s^r} \\
&= \lim_{0 \neq s \to 0} \Big(\frac{(\chi(\gamma)^s - 1)^r}{s^r} G_\rho(\chi(\gamma)^s - 1) \Big) \\
&= \Big(\lim_{0 \neq s \to 0} \frac{\chi(\gamma)^s - 1}{s} \Big)^r G_\rho(0) \\
&= (\log_p(\chi(\gamma)))^r f_\rho^*(0) \\
&= c_{\chi,\gamma}^r \mathcal{L}^*(\rho),
\end{aligned}
$$

where the last equality follows from Proposition 3.16. If $r < 0$ we do not have the interpretation as partial derivative but the same arguments prove the statement concerning the leading coefficient at $s = 0$. $\qquad \square$

REMARK 3.18. Of particular interest is the case $\chi = \chi_{\mathrm{cyc}}$ in which the above calculus can be interpreted as partial derivation in the 'cyclotomic' direction (cf. Remark 5.6)
.

3.5. GENERALIZED EULER-POINCARE CHARACTERISTICS. In this section we observe that the constructions made in §3.3 give rise to a natural extension of results from [10, 14, 36].

We fix a continuous representation $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ and a complex $A^\cdot \in \Sigma_{\mathrm{ss}-\rho}$ and in each degree i we set $\mathrm{H}^i_\mathfrak{B}(G, -) := H^i\big((\mathbb{H}_{-\cdot}(G, -), \mathfrak{B}_{-\cdot})\big)$. We then define the (generalized) additive, resp. multiplicative, Euler-Poincare characteristic of the complex $A^\cdot(\rho^*)$ by setting

$$
\chi_{\mathrm{add}}(G, A^\cdot(\rho^*)) := \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{length}_\mathcal{O}\big(\mathrm{H}^i_\mathfrak{B}(G, A^\cdot(\rho^*)) \big),
$$

resp.

$$
\chi_{\mathrm{mult}}(G, A^\cdot(\rho^*)) := (\#\kappa_L)^{\chi_{\mathrm{add}}(G, A^\cdot(\rho^*))}
$$

where κ_L is the residue class field of L. We recall that for single $\Lambda(G)$-module M or rather its Pontryagin-dual D similar Euler characteristics have also been studied by other authors (cf. [11, 40, 16]). Indeed, they use the Hochschild-Serre spectral sequence to construct differentials

$$
d^i : \mathrm{H}^i(G, D) \to \mathrm{H}^i(H, D)^\Gamma \to \mathrm{H}^i(H, D)_\Gamma \to \mathrm{H}^{i+1}(G, D)
$$

where the middle map is induced by the identity; then their generalized Euler characteristic is defined similar as above using the complex $(\mathrm{H}^\cdot(G, D), d^\cdot)$ instead of $(\mathbb{H}_{-\cdot}(G, -), \mathfrak{B}_{-\cdot})$. But by Lemma 3.13 (i) one sees immediately that the Pontryagin dual of d_i coincides with the Bockstein map $\mathfrak{B}_{i+1} : \mathbb{H}_{i+1}(G, P^\cdot) \to \mathbb{H}_i(G, P^\cdot)$ as computed in Lemma 3.1 with $A^\cdot = \Lambda(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(G)} P^\cdot$, where $P^\cdot$ denotes a projective solution of M. More precisely, in [11] a truncated version of this generalized Euler characteristic is used.

PROPOSITION 3.19. *Let* ord_L *denote the valuation of* L *which takes the value* 1
on any uniformizing parameter and $|-|_p$ *the* p*-adic absolute value, normalized
such that* $|p|_p = p^{-1}$.
If $A^\cdot \in \Sigma_{\mathrm{ss}-\rho}$ *and* $a : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(A^\cdot)$ *is any morphism, then for* $\mathcal{L} := [A^\cdot, a]$
one has

$$\chi_{\mathrm{add}}(G, A^\cdot(\rho^*)) = \mathrm{ord}_L(\mathcal{L}^*(\rho))$$

and

$$\chi_{\mathrm{mult}}(G, A^\cdot(\rho^*)) = |\mathcal{L}^*(\rho)|_p^{-[L:\mathbb{Q}_p]}.$$

Proof. First observe that using Lemma 3.13, property h) and the fact that $\mathcal{O}$
is regular we obtain canonical isomorphisms

$$\mathbf{1}_\mathcal{O} \xrightarrow{\mathcal{O}^n \otimes_{\Lambda(G)} a} \mathbf{d}_\mathcal{O}(\mathcal{O}^n \otimes^{\mathbb{L}}_{\Lambda(G)} A^\cdot) \quad \cong \quad \mathbf{d}_\mathcal{O}(\mathcal{O} \otimes^{\mathbb{L}}_{\Lambda_\mathcal{O}(G)} A^\cdot(\rho^*)))$$

$$\cong \quad \mathbf{d}_\mathcal{O}(\mathrm{H}_{\mathfrak{B}}^\cdot(G, A^\cdot(\rho^*)))$$

$$\cong \quad \prod_{i \in \mathbb{Z}} \mathbf{d}_\mathcal{O}\big(\mathrm{H}_{\mathfrak{B}}^i(G, A^\cdot(\rho^*))\big)^{(-1)^i}.$$

After tensoring with L and identifying then the factors at the end with the unit
object by acyclicity we recover the definition of the leading term $(A^\cdot, a)^*(\rho)$.
From this description it becomes clear that the latter can also be written mod-
ulo $\mathcal{O}^\times$ as the product over i of the following maps

$$(\mathbf{1}_\mathcal{O})_L \xrightarrow{def} \mathbf{d}_\mathcal{O}(\mathrm{H}_{\mathfrak{B}}^i(A^\cdot(\rho^*)))_L^{(-1)^i} \xrightarrow{acyc} \mathbf{1}_L .$$

The result follows now from Remark 2.4.   □

REMARK 3.20. In the special case that the complex $A^\cdot(\rho^*) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is acyclic,
the leading term $\mathcal{L}^*(\rho)$ is just the value of $\mathcal{L}$ at ρ (in the sense of Remark 3.15)
and so the result of Proposition 3.19 recovers the results of [10, thm. 3.6], [36,
prop. 6.3] and [14, rem. 4.1.13].

4. Global Zeta isomorphisms

In this section we recall the Tamagawa Number Conjecture as formulated by
Fukaya and Kato in [14].

4.1. GALOIS COHOMOLOGY. The main reference for this section is [14, §1.6],
but see also [6], here we use the same notation as in [37]. For simplicity we
assume that p is odd throughout this section. Let $U = \mathrm{spec}(\mathbb{Z}[\frac{1}{S}])$ be a dense
open subset of $\mathrm{spec}(\mathbb{Z})$ where S contains $S_p := \{p\}$ and $S_\infty := \{\infty\}$ (by
abuse of notation). We write G_S for the Galois group of the maximal outside
S unramified extension of $\mathbb{Q}$. Let X be a topological abelian group with a
continuous action of G_S. As usual we write $\mathrm{R}\Gamma(U, X)$ ($\mathrm{R}\Gamma_c(U, X)$) for global
Galois cohomology with restricted ramification (and compact support), for any
place v of $\mathbb{Q}$ we denote by $\mathrm{R}\Gamma(\mathbb{Q}_v, X)$ local Galois cohomology. Let L be a
finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and V a finite dimensional L-
vector space with continuous $G_{\mathbb{Q}_v}$- and $G_\mathbb{Q}$-action, respectively. Then the *finite*

parts of global and local cohomology are written as $\mathrm{R}\Gamma_f(\mathbb{Q}, V)$ and $\mathrm{R}\Gamma_f(\mathbb{Q}_v, V)$, respectively. There is a canonical exact triangle

$$(8) \qquad \mathrm{R}\Gamma_c(U, V) \longrightarrow \mathrm{R}\Gamma_f(\mathbb{Q}, V) \longrightarrow \bigoplus_S \mathrm{R}\Gamma_f(\mathbb{Q}_v, V) \longrightarrow \cdot$$

For any prime ℓ set $t_\ell(V) := D_{dR}(V)/D_{dR}^0(V)$ if $\ell = p$ and $t_\ell(V) = 0$ otherwise. For the definition of the canonical isomorphism in $\mathcal{C}_L$

$$(9) \qquad \eta_\ell(V): \quad \mathbf{1}_L \quad \to \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_l, V))\mathbf{d}_L(t_\ell(V)).$$

we refer the reader to [14, §2.4.4] or the appendix of [37].

4.2. K-MOTIVES OVER $\mathbb{Q}$. For the background on this material we refer the reader to [14, §2.2, 2.4], [6, §3] or the survey [37, §2]. Let K be a finite extension of $\mathbb{Q}$. As usual we write M_B, M_{dR} and M_λ for the Betti, de Rham and λ-adic realizations of a K-motive M where λ varies over all finite places of K. Also, we write $t_M = M_{dR}/M_{dR}^0$ for the tangent space of M. Henceforth, for any commutative ring R and $R[G(\mathbb{C}/\mathbb{R})]$-module X we denote by X^+ and X^- the R-submodule of X on which complex conjugation acts by $+1$ and -1, respectively. In our later calculations we will use the following isomorphisms

- The comparison between the Betti- and the λ-adic realization induces canonical isomorphisms of K_λ and K_l-modules

$$(10) \qquad g_\lambda^+ : K_\lambda \otimes_K M_B^+ \cong M_\lambda^+ \text{ and } g_\ell^+ : K_\ell \otimes_K M_B^+ \cong M_\ell^+.$$

- The comparison between de Rahm and Betti-cohomology induces the the $\mathbb{R}$-linear period map

$$(11) \qquad \mathbb{R} \otimes_\mathbb{Q} M_B^+ \xrightarrow{\alpha_M} \mathbb{R} \otimes_\mathbb{Q} t_M.$$

- The comparison between p-adic and de Rham cohomology induces an isomorphism of K_λ-vector spaces

$$(12) \qquad t_p(M_\lambda) = D_{dR}(M_\lambda)/D_{dR}^0(M_\lambda) \xrightarrow[\cong]{g_{dR}^t} K_\lambda \otimes_K t_M.$$

The *motivic cohomology* K-vector spaces $\mathrm{H}_f^0(M) := \mathrm{H}^0(M)$ and $\mathrm{H}_f^1(M)$ may be defined by algebraic K-theory or motivic cohomology a la Voevodsky. They are conjectured to be finite dimensional. For the definition we refer the reader to [6].

4.3. THE TAMAGAWA NUMBER CONJECTURE. Let us first recall from [14, §2.2.2] that, for each embedding $K \to \mathbb{C}$, the complex L-function attached to a K-motive M is defined as Euler product

$$L_K(M, s) = \prod_p P_p(M, p^{-s})^{-1}$$

for the real part of s large enough. We assume meromorphic continuation and write as usual $L_K^*(M) \in \mathbb{R}^\times$ and $r(M) \in \mathbb{Z}$ for its leading coefficient and order of vanishing at $s = 0$, respectively.

To establish a link between $L^*(M)$ and Galois cohomology one uses the *fundamental line*:

$$\Delta_K(M): \quad = \quad \mathbf{d}_K(\mathrm{H}^0_f(M))^{-1}\mathbf{d}_K(\mathrm{H}^1_f(M))\mathbf{d}_K(\mathrm{H}^0_f(M^*(1))^*)\mathbf{d}_K(\mathrm{H}^1_f(M^*(1))^*)^{-1}$$
$$\mathbf{d}_K(M^+_B)\mathbf{d}_K(t_M)^{-1}.$$

The conjecture formulated in [14, §2.2.7] induces a canonical isomorphism in $\mathcal{C}_{K_\mathbb{R}}$ (period-regulator map)

$$(13) \qquad\qquad\qquad \vartheta_\infty : K_\mathbb{R} \otimes_K \Delta_K(M) \cong \mathbf{1}_{K_\mathbb{R}}.$$

In addition, a standard conjecture on Cycle class and Chern class maps induces, for any place λ above p, the p-adic period-regulator isomorphism in $\mathcal{C}_{K_\lambda}$ (involving the maps η_l)

$$(14) \qquad\qquad\qquad \vartheta_\lambda : \Delta_K(M)_{K_\lambda} \cong \mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U, M_\lambda))^{-1}.$$

Now let F_∞ be a p-adic Lie extension of $\mathbb{Q}$ with Galois group $G = G(F_\infty/\mathbb{Q})$. By $\Lambda = \Lambda(G)$ we denote its Iwasawa algebra. For a $\mathbb{Q}$-motive M over $\mathbb{Q}$ we fix a $G_\mathbb{Q}$-stable $\mathbb{Z}_p$-lattice T_p of M_p and define a left Λ-module

$$\mathbb{T} := \Lambda \otimes_{\mathbb{Z}_p} T_p$$

on which Λ acts via multiplication on the left factor from the left while $G_\mathbb{Q}$ acts diagonally via $g(x \otimes y) = x\bar{g}^{-1} \otimes g(y)$, where $\bar{g}$ denotes the image of $g \in G_\mathbb{Q}$ in G.

Let λ a finite place of K, $\mathcal{O}_\lambda$ the ring of integers of the completion K_λ of K at λ and assume that $\rho : G \to GL_n(\mathcal{O}_\lambda)$ is a continuous representation of G which, for some suitable choice of a basis, is the λ-adic realization N_λ of a K-motive N. We also write ρ for the induced ring homomorphism $\Lambda(G) \to M_n(\mathcal{O}_\lambda)$ and we consider $\mathcal{O}^n_\lambda$ as a right $\Lambda(G)$-module via action by ρ^t on the left, viewing $\mathcal{O}^n_\lambda$ as set of column vectors (contained in K^n_λ.) Note that, setting $M(\rho^*) := N^* \otimes M$, we obtain an isomorphism of Galois representations

$$\mathcal{O}^n_\lambda \otimes_{\Lambda(G)} \mathbb{T} \cong T_\lambda(M(\rho^*)),$$

where $T_\lambda(\rho^*)$ is the $\mathcal{O}_\lambda$-lattice $\rho^* \otimes T_p$ of $M(\rho^*)_\lambda$ and ρ^* denotes the contragredient(=dual) representation of ρ.

CONJECTURE 4.1 (Fukaya/Kato [14, conj. 2.3.2] or [37, conj. 4.1]). *There exists a (unique) isomorphism in $\mathcal{C}_\Lambda$*

$$\zeta_\Lambda(M) := \zeta_\Lambda(\mathbb{T}) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{R}\Gamma_c(U, \mathbb{T}))^{-1}$$

with the following property: for all K, λ and ρ as above the (generalized) base change $K^n_\lambda \otimes_\Lambda -$ sends $\zeta_\Lambda(M)$ to

$$\mathbf{1}_{K_\lambda} \xrightarrow{\zeta_K(M(\rho^*))_{K_\lambda}} \Delta_K(M(\rho^*))_{K_\lambda} \xrightarrow{\vartheta_\lambda} \mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U, M(\rho^*)_\lambda))^{-1},$$

where $\zeta_K(M(\rho^*)) : \mathbf{1}_K \to \Delta_K(M(\rho^*))$ is the unique isomorphism such that, for every embedding $K \to \mathbb{C}$, the leading coefficient $L^*_K(M(\rho^*))$ is equal to the

composite

$$\mathbf{1}_{\mathbb{C}} \xrightarrow{\zeta_K(M(\rho^*))_{\mathbb{C}}} \Delta_K(M(\rho^*))_{\mathbb{C}} \xrightarrow{(\vartheta_\infty)_{\mathbb{C}}} \mathbf{1}_{\mathbb{C}}.$$

It is easily shown that this conjecture implies the Equivariant Tamagawa Number Conjecture formulated by Flach and the first named author in [6, conj. 4(iv)] and hence also implies the 'main conjecture of non-abelian Iwasawa theory' discussed by Huber and Kings in [17].

5. THE INTERPOLATION FORMULA FOR TATE MOTIVES

In this section we shall give a first explicit application of the formalism developed in §3. More precisely, we show that the 'p-adic Stark conjecture at $s = 1$', as formulated by Serre in [33] and discussed by Tate in [35, chap. VI, §5], can be naturally interpreted as an interpolation formula for the leading term (in the sense of Definition 3.14) of certain global Zeta isomorphisms that are predicted to exist by Conjecture 4.1 in terms of the leading terms (in the classical sense) of suitable p-adic Artin L-functions. Interested readers can find further explicit results concerning Conjecture 4.1 in the special case that we consider here in both [2] and [7].

Throughout this section we fix an odd prime p and a totally real Galois extension F_∞ of $\mathbb{Q}$ which contains the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\text{cyc}}$ of $\mathbb{Q}$ and is such that $G := \text{G}(F_\infty/\mathbb{Q})$ is a compact p-adic Lie group. We assume further that $F_\infty/\mathbb{Q}$ is unramified outside a finite set of prime numbers S (which therefore contains p). We set $H := \text{G}(F_\infty/\mathbb{Q}_{\text{cyc}})$ and $\Gamma := \text{G}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q}) \cong G/H$. We fix a subfield E of F_∞ which has finite degree over $\mathbb{Q}$ and, for simplicity, we assume throughout that the following condition is satisfied:

(15) $\qquad E \cap \mathbb{Q}_{\text{cyc}} = \mathbb{Q}$ and $E_w \cap \mathbb{Q}_{p,\text{cyc}} = \mathbb{Q}_p$ for all $w \in S_p(E)$.

We let $\mathbb{T}$ denote the $\Lambda(G)$-module $\Lambda(G)$ endowed with the following action of $\text{G}(\overline{\mathbb{Q}}/\mathbb{Q})$: each $\sigma \in \text{G}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathbb{T}$ as multiplication by the element $\chi_{\text{cyc}}(\bar\sigma)\bar\sigma^{-1}$ where $\bar\sigma$ denotes the image of σ in G and χ_{cyc} is the cyclotomic character $G \to \Gamma \to \mathbb{Z}_p^\times$. For each subfield F of F_∞ which is Galois over $\mathbb{Q}$ we let $\mathbb{T}_F$ denote the $\Lambda(\text{G}(F/\mathbb{Q}))$-module $\Lambda(\text{G}(F/\mathbb{Q})) \otimes_{\Lambda(G)} \mathbb{T}$. We also set $U := \text{Spec}(\mathbb{Z}[\frac{1}{S}])$ and note that for each such field F there is a natural isomorphism in $D^p(\Lambda(\text{G}(F/\mathbb{Q})))$ of the form

(16) $\qquad \Lambda(\text{G}(F/\mathbb{Q})) \otimes_{\Lambda(G)}^{\mathbb{L}} R\Gamma_c(U, \mathbb{T}) \cong R\Gamma_c(U, \mathbb{T}_F).$

We regard each character of $\bar{G} := \text{G}(E/\mathbb{Q})$ as a character of G via the natural projection $G \to \bar{G}$. For any field C we write $R_C(\bar{G})$ for the ring of finite dimensional C-valued characters of $\bar{G}$, and for each C-valued character ρ we fix a representation space V_ρ of character ρ. For any $\mathbb{Q}_p[\bar{G}]$-module N, resp. endomorphism α of a $\mathbb{Q}_p[\bar{G}]$-module N, we write N^ρ for the $\mathbb{C}_p$-module

$$\text{Hom}_{\bar{G}}(V_\rho, \mathbb{C}_p \otimes_{\mathbb{Q}_p} N) \cong ((V_{\rho^*})_{\mathbb{C}_p} \otimes_{\mathbb{Q}_p} N)_{\bar{G}},$$

resp. α^ρ for the induced endomorphism of N^ρ. We use similar notation for complex characters ρ and $\mathbb{Q}[\bar{G}]$-modules N.

For any abelian group A we write $A\hat{\otimes}\mathbb{Z}_p$ for its p-adic completion $\varprojlim_n A/p^n A$.

5.1. LEOPOLDT'S CONJECTURE. We recall that Leopoldt's conjecture (for the field E at the prime p) is equivalent to the injectivity of the natural localisation map

$$\lambda_p : \mathcal{O}_E\left[\frac{1}{p}\right]^\times \otimes \mathbb{Z}_p \to \prod_{w\in S_p(E)} E_w^\times \hat{\otimes}\mathbb{Z}_p,$$

where $S_p(E)$ denotes the set of places of E which lie above p. If $\rho \in R_{\mathbb{C}_p}(\bar{G})$, then we say that Leopoldt's conjecture 'is valid at ρ' if $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \ker(\lambda_p))^\rho = 0$. We set $c_\gamma := c_{\chi_{\mathrm{cyc}},\gamma} \in \mathbb{Q}_p^\times$ and for each $\rho \in R_{\mathbb{C}_p}(\bar{G})$ we define

$$\langle \rho, 1\rangle := \dim_{\mathbb{C}_p}(H^0(\bar{G}, V_\rho)) = \dim_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho).$$

LEMMA 5.1. *We fix $\rho \in R_{\mathbb{C}_p}(\bar{G})$ and assume that Leopoldt's conjecture is valid at ρ.*

(i) *There are canonical isomorphisms*

$$(H_c^i(U, \mathbb{T}_E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\rho \cong \begin{cases} (\mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\rho, & \text{if } i = 2 \\ (\mathbb{Q}_p)^\rho, & \text{if } i = 3 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *$R\Gamma_c(U, \mathbb{T}) \in \Sigma_{\mathrm{ss}-\rho}$ and $r_G(R\Gamma_c(U, \mathbb{T}))(\rho) = \langle \rho, 1\rangle$.*

(iii) *For each $w \in S_p(E)$ we write $\mathrm{N}_{E_w/\mathbb{Q}_p}$ for the morphism $E_w^\times \hat{\otimes}\mathbb{Z}_p \to \mathbb{Q}_p^\times \hat{\otimes}\mathbb{Z}_p$ that is induced by the field theoretic norm map. Then, with respect to the identifications given in claim (i), the morphism*

$$(\mathfrak{B}^2)^\rho : (H_c^2(U, \mathbb{T}_E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\rho \to (H_c^3(U, \mathbb{T}_E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\rho,$$

is induced by the map

$$\log_{p,\gamma,E} : \prod_{w\in S_p(E)} E_w^\times \hat{\otimes}\mathbb{Z}_p \to \mathbb{Z}_p$$

which sends each element $(e_w)_w$ to $-\sum_w c_\gamma^{-1} \log_p(\mathrm{N}_{E_w/\mathbb{Q}_p}(e_w))$.

Proof. Claim (i) can be verified by combining the exact cohomology sequence of the tautological exact triangle

(17) $$R\Gamma_c(U, \mathbb{T}_E) \to R\Gamma(U, \mathbb{T}_E) \to \bigoplus_{\ell \in S} R\Gamma(\mathbb{Q}_\ell, \mathbb{T}_E) \to R\Gamma_c(U, \mathbb{T}_E)[1]$$

together with the canonical identifications $H^i(U, \mathbb{T}_E) \cong H^i(\mathcal{O}_E[\frac{1}{S}], \mathbb{Z}_p(1))$ and $H^i(\mathbb{Q}_\ell, \mathbb{T}_E) \cong \bigoplus_{w\in S_\ell(E)} H^i(E_w, \mathbb{Z}_p(1))$ and an explicit computation of each of the groups $H^i(\mathcal{O}_E[\frac{1}{S}], \mathbb{Z}_p(1))$ and $H^i(E_w, \mathbb{Z}_p(1))$. As this is routine we leave explicit details to the reader except to note that $H_c^2(U, \mathbb{T}_E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is canonically isomorphic to $\mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (independently of Leopoldt's conjecture), whilst the fact that E is totally real implies the vanishing of $(H_c^1(U, \mathbb{T}_E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\rho$ is equivalent to that of $(\ker(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\rho$.

To prove claim (ii) and (iii) we write E_{cyc}, $E_{w,\mathrm{cyc}}$ for each $w \in S_p(E)$ and $\mathbb{Q}_{p,\mathrm{cyc}}$ for the cyclotomic $\mathbb{Z}_p$-extensions of E, E_w and $\mathbb{Q}_p$. Then (15) induces

a direct product decomposition $\mathrm{G}(E_{\mathrm{cyc}}/\mathbb{Q}) \cong \Gamma \times \bar{G}$ and hence allows us to identify Γ with each of $\mathrm{G}(E_{\mathrm{cyc}}/E)$, $\mathrm{G}(E_{w,\mathrm{cyc}}/E_w)$ and $\mathrm{G}(\mathbb{Q}_{p,\mathrm{cyc}}/\mathbb{Q}_p)$. We note also that, in terms of the notation of Lemma 3.13, the isomorphism (16) (with $F = E_{\mathrm{cyc}}$) induces a canonical isomorphism in $D^p(\Lambda(\Gamma))$ of the form

$$(18) \qquad R\Gamma_c(U, \mathbb{T})_\rho \cong \mathcal{O}^n \otimes_{\mathbb{Z}_p[\bar{G}]} R\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}),$$

where Γ acts naturally on the right hand factor in the tensor product.

From Lemma 3.13(i), we may therefore deduce that $R\Gamma_c(U, \mathbb{T}) \in \Sigma_{\mathrm{ss}-\rho}$ if and only if $\mathcal{O}^n \otimes_{\mathbb{Z}_p[\bar{G}]} R\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}) \in \Sigma_{\mathrm{ss}}$. But the latter containment can be easily verified by using the criterion of Remark 3.3(ii): indeed, one need only note that $H_c^i(U, \mathbb{T}_{E_{\mathrm{cyc}}})$ is finite if $i \notin \{2, 3\}$, that $H_c^3(U, \mathbb{T}_{E_{\mathrm{cyc}}})$ identifies with $\mathbb{Z}_p$ (as a Γ-module) and that the exact sequences of (3) combine with the descriptions of claim (i) to imply that $((\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^1(U, \mathbb{T}_{E_{\mathrm{cyc}}}))^\rho)^\Gamma$ and $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^1(U, \mathbb{T}_{E_{\mathrm{cyc}}}))_\Gamma^\rho$ both vanish. In addition, the same observations combine with Lemma 3.13(ii) to imply that $r_G(R\Gamma_c(U, \mathbb{T}))(\rho) = \dim_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho)$.

Regarding claim (iii), the isomorphism (18) combines with Lemma 3.13(i) and Lemma 3.1 to imply that $(\mathfrak{B}^2)^\rho = (\hat{\mathfrak{B}}^2)^\rho$ where $\hat{\mathfrak{B}}^2$ is the Bockstein morphism of the pair $(R\Gamma_c(U, \mathbb{T}_{E_{\mathrm{cyc}}}), T)$. Also, the cohomology sequence of the triangle (17) gives rise to a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{w \in S_p(E)} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(E_w, \mathbb{Z}_p(1)) & \longrightarrow & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^2(U, \mathbb{T}_E) \\
{\scriptstyle (-1) \times (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{B}_w^1)_w} \downarrow & & \downarrow {\scriptstyle \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{\mathfrak{B}}^2} \\
\bigoplus_{w \in S_p(E)} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^2(E_w, \mathbb{Z}_p(1)) & \longrightarrow & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^3(U, \mathbb{T}_E)
\end{array}$$

where the upper row is the (tautological) surjection induced by the canonical identifications $H^1(E_w, \mathbb{Z}_p(1)) \cong E_w^\times \hat{\otimes} \mathbb{Z}_p$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_c^2(U, \mathbb{T}_E) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_p)$, the lower row is the surjection induced by the canonical identifications $H^2(E_w, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ and $H_c^3(U, \mathbb{T}_E) \cong \mathbb{Z}_p$ together with the identity map on $\mathbb{Z}_p$, $\mathfrak{B}_w^1$ is the Bockstein morphism of the pair $(R\Gamma(E_{w,\mathrm{cyc}}, \mathbb{Z}_p(1)), T)$ and the factor -1 occurs on the left hand vertical arrow because the diagram compares Bockstein morphisms in degrees 1 and 2.

Further, for each $w \in S_p(E)$ the natural isomorphism (in $D^p(\mathbb{Z}_p)$)

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_p[\mathrm{G}(E_w/\mathbb{Q}_p)]}^{\mathbb{L}} R\Gamma(E_w, \mathbb{Z}_p(1)) \cong R\Gamma(\mathbb{Q}_p, \mathbb{Z}_p(1))$$

induces a commutative diagram

$$\begin{array}{ccc}
H^1(E_w, \mathbb{Z}_p(1)) & \longrightarrow & H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \\
{\scriptstyle \mathfrak{B}_w^1} \downarrow & & \downarrow {\scriptstyle \mathfrak{B}_p^1} \\
H^2(E_w, \mathbb{Z}_p(1)) & \longrightarrow & H^2(\mathbb{Q}_p, \mathbb{Z}_p(1))
\end{array}$$

where the upper horizontal arrow is induced by the canonical identifications $H^1(E_w, \mathbb{Z}_p(1)) \cong E_w^\times \hat{\otimes} \mathbb{Z}_p$ and $H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Q}_p^\times \hat{\otimes} \mathbb{Z}_p$ together with the map $\mathrm{N}_{E_w/\mathbb{Q}_p}$, the lower horizontal arrow is induced by the canonical identifications $H^2(E_w, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ and $H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ together with the identity map on $\mathbb{Z}_p$, and $\mathfrak{B}_p^1$ is the Bockstein morphism of the pair $(R\Gamma(\mathbb{Q}_{p,\mathrm{cyc}}, \mathbb{Z}_p(1))), T)$.

To prove claim (iii) it thus suffices to recall that, with respect to the identifications $H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Q}_p^\times \hat{\otimes} \mathbb{Z}_p$ and $H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$, the map $\mathfrak{B}_p^1$ is equal to $c_\gamma^{-1} \cdot \log_p$ (see, for example, [8, p. 352]).   □

5.2. THE p-ADIC STARK CONJECTURE AT $s = 1$. For each character $\chi \in R_\mathbb{C}(\bar{G})$ we write $L_S(s, \chi)$ for the Artin L-function of χ that is truncated by removing the Euler factors attached to primes in S (cf. [35, Chap. 0, §4]). Then, for each character $\rho \in R_{\mathbb{C}_p}(G)$ there exists a unique p-adic meromorphic function $L_{p,S}(\cdot, \rho) : \mathbb{Z}_p \to \mathbb{C}_p$ such that for each strictly negative integer n and each isomorphism $j : \mathbb{C}_p \cong \mathbb{C}$ one has

$$L_{p,S}(n, \rho)^j = L_S(n, (\rho \cdot \omega^{n-1})^j)$$

where $\omega : \mathrm{G}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ is the Teichmüller character. Henceforth we will fix such an isomorphism j and often omit it from the notation. This function is the '(S-truncated) p-adic Artin L-function' of ρ that is constructed by Greenberg in [15] by combining techniques of Brauer induction with the fundamental results of Deligne and Ribet [13] and Cassou-Noguès [9].

In this section we recall a conjecture of Serre regarding the 'leading term at $s = 1$' of $L_{p,S}(s, \rho)$. To this end we set $E_\infty := \mathbb{R} \otimes_\mathbb{Q} E \cong \prod_{\mathrm{Hom}(E,\mathbb{C})} \mathbb{R}$ and write $\log_\infty(\mathcal{O}_E^\times)$ for the inverse image of $\mathcal{O}_E^\times \hookrightarrow E_\infty^\times$ under the (componentwise) exponential map $\exp_\infty : E_\infty \to E_\infty^\times$. We set $E_0 := \{x \in E : \mathrm{Tr}_{E/\mathbb{Q}}(x) = 0\}$. Then $\log_\infty(\mathcal{O}_E^\times)$ is a lattice in $\mathbb{R} \otimes_\mathbb{Q} E_0$ and so there is a canonical isomorphism of $\mathbb{C}[\bar{G}]$-modules $\mu_\infty : \mathbb{C} \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{C} \otimes_\mathbb{Q} E_0$. This implies that the $\mathbb{Q}[\bar{G}]$-modules E_0 and $\mathcal{O}_E^\times \otimes_\mathbb{Z} \mathbb{Q}$ are (non-canonically) isomorphic. We note also that the composite morphism

$$(19) \quad \log_\infty(\mathcal{O}_E^\times) \xrightarrow{\exp_\infty} \mathcal{O}_E^\times \xrightarrow{\lambda_p} \prod_{w \in S_p(E)} U_{E_w}^1$$
$$\xrightarrow{(u_w)_w \mapsto (\log_p(u_w))_w} \prod_{w \in S_p(E)} E_w \cong \mathbb{Q}_p \otimes_\mathbb{Q} E,$$

factors through the inclusion $\mathbb{Q}_p \otimes_\mathbb{Q} E_0 \subset \mathbb{Q}_p \otimes_\mathbb{Q} E$ and hence induces an isomorphism of $\mathbb{Q}_p[\bar{G}]$-modules $\mu_p : \mathbb{Q}_p \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{Q}_p \otimes_\mathbb{Q} E_0$.

CONJECTURE 5.2. *For each $\rho \in R_{\mathbb{C}_p}(\bar{G})$ we set*

$$L_{p,S}^*(1, \rho) := \lim_{s \to 1} (s - 1)^{\langle \rho, 1 \rangle} \cdot L_{p,S}(s, \rho).$$

Then $L_{p,S}^*(1, \rho)$ is equal to the leading term of $L_{p,S}(s, \rho)$ at $s = 1$, and for each choice of isomorphism of $\mathbb{Q}[\bar{G}]$-modules $g : E_0 \to \mathcal{O}_E^\times \otimes_\mathbb{Z} \mathbb{Q}$ one has

$$\frac{L_{p,S}^*(1, \rho)}{\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mu_p) \circ (\mathbb{C}_p \otimes_\mathbb{Q} g))^\rho} = \frac{L_S^*(1, \rho)}{\det_\mathbb{C}(\mu_\infty \circ (\mathbb{C} \otimes_\mathbb{Q} g))^\rho}.$$

REMARK 5.3. This conjecture is the 'p-adic Stark conjecture at $s = 1$' as discussed by Tate in [35, Chap. VI, §5] (where it is attributed to Serre [33]). More precisely, there are some imprecisions in the statement of [35, Chap. VI,

§5] (for example, and as already noted by Solomon in [34, §3.3], the intended meaning of the symbols 'log U' and 'μ_p' in [35, p. 137] is unclear) and Conjecture 5.2 represents a natural clarification of the presentation given in loc. cit.

REMARK 5.4. We fix a subgroup J of $\bar{G}$ and write 1_J for the trivial character of J. If $\rho = \mathrm{Ind}_J^{\bar{G}} 1_J$, then the inductive behaviour of L-functions combines with the analytic class number formula for E^J to show that Conjecture 5.2 is valid for ρ if and only if the residue at $s = 1$ of the p-adic zeta function of the field E^J is non-zero and equal to $2^{[E^J:\mathbb{Q}]-1}hR_pe_p/\sqrt{|d|}$ where h, R_p and d are the class number, p-adic regulator and absolute discriminant of E^J respectively and $e_p := \prod_{v \in S_p(E^J)}(1 - \mathrm{N}v^{-1})$ (cf. [35, rem., p. 138]). From the main result of Colmez in [12] one may thus deduce that Conjecture 5.2 is valid for $\rho = \mathrm{Ind}_J^G 1_J$ if and only if Leopoldt's conjecture is valid for E^J. We note also that if Leopoldt's conjecture is valid for E, then it is valid for all such intermediate fields E^J.

5.3. THE INTERPOLATION FORMULA. We now reinterpret the equality of Conjecture 5.2 as an interpolation formula for the Zeta isomorphism $\zeta_{\Lambda(G)}(\mathbb{T})$.

THEOREM 5.5. *If Conjecture 5.2 is valid, then for each $\rho \in R_{\mathbb{C}_p}(\bar{G})$ one has $R\Gamma_c(U, \mathbb{T}) \in \Sigma_{\mathrm{ss}-\rho}$, $r_G(R\Gamma_c(U, \mathbb{T}))(\rho) = \langle\rho, 1\rangle$ and*

$$(20) \qquad\qquad c_\gamma^{\langle\rho,1\rangle} \cdot \zeta_{\Lambda(G)}(\mathbb{T})^*(\rho) = L_{p,S}^*(1, \rho).$$

REMARK 5.6. One can naturally interpret (20) as an equality of leading terms of p-adic meromorphic functions. Indeed, whilst Conjecture 5.2 predicts that $L_{p,S}^*(1, \rho)$ is the leading term at $s = 1$ of $L_{p,S}(s, \rho)$, Lemma 3.17 interprets the left hand side of (20) as the leading term at $s = 0$ of the function $f_{\mathcal{L}}(\rho\chi_{\mathrm{cyc}}^s)$ with $\mathcal{L} := [R\Gamma_c(U, \mathbb{T}), \zeta_{\Lambda(G)}(\mathbb{T})] \in K_1(\Lambda(G), \Sigma_{\mathrm{ss}-\rho})$.

Proof. We note first that if Conjecture 5.2 is valid, then Remark 5.4 implies that Leopoldt's conjecture is valid for E and so Lemma 5.1(ii) implies that $R\Gamma_c(U, \mathbb{T}) \in \Sigma_{\mathrm{ss}-\rho}$ and $r_G(R\Gamma_c(U, \mathbb{T}))(\rho) = \langle\rho, 1\rangle$ for each $\rho \in R_{\mathbb{C}_p}(\bar{G})$.
We now fix $\rho \in R_{\mathbb{C}_p}(\bar{G})$ and a number field K over which the character ρ can be realised. We fix an embedding $K \hookrightarrow \mathbb{C}$ and write λ for the place of K which is induced by j. We set $M := h^0(\mathrm{Spec}\, E)(1)$ and note that $M([\rho]^*) := M \otimes [\rho]^*$ is a K-motive, where $[\rho]^*$ denotes the dual of the Artin motive corresponding to ρ.
To evaluate $\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)$ we need to make Definition 3.14 explicit. To do this we use the observations of [5, §1.1, §1.3] to explicate the isomorphism $\zeta_K(M([\rho]^*))_{K_\lambda}$ which occurs in Conjecture (4.1). Indeed one has $H_f^1(M) = \mathcal{O}_E^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, $H_f^0(M^*(1)) = \mathbb{Q}$, $t_M := E$ and $H_f^0(M) = H_f^1(M^*(1)) = M_B^+ = 0$ (the latter since E is totally real) so that

$$\mathbb{C} \otimes_K \Delta_K(M([\rho]^*)) = \mathbf{d}_{\mathbb{C}}((\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)_\rho)\mathbf{d}_{\mathbb{C}}((\mathbb{Q})_\rho)\mathbf{d}_{\mathbb{C}}((E)_\rho)^{-1}$$

and $\zeta_K(M([\rho]^*))_{K_\lambda}$ is equal to the composite morphism

$$(21) \quad \mathbf{1}_{\mathbb{C}_p} \to \mathbf{1}_{\mathbb{C}_p}$$
$$\to \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)^\rho) \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho) \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E)^\rho)^{-1}$$
$$\to \mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} H_c^2(U, M([\rho]^*)_\lambda))^{-1} \mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} H_c^3(U, M([\rho]^*)_\lambda))$$
$$\to \mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} R\Gamma_c(U, M([\rho]^*)_\lambda))^{-1}.$$

In this displayed formula we have used the following notation: the first map corresponds to multiplication by $L_S^*(1, \rho)$; the second map is induced by applying $(\mathbb{C}_p \otimes_{\mathbb{R}, j^{-1}} -)^\rho$ to both the natural isomorphism $E \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\mathrm{Hom}(E,\mathbb{C})} \mathbb{R}$ and also the exact sequence

$$(22) \quad 0 \to \mathcal{O}_E^\times \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{(\log \circ \sigma)_\sigma} \prod_{\sigma \in \mathrm{Hom}(E,\mathbb{C})} \mathbb{R} \xrightarrow{(x_\sigma)_\sigma \mapsto \sum_\sigma x_\sigma} \mathbb{R} \to 0;$$

the third map is induced by Lemma 5.1(i) and the inverse of the isomorphism

$$(23) \quad \prod_{w \in S_p(E)} U_{E_w}^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{(u_w)_w \mapsto (\log_p(u_w))_w} \prod_{w \in S_p(E)} E_w \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} E;$$

the last map is induced by property h) as described in §2.1 (with $R = \mathbb{C}_p$). Also, from Lemma 5.1(iii) we know that $\mathbb{C}_p \otimes_{K_\lambda} t(R\Gamma_c(U, \mathbb{T})(\rho^*))$ is equal to the composite

$$(24) \quad \mathbf{d}_{\mathbb{C}_p}(\mathbb{C}_p \otimes_{K_\lambda} R\Gamma_c(U, M([\rho]^*)_\lambda)^{-1}$$
$$\to \mathbf{d}_{\mathbb{C}_p}(H_c^2(U, M([\rho]^*)_\lambda)^{-1} \mathbf{d}_{\mathbb{C}_p}(H_c^3(U, M([\rho]^*)_\lambda)$$
$$\to \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho)^{-1} \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p)^\rho) = \mathbf{1}_{\mathbb{C}_p}$$

where the first arrow is induced by property h) and the second by Lemma 5.1(i) and the map $-c_\gamma^{-1} \log_{p,\gamma,E}$.

Now, after taking account of Lemma 5.1(ii), $\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)$ is defined to be $(-1)^{\langle \rho, 1 \rangle}$ times the element of $\mathbb{C}_p^\times$ which corresponds to the composite of (21) and (24) (cf. Definition 3.14). Thus, after noting that there is a commutative diagram of the form

$$\begin{array}{ccc}
\prod_{w \in S_p(E)} U_{E_w}^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \longrightarrow & \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\
{\scriptstyle (23)} \downarrow & & \downarrow {\scriptstyle (-1) \times \log_{p,\gamma,E}} \\
E \otimes_{\mathbb{Q}} \mathbb{Q}_p & \xrightarrow{\mathrm{Tr}_{E/\mathbb{Q}}} & \mathbb{Q}_p
\end{array}$$

where the upper horizontal arrow is the tautological projection, the observations made above imply that

$$(25) \quad c_\gamma^{\langle \rho, 1 \rangle} \cdot \zeta_{\Lambda(G)}(\mathbb{T})^*(\rho) = L_S^*(1, \rho) \cdot \xi$$

where ξ is the element of $\mathbb{C}_p^\times$ which corresponds to the composite

$$
\begin{aligned}
(26) \qquad \mathbf{1}_{\mathbb{C}_p} =& \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)^\rho)^{-1} \\
&\to \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E_0)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times)^\rho)^{-1} \\
&\to \mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E_0)^\rho)\mathbf{d}_{\mathbb{C}_p}((\mathbb{Q}_p \otimes_{\mathbb{Q}} E_0)^\rho)^{-1} = \mathbf{1}_{\mathbb{C}_p}
\end{aligned}
$$

where the first arrow is induced by applying $\mathbb{C}_p \otimes_{\mathbb{R},j^{-1}} -$ to the isomorphism $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_E^\times \cong \mathbb{R} \otimes_{\mathbb{Q}} E_0$ coming from the map $(\log \circ \sigma)_\sigma$ in (22) and the second by the isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathcal{O}_E^\times \cong \mathbb{Q}_p \otimes_{\mathbb{Q}} E_0$ coming from the second and third arrows in (19). (Note that the factor $(-1)^{\langle \rho,1 \rangle}$ in the definition of $\zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)$ cancels against the factor -1 which occurs in the commutative diagram above, and hence does not occur in the formula (25)).
But, upon comparing the definitions of μ_∞ and μ_p in §5.2 with the maps involved in (26), one finds that ξ is equal to

$$
\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mu_p) \circ (\mathbb{C}_p \otimes_{\mathbb{C},j^{-1}} \mu_\infty)^{-1})^\rho = \frac{\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mu_p) \circ (\mathbb{C}_p \otimes_{\mathbb{Q}} g))^\rho}{(\det_{\mathbb{C}}(\mu_\infty \circ (\mathbb{C} \otimes_{\mathbb{Q}} g))^\rho)}
$$

and hence (25) implies that

$$
\frac{c_\gamma^{\langle \rho,1 \rangle} \cdot \zeta_{\Lambda(G)}(\mathbb{T})^*(\rho)}{\det_{\mathbb{C}_p}((\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mu_p) \circ (\mathbb{C}_p \otimes_{\mathbb{Q}} g))^\rho} = \frac{L_S^*(1,\rho)}{\det_{\mathbb{C}}(\mu_\infty \circ (\mathbb{C} \otimes_{\mathbb{Q}} g))^\rho}.
$$

The claimed equality (20) now follows immediately upon comparing this equality to that of Conjecture 5.2. $\qquad \square$

COROLLARY 5.7. *If Leopoldt's conjecture is valid for E at p, then for every $\mathbb{Q}_p$-rational character ρ of $\bar{G}$ there exists a natural number n_ρ such that*

$$
(c_\gamma^{\langle \rho,1 \rangle} \cdot \zeta_{\Lambda(G)}(\mathbb{T})^*(\rho))^{n_\rho} = L_{p,S}^*(1,\rho)^{n_\rho}.
$$

Further, if ρ is a permutation character, then one can take $n_\rho = 1$.

Proof. If ρ is $\mathbb{Q}_p$-rational, then Artin's Induction Theorem implies the existence of a natural number n_ρ such that in $R_{\mathbb{C}_p}(G)$ one has $n_\rho \cdot \rho = \sum_H n_H \cdot \text{ind}_H^G 1_H$ where H runs over the set of subgroups of $\bar{G}$ and each n_H is an integer (cf. [35, chap. II, thm. 1.2]). Further, ρ is said to be a permutation character if and only if there exists such a formula with $n_\rho = 1$. The stated result thus follows by combining Theorem 5.5 with Remark 5.4 and the fact that each side of (20) is both additive and inductive in ρ. $\qquad \square$

6. The interpolation formula for critical motives

As a second application of the formalism introduced in §3, in this section we prove an interpolation formula for the leading terms (in the sense of Definition 3.14) of the p-adic L-functions that Fukaya and Kato conjecture to exist for any critical motive which has good ordinary reduction at all places above p. (We recall that a motive M is said to be 'critical' if the map (11) is bijective). To study these p-adic L-functions we must combine Conjecture 4.1 together with a local analogue of this conjecture (which is also due to Fukaya and Kato, and

is recalled as Conjecture 6.1 below) and aspects of Nekovar's theory of Selmer complexes and of the theory of p-adic height pairings.

6.1. LOCAL EPSILON ISOMORPHISMS. We fix once and for all the p-adic period $t = "2\pi i"$, i.e. a generator of $\mathbb{Z}_p(1)$. Further we write $\epsilon_p(V) := \epsilon(D_{pst}(V))$ for Deligne's ϵ-factor at p where $D_{pst}(V)$ is endowed with the linearized action of the Weil-group and thereby considered as a representation of the Weil-Deligne group, see [14, §3.2] or [27, appendix C] (Here we suppress the dependence of the choice of a Haar measure and of $t = 2\pi i$ in the notation. The above choice of $t = (t_n) \in \mathbb{Z}_p(1)$ determines a homomorphism $\psi_p : \mathbb{Q}_p \to \bar{\mathbb{Q}}_p^{\times}$ with $\ker(\psi_p) = \mathbb{Z}_p$ sending $\frac{1}{p^n}$ to $t_n \in \mu_{p^n}$). Note that $(B_{dR})^{I_p} = \widehat{\mathbb{Q}_p^{nr}}$, the completion of the maximal unramified extension $\mathbb{Q}_p^{nr}$ of $\mathbb{Q}_p$. Let L be any finite extension of $\mathbb{Q}_p$ and V any finite dimensional L-vector space with continuous $G_{\mathbb{Q}_p}$-action. We set $\widetilde{L} := \widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} L$,

$$\Gamma^*(-j) := \begin{cases} \Gamma(j) = (j-1)!, & \text{if } j > 0, \\ \lim_{s \to j}(s-j)\Gamma(s) = (-1)^j((-j)!)^{-1}, & \text{if } j \leq 0, \end{cases}$$

and $\Gamma_L(V) := \prod_{j \in \mathbb{Z}} \Gamma^*(j)^{-h(-j)}$, where $h(j) := \dim_L gr^j D_{dR}(V)$.
In [14, prop. 3.3.5] the existence of a map

$$\epsilon_{dR}(V) : \mathbf{1}_{\widetilde{L}} \to \mathbf{d}_{\widetilde{L}}(V)\mathbf{d}_{\widetilde{L}}(D_{dR}(V))^{-1}$$

with certain properties was shown. Using this we define an isomorphism

$$\epsilon_{p,L}(V) : \mathbf{1}_{\widetilde{L}} \to \left(\mathbf{d}_L(R\Gamma(\mathbb{Q}_p, V))\mathbf{d}_L(V)\right)_{\widetilde{L}}$$

as product of $\Gamma_L(V)$, $\eta_l(V) \cdot \overline{(\eta_l(V^*(1))^*)}$ and $\epsilon_{dR}(V)$.
Now let T be a Galois stable $\mathcal{O} := \mathcal{O}_L$-lattice of V and set $\widetilde{\mathcal{O}} := W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}$, where $W(\overline{\mathbb{F}_p})$ denotes the Wittring of $\overline{\mathbb{F}_p}$.
We define

$$\widetilde{\Lambda} := W(\overline{\mathbb{F}_p})[\![G]\!] = \varprojlim_n \left(W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G/G_n]\right),$$

we assume $L = \mathbb{Q}_p$ and set as before $\mathbb{T} := \Lambda \otimes_{\mathbb{Z}_p} T$. We write $T(\rho^*)$ for the $\mathcal{O}$-lattice $\rho^* \otimes T$ of $\rho^* \otimes V$, which we assume to be de Rham. The following conjecture is a local integrality statement

CONJECTURE 6.1 (Fukaya/Kato [14, conj. 3.4.3], see also [37, conj. 5.9]). *There exists a (unique) isomorphism in $\mathcal{C}_{\widetilde{\Lambda}}$*

$$\epsilon_{p,\Lambda}(\mathbb{T}) : \mathbf{1}_{\widetilde{\Lambda}} \to \left(\mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}))\mathbf{d}_\Lambda(\mathbb{T})\right)_{\widetilde{\Lambda}}$$

such that for all $\rho : G \to GL_n(\mathcal{O}) \subseteq GL_n(L)$, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, we have

$$L^n \otimes_\Lambda \epsilon_{p,\Lambda}(\mathbb{T}) = \epsilon_{p,L}(T(\rho^*)).$$

6.2. p-ADIC HEIGHT PAIRINGS. To prepare for our derivation of the interpolation formula in this section we discuss certain preliminaries regarding height pairings.

We fix a continuous finite-dimensional L-linear representation W of $G_{\mathbb{Q}}$ which satisfies the following 'condition of Dabrowski-Panchiskin':

(DP) W is de Rham and there exist a $G_{\mathbb{Q}_p}$-subrepresentation $\hat{W}$ of W (restricted to $G_{\mathbb{Q}_p}$) such that $D_{dR}^0(\hat{W}) = t_p(W) := D_{dR}(W)/D_{dR}^0(W)$.

Thus we have an exact sequence of $G_{\mathbb{Q}_p}$-representations

$$0 \to \hat{W} \to W \to \tilde{W} \to 0$$

such that $D_{dR}^0(\hat{W}) = t_p(\tilde{W}) = 0$ (cf. [21, prop. 1.28]). Setting $Z := W^*(1)$, $\hat{Z} := \tilde{W}^*(1)$, and $\tilde{Z} := \hat{W}^*(1)$, we obtain by Kummer duality the analogous exact sequence

$$0 \to \hat{Z} \to Z \to \tilde{Z} \to 0$$

and Z satisfies the condition (DP).

Let S be a finite set of places of $\mathbb{Q}$ containing the places $S_\infty := \{\infty\}$ as well as $S_p := \{p\}$ and such that W (and thus Z) are also representations of $G_S(\mathbb{Q})$, the Galois group of the maximal outside S unramified extension of $\mathbb{Q}$. Furthermore, we set $S_f := S \setminus S_\infty$.

The Selmer complex $SC_U(\hat{W}, W)$ is by definition the mapping fibre
(27)

$$SC_U(\hat{W}, W) \longrightarrow R\Gamma(U, W) \longrightarrow R\Gamma(\mathbb{Q}_p, W/\hat{W}) \oplus \bigoplus_{S_f \setminus S_p} R\Gamma(\mathbb{Q}_\ell, W) \longrightarrow$$

while $SC(\hat{W}, W)$ is the mapping fibre
(28)

$$SC(\hat{W}, W) \longrightarrow R\Gamma(U, W) \longrightarrow R\Gamma(\mathbb{Q}_p, W/\hat{W}) \oplus \bigoplus_{S_f \setminus S_p} R\Gamma_{/f}(\mathbb{Q}_\ell, W) \longrightarrow .$$

Here $R\Gamma_{/f}(\mathbb{Q}_\ell, W)$ is defined as mapping cone

$$(29) \qquad R\Gamma_f(\mathbb{Q}_\ell, W) \longrightarrow R\Gamma(\mathbb{Q}_\ell, W) \longrightarrow R\Gamma_{/f}(\mathbb{Q}_\ell, W) \longrightarrow$$

For any $G_{\mathbb{Q}_p}$-representation V and any prime number ℓ we define an element of $L[u]$ by setting

$$P_\ell(V, u) := P_{L,\ell}(V, u) := \begin{cases} \det_L(1 - \varphi_\ell u | V^{I_\ell}), & \text{if } \ell \neq p \\ \det_L(1 - \varphi_p u | D_{cris}(V)), & \text{if } \ell = p, \end{cases}$$

where φ_l denotes the geometric Frobenius automorphism at l.

The following three conditions are easily seen to be equivalent

(A$_1$) $P_l(W, 1)P_l(Z, 1) \neq 0$ for all $l \in S_f \setminus S_p$,
(A$_2$) $H^0(\mathbb{Q}_\ell, W) = H^0(\mathbb{Q}_\ell, Z) = 0$ for all $l \in S_f \setminus S_p$,
(A$_3$) $R\Gamma_f(\mathbb{Q}_\ell, W)$ is quasi-null for all $l \in S_f \setminus S_p$.

We also consider the following group of conditions

(B$_1$) $P_p(W, 1)P_p(Z, 1) \neq 0$
(B$_2$) $D_{cris}(W)^{\varphi_p - 1} = D_{cris}(Z)^{\varphi_p - 1} = 0$

(B_3) $H^0(\mathbb{Q}_p, W) = H^0(\mathbb{Q}_p, Z) = 0$

Then (B_1) is equivalent to (B_2) and (B_3) implies (B_2) by [21, thm. 1.15].
Finally we consider the following conditions

(C_1) $P_p(\tilde{W}, 1)P_p(\tilde{Z}, 1) \neq 0$
(C_2) $D_{cris}(\tilde{W})^{\varphi_p - 1} = D_{cris}(\tilde{Z})^{\varphi_p - 1} = 0$
(C_3) $H^0(\mathbb{Q}_p, \tilde{W}) = H^0(\mathbb{Q}_p, \tilde{Z}) = 0$.

These conditions are again easily seen to be equivalent (to see that (C_2) is
equivalent to (C_3) use (loc. cit.) and the fact that $t_p(\tilde{W}) = t_p(\tilde{Z}) = 0$).

LEMMA 6.2. *Let X denote either W or Z.*

(i) *If condition (A_1) holds, then for every $\ell \in S_f \setminus S_p$ the following com-*
plexes are quasi-null

$$R\Gamma(\mathbb{Q}_\ell, X) \cong R\Gamma_f(\mathbb{Q}_\ell, X) \cong R\Gamma_{/f}(\mathbb{Q}_\ell, X) \cong 0.$$

(ii) *If condition (C_1) is satisfied, then there are isomorphisms*

$$R\Gamma_{/f}(\mathbb{Q}_p, X) \cong R\Gamma(\mathbb{Q}_p, \tilde{X}) \quad and \quad R\Gamma_f(\mathbb{Q}_p, X) \cong R\Gamma(\mathbb{Q}_p, \hat{X})$$

in $D^p(L)$.

(iii) *If conditions (A_1) and (C_1) are both satisfied, then we have a quasi-*
isomorphism in $D^p(L)$

$$SC_U(\hat{W}, W) \cong R\Gamma_f(\mathbb{Q}, W).$$

Proof. We assume (A_1). Then by local duality and the local Euler characteristic
formula it follows immediately that $R\Gamma(\mathbb{Q}_\ell, X)$ is quasi-null. The other state-
ments in (i) are obvious. To prove (ii) we assume (C_1). Then, since every com-
plex of vector spaces is quasi-isomorphic to its cohomology, considered as com-
plex with zero differential, we have $R\Gamma(\mathbb{Q}_p, \hat{X}) \cong R\Gamma_f(\mathbb{Q}_p, \hat{X}) \cong R\Gamma_f(\mathbb{Q}_p, X)$
by [14, lem. 4.1.7]. Thus the exact triangles

$$R\Gamma(\mathbb{Q}_p, \hat{X}) \to R\Gamma(\mathbb{Q}_p, X) \to R\Gamma(\mathbb{Q}_p, \tilde{X}) \to$$

and

$$R\Gamma_f(\mathbb{Q}_p, X) \to R\Gamma(\mathbb{Q}_p, X) \to R\Gamma_{/f}(\mathbb{Q}_p, X) \to$$

are naturally isomorphic in $D^p(L)$. Finally, we note that claim (iii) follows
immediately from claims (i) and (ii) and the definitions of $SC_U(\hat{W}, W)$ and
$R\Gamma_f(\mathbb{Q}, W)$. □

Let M be any motive over $\mathbb{Q}$, $V = M_p$ its p-adic realization, ρ an Artin rep-
resentation defined over the number field K and $[\rho]$ its corresponding Artin
motive. Then, for any place λ of K above p, the λ-adic realisation

(30) $$W := N_\lambda = V \otimes_{\mathbb{Q}_p} [\rho]_\lambda^*$$

of the motive $N := M(\rho^*) := M \otimes [\rho]^*$ is an $L := K_\lambda$-adic representation. We
assume that V (and hence, since $[\rho]^*$ is pure of weight zero, also W) satisfies the
condition (DP). We fix a Galois stable lattice T of V and set $T_\rho := T \otimes_{\mathbb{Z}_p} \mathcal{O}^n$,
a Galois stable lattice in W (where we assume that without loss of generality
$[\rho]_\lambda^*$ is given as $\rho^* : G_\mathbb{Q} \to GL_n(\mathcal{O})$). Similarly we fix a $G_{\mathbb{Q}_p}$-stable lattice $\hat{T}$ of

$\hat{V}$ and take as $\tilde{T}$ the lattice in $\tilde{V}$ induced from T. Finally we set $\hat{T}_\rho := \hat{T} \otimes_{\mathbb{Z}_p} \mathcal{O}^n$ and $\tilde{T}_\rho := \tilde{T} \otimes_{\mathbb{Z}_p} \mathcal{O}^n$, they are Galois stable $\mathcal{O}$-lattices of $\hat{W}$ and $\tilde{W}$, respectively.

EXAMPLE 6.3. Let A be an abelian variety that is defined over $\mathbb{Q}$ and set $M := h^1(A)(1)$. If A has good ordinary reduction at p, then $W := N_\lambda$ satisfies the conditions (DP), (A_1), (B_1) and (C_1) (the last three for weight reasons - Weil conjectures, while more general, the condition (DP) holds for any motive which has good ordinary reduction at p (see [26])). However, if, for example, A is an elliptic curve with (split) multiplicative reduction at p, then the condition (B_1) is not satisfied.

Now we define a $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-lattice

$$\hat{T} := T \cap \hat{V},$$

of $\hat{V}$. As before let $\mathbb{T}$ denote the big Galois representation $\Lambda \otimes_{\mathbb{Z}_p} T$ and put $\hat{\mathbb{T}} := \Lambda \otimes_{\mathbb{Z}_p} \hat{T}$ similarly. Then $\hat{\mathbb{T}}$ is $G_{\mathbb{Q}_p}$-stable sub-Λ-module of $\mathbb{T}$. In fact, it is a direct summand of $\mathbb{T}$ and we have an isomorphism in $\mathcal{C}_{\tilde{\Lambda}}$

(31) $$\beta : \mathbf{d}_\Lambda(\mathbb{T}^+)_{\tilde{\Lambda}} \cong \mathbf{d}_\Lambda(\hat{\mathbb{T}})_{\tilde{\Lambda}}.$$

Now the Selmer complexes $SC_U(\hat{\mathbb{T}}, \mathbb{T})$ and $SC(\hat{\mathbb{T}}, \mathbb{T})$ are defined analogously as for W above.

We point out that $SC_U(\hat{X}, X)$ coincides with the Selmer complex $\widetilde{R\Gamma}_f(X)$ in [22, (11.3.1.5)] for $X \in \{W, Z\}$. More generally, if we define

$$\mathbb{T}_{cyc,\rho} := \Lambda(\Gamma) \otimes T_\rho$$

and similarly $\hat{\mathbb{T}}_{cyc,\rho}$ and $\tilde{\mathbb{T}}_{cyc,\rho}$, then $SC_U(\hat{\mathbb{T}}_{cyc,\rho}, \mathbb{T}_{cyc,\rho})$ identifies with the Selmer complex $\widetilde{R\Gamma}_{f,IW}(\mathbb{Q}_{cyc}/\mathbb{Q}, T_\rho)$ defined in [22, (8.8.5)] (with Nekovar's local conditions induced by $T_\ell^+ = \hat{\mathbb{T}}_{cyc}(\rho)$, if $\ell = p$, and 0 otherwise, and taking S_f for *his* Σ). Here Γ is the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{cyc}$ of $\mathbb{Q}$. Thus we obtain a pairing

$$h_p(W) : \mathrm{H}^1_f(\mathbb{Q}, W) \times \mathrm{H}^1_f(\mathbb{Q}, Z) \to L$$

from [22, §11] where $h_p(W)$ is called $\tilde{h}_{\pi,1,1}$. By [22, thm. 11.3.9] the pairing $h_p(W)$ coincides up to sign with the height-pairing constructed by Schneider [30] (in the case of abelian varieties), Perrin-Riou [24] (for semi-stable representations) and those constructed earlier by Nekovar [21]: see also §8.1 in (loc. cite.), Mazur-Tate [20] and Zarhin [39] for alternative definitions of related height pairings.

It follows from the construction of Nekovar's height pairing (cf. [22, the sentence after (11.1.3.2)]) that the induced map

(32) $$\mathrm{ad}(h_p(W)) : \mathrm{H}^1_f(\mathbb{Q}, W) \to \mathrm{H}^1_f(\mathbb{Q}, Z)^*$$

is equal to the composite

(33) $$\mathrm{H}^1_f(\mathbb{Q}, W) \cong \mathrm{H}^1(SC_U(\hat{W}, W)) \xrightarrow{\mathfrak{B}} \mathrm{H}^2(SC_U(\hat{W}, W))$$
$$\cong \mathrm{H}^2_f(\mathbb{Q}, W) \cong \mathrm{H}^1_f(\mathbb{Q}, Z)^*$$

where the first and third maps are by Lemma 6.2(iii), $\mathfrak{B}$ denotes the Bockstein morphism for $SC_U(\hat{\mathbb{T}}_{cyc,\rho}, \mathbb{T}_{cyc,\rho})$ and the last map is by global duality.

For the evaluation at representations in the next subsection we need the following descent properties. Let Υ be the set of all $l \neq p$ such that the ramification index of l in $F_\infty/\mathbb{Q}$ is infinite. Note that Υ is empty if G has a commutative open subgroup.

PROPOSITION 6.4. [14, prop. 1.6.5] *With the above notation we have canonical isomorphisms (for all l)*

$$L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma(U,\mathbb{T}) \cong \mathrm{R}\Gamma(U,W), \qquad L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma_c(U,\mathbb{T}) \cong \mathrm{R}\Gamma_c(U,W),$$

$$L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma(\mathbb{Q}_\ell,\mathbb{T}) \cong \mathrm{R}\Gamma(\mathbb{Q}_\ell,W), \quad L^n \otimes_{\Lambda,\rho} SC_U(\hat{\mathbb{T}},\mathbb{T}) \cong SC_U(\hat{W},W).$$

For $l \notin \Upsilon \cup S_p$ we also have: $L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma_f(\mathbb{Q}_\ell,\mathbb{T}) \cong \mathrm{R}\Gamma_f(\mathbb{Q}_\ell,W)$.

But note that the complex $\mathrm{R}\Gamma_f(\mathbb{Q}_\ell,\hat{\mathbb{T}})$ for $l \in \Upsilon$ and thus $SC(\hat{\mathbb{T}},\mathbb{T})$ does *not* descend like this in general. Instead, according to [14, prop. 4.2.17] one has a distinguished triangle
(34)

$$L^n \otimes_{\Lambda,\rho} SC(\hat{\mathbb{T}},\mathbb{T}) \longrightarrow SC(\hat{W},W) \longrightarrow \bigoplus_{l \in \Upsilon} \mathrm{R}\Gamma_f(\mathbb{Q}_\ell,W) \longrightarrow .$$

6.3. THE INTERPOLATION FORMULA. In this section we assume that the motive $N := M(\rho^*)$ is critical. Then, assuming [37, conj. 3.3] of Fontaine and Perrin-Riou it follows that the motivic cohomology groups

(D_1) $\mathrm{H}_f^0(N) = \mathrm{H}_f^0(N^*(1)) = 0$

both vanish. Assuming also the conjecture [37, conj. 3.6] on p-adic regulator maps, this is equivalent to the condition

(D_2) $\mathrm{H}_f^0(\mathbb{Q},W) = \mathrm{H}_f^0(\mathbb{Q},Z) = 0$.

We also consider the condition

(F) The pairing $h_p(W)$ is non-degenerate.

EXAMPLE 6.5. If A is an abelian variety over $\mathbb{Q}$, then the conditions (D_1) and (D_2) are both satisfied for the motive $M = h^1(A)(1)$. However, very little is known about the non-degeneracy of the p-adic height pairing in the ordinary case. Indeed, as far as we are aware, the only theoretical evidence for non-degeneracy is an old result of Bertrand [1] that for an elliptic curve with complex multiplication, the height of a point of infinite order is non-zero (even this is unknown in the non CM case). Computationally, there has been a lot of work done recently by Stein and Wuthrich [38]. We are grateful to J. Coates, P. Schneider and C. Wuthrich for providing us with these examples.

PROPOSITION 6.6. *Assume that the conditions (A_1), (C_1) and (D_2) are satisfied. Then*

(i) $SC_U(\hat{\mathbb{T}}_{cyc,\rho}, \mathbb{T}_{cyc,\rho}) \in \Sigma_{ss}$ *if and only if the condition (F) holds.*
(i) *if (F) holds, we have $r_\Gamma(SC_U(\hat{\mathbb{T}}_{cyc,\rho}, \mathbb{T}_{cyc,\rho})) = \dim_L \mathrm{H}_f^1(\mathbb{Q},W)$.*

Proof. By assumption (D_2) implies that $\mathrm{H}^i(SC_U(\hat{W}, W)) = 0$ for $i \neq 1, 2$. Thus the claim follows from the fact that (32) and (33) coincide. $\qquad\square$

Now let F_∞ be as before a p-adic Lie extension of $\mathbb{Q}$ which contains $\mathbb{Q}_{cyc}$ and G its Galois group (with quotient Γ). We set $\Lambda := \Lambda(G)$. Since by [14, 4.1.4(2)] we have a canonical identification

(35) $\qquad (\Lambda_\mathcal{O}(\Gamma) \otimes_\mathcal{O} \mathcal{O}^n) \otimes^{\mathbb{L}}_{\Lambda(G)} SC_U(\hat{\mathbb{T}}, \mathbb{T}) \cong SC_U(\hat{\mathbb{T}}_{cyc,\rho}, \mathbb{T}_{cyc,\rho})$

we conclude that $SC_U(\hat{\mathbb{T}}, \mathbb{T}) \in \Sigma_{\mathrm{ss}-\rho}$ if and only (F) holds for W.
In [14] two p-adic L-functions

(36) $\qquad\qquad\qquad \mathcal{L}_U := \mathcal{L}_U(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(SC_U(\hat{\mathbb{T}}, \mathbb{T}))$

and

(37) $\qquad\qquad\qquad\quad \mathcal{L} := \mathcal{L}(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(SC(\hat{\mathbb{T}}, \mathbb{T}))$

were defined, modulo the validity of Conjecture 4.1 and Conjecture 6.1. Recall the definition of Σ_{SC_U} or Σ_{SC} at the end of section 2 where we abbreviate $SC_U = SC_U(\hat{\mathbb{T}}, \mathbb{T})$ and $SC = SC(\hat{\mathbb{T}}, \mathbb{T})$. Now $\mathcal{L}_U$ and $\mathcal{L}$ induce classes $[SC_U, \mathcal{L}_U]$ and $[SC, \mathcal{L}]$ in $K_1(\Lambda(G), \Sigma_{SC_U})$ and $K_1(\Lambda(G), \Sigma_{SC})$, which for simplicity we again call $\mathcal{L}_U$ and $\mathcal{L}$, respectively. However, we note that for comparison purposes, it would perhaps be more convenient to define both $\mathcal{L}$ and $\mathcal{L}_U$ as elements of the group $K_1(\Lambda(G), \Sigma'_{SC})$ where Σ'_{SC} is the smallest full subcategory of $C^p(\Lambda)$ which satisfies conditions (i)-(iii) and (iv') from section 2 and contains SC_U as well as the Euler factors $\mathrm{R}\Gamma_f(\mathbb{Q}_\ell, \hat{\mathbb{T}})$ for all $l \in S_f \setminus S_p$.

THEOREM 6.7. *We assume that W satisfies the conditions (A_1), (B_1), (C_1), (D_1) and (F) and that the isomorphisms $\zeta_\Lambda(M)$ and $\epsilon_{p,\Lambda}(\hat{\mathbb{T}})$ that are described in Conjectures 4.1 and 6.1 both exist. Then both $SC_U(\hat{\mathbb{T}}, \mathbb{T})$ and $SC(\hat{\mathbb{T}}, \mathbb{T})$ belong to Σ_{ss}, $r := r_G(SC_U(\hat{\mathbb{T}}, \mathbb{T}))(\rho) = r_G(SC(\hat{\mathbb{T}}, \mathbb{T}))(\rho) = \dim_L \mathrm{H}^1_f(\mathbb{Q}, W)$ and the leading term $\mathcal{L}^*(\rho)$ (respectively $\mathcal{L}_U^*(\rho)$) is equal to*
(38)
$$(-1)^r \frac{L^*_{K,B}(M(\rho^*))}{\Omega_\infty(M(\rho^*))R_\infty(M(\rho^*))} \cdot \Omega_p(M(\rho^*))R_p(M(\rho^*)) \cdot \Gamma(\hat{V})^{-1} \cdot \frac{P_{L,p}(\hat{W}^*(1), 1)}{P_{L,p}(\hat{W}, 1)},$$

where $L^*_{K,B}(M(\rho^*))$ is the leading coefficient of the complex L-function of N, truncated by removing Euler factors for all primes in $B := \Upsilon \cup S_p$ (respectively $B := S \setminus S_\infty$), and the regulators $R_\infty(M(\rho^*))$ and $R_p(M(\rho^*))$ and periods $\Omega_\infty(M(\rho^*))$ and $\Omega_p(M(\rho^*))$ are as defined in the course of the proof given below.

REMARK 6.8. The formulas of Theorem 6.7 generalize the formulas obtained by Perrin-Riou in [27, 4.2.2,4.3.6]. Further, by slightly altering the definition of the complex L-function an analogous formula can be proved even in the case that (B_1) is not satisfied. Indeed, if (B_1) fails, then one can have $P_{L,p}(W, 0) = 0$ and so the order of vanishing of $L_{K,B}(M(\rho^*), s)$ and $L_K(M(\rho^*), s)$ may differ. However, to avoid this problem, in formula (38) one need only replace

$P_{L,p}(\hat{W}, 1)$ by the leading coefficient of $P_{L,p}(\hat{W}, p^s)$ at $s = 0$, or equivalently one can replace the term $\frac{L^*_{K,B}(M(\rho^*))}{P_{L,p}(\hat{W},1)}$ by $\frac{L^*_{K,B\setminus\{p\}}(M(\rho^*))}{\{P_{L,p}(W,u)^{-1}P_{L,p}(\hat{W},u)\}_{u=1}}$.

Proof. This proof is closely modelled on that of [14, thm. 4.2.26] (as amplified in [37, proof of thm. 6.4]). At the outset we let $\gamma = (\gamma_i)_i$ and $\delta = (\delta_i)_i$ be a choice of 'good bases (in the sense of [14, 4.2.24]) of M^+_B and t_M for $\hat{\mathbb{T}}$ and let γ' and δ' be the induced K-bases of N^+_B and t_N, respectively. This induces a map

$$(39) \qquad can_{\gamma'.\delta'} : \mathbf{1}_K \to \mathbf{d}_K(N^+_B)\mathbf{d}_K(t_N)^{-1}.$$

Furthermore, let $P^\vee = (P^\vee_1, \ldots, P^\vee_{d(N)})$ and $P = (P_1, \ldots, P_{d(N)})$ be K-bases of $\mathrm{H}^1_f(N)$ and $\mathrm{H}^1_f(N^*(1))$, respectively. Setting $P^d = (P^d_1, \ldots, P^d_{d(N)})$ the dual basis of P we obtain similarly

$$(40) \qquad can_{P^\vee,P} : \mathbf{1}_K \to \mathbf{d}_K(\mathrm{H}^1_f(N))\mathbf{d}_K(\mathrm{H}^1_f(N^*(1))^*)^{-1}.$$

Then $can := can_{\gamma'.\delta'} \cdot can_{P^\vee,P}$ is an isomorphism
$$(41)$$
$$can : \mathbf{1}_K \to \Delta_K(N) = \mathbf{d}_K(N^+_B)\mathbf{d}_K(t_N)^{-1}\mathbf{d}_K(\mathrm{H}^1_f(N))\mathbf{d}_K(\mathrm{H}^1_f(N^*(1))^*)^{-1}.$$

We fix an embedding of K into $\mathbb{C}$. Now let $\Omega_\infty(N)$ and $R_\infty(N)$ denote the determinant of the isomorphism

$$(42) \qquad \alpha_N : (N^+_B)_\mathbb{C} \to (t_N)_\mathbb{C}$$

with respect to γ' and δ', and the inverse of

$$(43) \qquad h_\infty : \left(\mathrm{H}^1_f(N^*(1))^*\right)_\mathbb{C} \to \mathrm{H}^1_f(N)_\mathbb{C}$$

with respect to P^d and P, respectively. In other words, we have

$$\Omega_\infty(N): \quad \mathbf{1}_\mathbb{C} \xrightarrow{(can_{\gamma',\delta'})_\mathbb{C}} \mathbf{d}_K(N^+_B)_\mathbb{C}\mathbf{d}_K(t_N)^{-1}_\mathbb{C} \xrightarrow{\mathbf{d}(\alpha_N)\cdot\mathrm{id}} \mathbf{1}_\mathbb{C}$$

and

$$R_\infty(N): \quad \mathbf{1}_\mathbb{C} \xrightarrow{(can_{P^\vee,P})_\mathbb{C}} \mathbf{d}_K(\mathrm{H}^1_f(N))_\mathbb{C}\mathbf{d}_K(\mathrm{H}^1_f(N^*(1))^*)^{-1}_\mathbb{C} \xrightarrow{\mathrm{id}\cdot\mathbf{d}(h_\infty)^{-1}} \mathbf{1}_\mathbb{C}.$$

Note that we obtain

$$\Omega_\infty(N)R_\infty(N): \quad \mathbf{1}_\mathbb{C} \xrightarrow{can} \Delta_K(N)_\mathbb{C} \xrightarrow{(\vartheta_\infty)_\mathbb{C}} \mathbf{1}_\mathbb{C}.$$

Upon comparing this with

$$L^*_K(M): \quad \mathbf{1}_\mathbb{C} \xrightarrow{\zeta_K(N)_\mathbb{C}} \Delta_K(N)_\mathbb{C} \xrightarrow{(\vartheta_\infty)_\mathbb{C}} \mathbf{1}_\mathbb{C}.$$

we deduce that $\zeta_K(N) : \mathbf{1}_K \to \Delta_K(N)$ coincides with

$$\frac{L^*_K(M)}{\Omega_\infty(N)R_\infty(N)} \cdot can : \mathbf{1}_K \to \Delta_K(N).$$

Also, $\mathcal{L}_U^*(\rho)$ is defined (in Definition 3.14) to be the isomorphism

$$\mathbf{1}_{\tilde{L}} \xrightarrow{\zeta_\Lambda(M)(\rho)_{\tilde{L}}} \mathbf{d}_L(\mathrm{R}\Gamma_c(U,W))_{\tilde{L}}^{-1} \xrightarrow{\beta(\rho)\epsilon(\hat{\mathbb{T}})^{-1}(\rho)}$$

$$\mathbf{d}_L(SC_U(\hat{W},W))_{\tilde{L}}^{-1} \xrightarrow{t(SC_U(\rho^*))_{\tilde{L}}} \mathbf{1}_{\tilde{L}}$$

where we set $\zeta_\Lambda(M)(\rho) := L^n \otimes_\Lambda \zeta_\Lambda(N)$, $\beta(\rho) := L^n \otimes_\Lambda \beta$ and $\epsilon(\hat{\mathbb{T}})(\rho) := L^n \otimes_\Lambda \epsilon_{p,\Lambda}(\hat{\mathbb{T}})$. Using that $\zeta_\Lambda(M)(\rho)$ equals

$$\mathbf{1}_{\tilde{L}} \xrightarrow{\zeta_K(N)_{\tilde{L}}} \Delta_K(N)_{\tilde{L}} \xrightarrow{\vartheta_\lambda} \mathbf{d}_L(\mathrm{R}\Gamma_c(U,W))_{\tilde{L}}^{-1}$$

by [14, conj. 2.3.2] (or [37, conj. 3.7, Conj. 4.1]), one sees easily that $\mathcal{L}_U^*(\rho)$ is the product of the following terms (44)-(50):

(44)
$$\frac{L_K^*(N)}{\Omega_\infty(M(\rho^*))R_\infty(N)},$$

(45)
$$\Gamma_L(\hat{W})^{-1} = \Gamma_{\mathbb{Q}_p}(\hat{V})^{-1},$$

(46) $\quad \Omega_p(M(\rho^*)): \mathbf{d}_L(\hat{W})_{\tilde{L}} \xrightarrow{\cdot \epsilon_{dR}(\hat{W})^{-1}} \mathbf{d}_L(D_{dR}(\hat{W}))_{\tilde{L}} \xrightarrow{\mathbf{d}(g_{dR}^t)} \mathbf{d}_K(t_{M(\rho^*)})_{\tilde{L}} \xrightarrow{\cdot can_{\gamma,\delta}}$

$$\mathbf{d}_K\big(M(\rho^*)_B^+\big)_{\tilde{L}} \xrightarrow{\mathbf{d}(g_\lambda^+)} \mathbf{d}_L(W^+)_{\tilde{L}} \xrightarrow{\beta(\rho)} \mathbf{d}_L(\hat{W})_{\tilde{L}}$$

where we apply Remark 2.2 to obtain an automorphism of $\mathbf{1}_{\tilde{L}}$.

(47) $\quad \displaystyle\prod_{S\backslash\{p,\infty\}} P_{L,l}(W,1): \mathbf{1}_L \xrightarrow{\prod \eta_l(W)} \prod \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_\ell,W)) \xrightarrow{acyc} \mathbf{1}_L$

where the first map comes from the trivialization by the identity and the second from the acyclicity,

(48) $\quad \{P_{L,p}(W,u)P_{L,p}(\hat{W},u)^{-1}\}_{u=1}:$

$$\mathbf{1}_L \xrightarrow{\eta_W \cdot \eta_{\hat{W}}^{-1}} \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_p,W))\mathbf{d}_L(\mathrm{R}\Gamma(\mathbb{Q}_p,\hat{W}))^{-1} \xrightarrow{quasi} \mathbf{1}_L,$$

where we use that $t_p(W) = D_{dR}(\hat{W}) = t_p(\hat{W})$ and the quasi-isomorphism mentioned in Lemma 6.2(ii),

(49) $\quad P_{L,p}(\hat{W}^*(1),1): \mathbf{1}_L \xrightarrow{\overline{(\eta_{\hat{W}^*(1)})^*}} \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_p,\hat{W}^*(1))) \xrightarrow{acyc} \mathbf{1}_L,$

where we use that $t_p(\hat{W}^*(1)) = D^0_{dR}(\hat{W}) = 0$, and the determinant over L of the isomorphism $ad(h_p(W))$ with respect to the bases $P^\vee$ and P:

$$(50) \quad R_p(N): \quad \mathbf{1}_L \xrightarrow{(can_{P^\vee,P})_L} \mathbf{d}_K(\mathrm{H}^1_f(N))_L \mathbf{d}_K(\mathrm{H}^1_f(N^*(1))^*)^{-1}_L \xrightarrow{\cong}$$

$$\mathbf{d}_K(\mathrm{H}^1_f(\mathbb{Q},W))_L \mathbf{d}_K(\mathrm{H}^1_f(\mathbb{Q},Z)^*)^{-1}_L \xrightarrow{h_p(W)} \mathbf{1}_L.$$

Indeed, as remarked above $\zeta_\Lambda(M)(\rho)$ decomposes up to the comparison isomorphism $\mathbf{d}(g^+_\lambda)$, which contributes to factor (46), into $\zeta_K(N)_L$ and ϑ_λ. While $\zeta_K(N)_L$ gives the full factor (44) and contributes with $can_{\gamma,\delta}$ and $can_{P^\vee,P}$ to the factors (46) and (50), respectively, the second part ϑ_λ gives the full factor (47), the half factor (48) in form of $\eta(W)$ and contributes $\mathbf{d}(g^+_{dR})$ to factor (46). Further, $\beta(\rho)$ contributes to factor (46), while according to [14, §3.3] $\epsilon(\hat{\mathbb{T}})^{-1}(\rho) = \epsilon_{p,L}(\hat{W})^{-1}$ gives the full factors (45) and (49), the other half of (48) in form of $\eta^{-1}_{\hat{W}}$ and adds $\epsilon_{dR}(\hat{W})$ to factor (46). Finally, we had observed at the end of §6.2 that $t(SC_U(\rho^*))$ equals $h_p(W)$.

In order to derive the corresponding formula for $\mathcal{L}^*(\rho)$ we use the exact triangle

$$SC_U(\hat{W},W) \to L^n \otimes^{\mathbb{L}}_\Lambda SC(\hat{\mathbb{T}},\mathbb{T}) \to \bigoplus_{S_f \backslash (S_p \cup \Upsilon)} \mathrm{R}\Gamma_f(\mathbb{Q}_\ell,W) \to$$

and the equality

$$L^*_{K,\Upsilon'}(N) = L^*_{K,B'}(N) \prod_{B \backslash \Upsilon} P_{L,l}(W,1)^{-1}$$

with $\Upsilon' = \Upsilon \cup \{p\}$ and $B' = S \setminus S_\infty$. $\qquad\square$

EXAMPLE 6.9. Let E be an elliptic curve defined over $\mathbb{Q}$, $M = h^1(E)(1)$ and $F_\infty := \mathbb{Q}(E(p))$ where $E(p) \subseteq E(\overline{\mathbb{Q}})$ denotes the group of p-power torsion points of E. It is conjectured that $SC_U(\hat{\mathbb{T}},\mathbb{T})$ always belongs to Σ_{S^*} (cf. [10, conj. 5.1] and [14, 4.3.5 and prop. 4.3.7]). As was shown in [14] the existence of $\mathcal{L}(M)$ implies the existence of $\mathcal{L}_E \in K_1(\Lambda(G)_{S^*})$ as conjectured in [10, conj. 5.7] to exist with a prediscribed interpolation property $\mathcal{L}_E(\rho)$ for $r_G(SC(\hat{\mathbb{T}},\mathbb{T}))(\rho) = 0$. Now the above formula (38) predicts the leading terms $\mathcal{L}^*_E(\rho)$ for all Artin characters with non-degenerate archimedean and p-adic height pairing.

REFERENCES

[1] D. Bertrand, Valuers de fonctions theta et hauteur p-adiques, Seminar on Number Theory, Paris, 1980/81, Progress in Mathematics 22 (Birkhäuser, Boston, 1982) 111. 6.5

[2] M. Breuning and D. Burns, On the leading terms of Artin L-functions at $s = 0$ and $s = 1$, manuscript submitted. 5

[3] D. Burns, On the values of equivariant Zeta functions of curves over finite fields, Documenta Math. 9 (2004) 357-399. 1

[4] D. Burns, Algebraic p-adic L-functions and main conjectures in non-commutative Iwasawa theory, manuscript in preparation. 1

[5] D. Burns and M. Flach, Motivic L-functions and Galois module structures, Math. Ann. 305 (1996) 65-102. 5.3

[6] D. Burns and M. Flach, Equivariant Tamagawa numbers for motives with (non-commutative) coefficients, Documenta Math. 6 (2001) 501-570. 1, 4.1, 4.2, 4.2, 4.3

[7] D. Burns and M. Flach, On the equivariant Tamagawa number conjecture for Tate motives, Part II preprint 2005. 5

[8] D. Burns and C. Greither, On the equivariant Tamagawa number conjecture for Tate motives, Invent. Math. 153 (2003), no. 2, 303–359. 1, 5.1

[9] P. Cassou-Noguès, Valeurs aux entiers négatif des fonctions zêta et fonctions zêta p-adiques, Invent. Math. 51 (1979) 29-59. 5.2

[10] J. Coates, T. Fukaya, K. Kato, R. Sujatha, and O. Venjakob, The GL_2 main conjecture for elliptic curves without complex multiplication, to appear in Publ. Math. I.H.E.S. 1, 3, 3.4, 3.4.1, 3.5, 3.20, 6.9

[11] J. Coates, P. Schneider, and R. Sujatha, Links between cyclotomic and GL_2 Iwasawa theory, Documenta Math., Extra Volume: Kazuya Kato's Fiftieth Birthday, (2003), 187-215. 3.5

[12] P. Colmez, Résidu en $s = 1$ des fonctions zêta p-adiques, Inventiones Math. 91 (1988) 371-389. 5.4

[13] P. Deligne and K. Ribet, Values of Abelian L-functions at negative integers over totally real fields, Invent. Math. 59 (1980) 227-286. 5.2

[14] T. Fukaya and K. Kato, A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory, to appear in Proc. St . Petersburg Math. Soc. 11 (2005). 1, 2.2, 2.6, 2.2, 3.15, 3.4.2, 3.5, 3.20, 4, 4.1, 4.1, 4.2, 4.3, 4.1, 6.1, 6.1, 6.2, 6.4, 6.2, 6.3, 6.3, 6.3, 6.3, 6.3, 6.9

[15] R. Greenberg, On p-adic Artin L-functions, Nagoya Math. J. 89 (1983) 77-87. 5.2

[16] Y. Hachimori and O. Venjakob, Completely faithful Selmer groups over Kummer extensions, Documenta Math., Extra Volume: Kazuya Kato's Fiftieth Birthday, (2003), 443–478. 3.5

[17] A. Huber and G. Kings, Equivariant Bloch-Kato conjecture and non-abelian Iwasawa main conjecture, *in* Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 149–162, Higher Ed. Press, Beijing, 2002; 1, 4.3

[18] F. Knudsen, Determinant functors on exact categories and their extensions to categories of bounded complexes, Michigan Math. J. 50 (2002), 407-444. 2.1

[19] B. Mazur and K. Rubin, Finding large Selmer groups, preprint 2005. 1

[20] B. Mazur and J. Tate, Canonical height pairings via biextensions, in *Arithmetic and geometry, Vol. I*, 195–237, Progr. Math., 35, Birkhäuser, 1983. 6.2

[21] J. Nekovar, On p-adic height pairings, in *Séminaire de Théorie des Nombres, Paris, 1990–91*, 127–202, Progr. Math., 108, Birkhäuser, 1993. 6.2, 6.2, 6.2

[22] J. Nekovar, *Selmer complexes*, preprint (2001). 1, 6.2

[23] J. Neukirch, A. Schmidt, and K. Wingberg, Cohomology of number fields, Grundlehren der mathematischen Wissenschaften, vol. 323, Springer, 2000. 3.1

[24] B. Perrin-Riou, Théorie d'Iwasawa et hauteurs p-adiques Invent. Math. 109 (1992), no. 1, 137–185. 6.2

[25] B. Perrin-Riou, Théorie d'Iwasawa et hauteurs p-adiques (cas des variétés abéliennes), Séminaire de Théorie des Nombres, Paris, 1990–91, Progr. Math., vol. 108, Birkhäuser Boston, Boston, MA, 1993, pp. 203–220. 1

[26] B. Perrin-Riou, Representations p-adiques ordinaires, Asterisque 223 (1994), 185–220. 6.3

[27] B. Perrin-Riou, p-adic L-functions and p-adic representations, SMF/AMS Texts and Monographs, vol. 3, American Mathematical Society, Providence, RI, 2000. 1, 6.1, 6.8

[28] M. Rapoport, Th. Zink, Über die lokale Zetafunktion von Shimuravarietäten. Mon-odromiefiltration und verschwindende Zyklen in ungleicher Charakteristik, Invent. math. 68 (1982) 21-101. 3.2

[29] P. Schneider, Height pairings in the Iwasawa theory of abelian varieties, in *Seminar on Number Theory, Paris 1980-81 (Paris, 1980/1981)*, 309–316, Progr. Math., 22, Birkhäuser, 1982. 1

[30] P. Schneider, p-adic height pairings. I., Invent. Math. 69 (1982), 401–409. 1, 6.2

[31] P. Schneider, Iwasawa L-functions of varieties over algebraic number fields. A first approach, Invent. Math. 71 (1983), no. 2, 251–293. 1

[32] P. Schneider, p-adic height pairings. II., Invent. Math. 79 (1985), 329–374. 1

[33] J-P. Serre, Sur le résidue de la fonction zêta p-adique d'un corps de nombres, C.R. Acad. Sci. Paris 278 (1978) 183-188. 1, 5, 5.3

[34] D. Solomon, p-adic abelian Stark conjecture at $s = 1$, Ann. Inst. Fourier 52 (2002) 379-417. 5.3

[35] J. Tate, Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$ (notes par D.Bernardi et N. Schappacher), Progress in Math., 47, Birkhäuser, Boston, 1984. 1, 5, 5.2, 5.3, 5.4, 5.3

[36] O. Venjakob, Characteristic Elements in Noncommutative Iwasawa Theory, J. reine angew. Math., 583, 2005. 3.5, 3.20

[37] O. Venjakob, From the Birch & Swinnerton-Dyer Conjecture over the Equivariant Tamagawa Number Conjecture to non-commutative Iwasawa theory - a survey, arXiv:math.NT/0507275 (2005). 1, 1, 4.1, 4.1, 4.2, 4.1, 6.1, 6.3, 6.3, 6.3

[38] C. Wuthrich On p-adic heights in families of elliptic curves, J. London Math. Soc. (2) 70 (2004), no. 1 , 23–40. 6.5

[39] Y. Zarhin, Neron coupling and quasicharacters, Izv. Akad. Nauk SSSR Ser. Mat. 36 (1972), 497–509. 6.2

[40] S. Zerbes, Selmer groups over p-adic Lie extensions. I, J. London Math. Soc. (2) 70 (2004), no. 3, 586–608. 3.5

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